Conditions for coherence transformations under incoherent operations

Shuanping Du,[1,*] Zhaofang Bai,[1,†] and Yu Guo[2,‡]
[1]*School of Mathematical Sciences, Xiamen University, Xiamen, Fujian, 361000, China*
[2]*School of Mathematics and Computer Science, Shanxi Datong University, Datong, Shanxi, 037009, China*
(Received 17 March 2015; published 27 May 2015)

We build in this paper the counterpart of the celebrated Nielsen theorem for coherence manipulation. This offers an affirmative answer to the open question: whether, given two states ρ and σ, either ρ can be transformed into σ or vice versa under incoherent operations [Baumgratz *et al.*, Phys. Rev. Lett. **113**, 140401 (2014)]. As a consequence, we find that there exist essentially different types of coherence. Moreover, incoherent operations can be enhanced in the presence of certain coherent states. These extra states are coherent catalysts: they allow uncertain incoherent operations to be realized without being consumed in any way. Our main result also sheds light on the construction of coherence measures.

DOI: 10.1103/PhysRevA.91.052120 PACS number(s): 03.65.Ud, 03.67.Mn, 03.65.Ta

Introduction. Superposition is a critical property of quantum systems resulting in quantum coherence and quantum entanglement. Quantum coherence and also entanglement provide important resources for quantum information processing; for example, the Deutschs algorithm, the Shors algorithm, teleportation, superdense coding, and quantum cryptography [1]. As with any such resource, there arises naturally the question of how it can be quantified and manipulated. Attempts have been made to find meaningful measures of entanglement [2–6], and also to uncover the fundamental laws of its behavior under local quantum operations and classical communication (LOCC) [2–12]. The celebrated Nielsen theorem finds possible entanglement manipulation between bipartite entangled states by LOCC [7]. Let $|\psi\rangle = \sum_{i=1}^{d}\sqrt{\psi_j}|jj\rangle$ and $|\phi\rangle = \sum_{i=1}^{d}\sqrt{\phi_j}|jj\rangle$ be two bipartite states whose Schmidt coefficients are ordered in decreasing order, $\psi_1 \geqslant \psi_2 \geqslant \cdots \geqslant \psi_d$, $\phi_1 \geqslant \phi_2 \geqslant \cdots \geqslant \phi_d$. Then $|\psi\rangle \to |\phi\rangle$ by LOCC if and only if $(\psi_1,\psi_2,\ldots,\psi_d) \prec (\phi_1,\phi_2,\ldots,\phi_d)$. This reveals a partial ordering of the entangled states and connects quantum entanglement to the algebraic theory of majorization.

In [13], the researchers establish a rigorous framework for the quantification of coherence as a resource following the viewpoints that have been established for entanglement in [6]. The setting of single copies of coherent states is of considerable interest from the practical point of view as this is most readily accessible in the laboratory. It is expected that a theory of coherence manipulation can be established that proceeds along the lines of analogous developments in entanglement theory [13]. The aim of this paper is to build the counterpart of Nielsen theorem for coherence manipulation. What is surprising is that majorization is also a key ingredient. It provides the relevant structure that determines the interconvertibility of coherent states.

Majorization is an active research area in linear algebra. We use Chap. 2 of [14] as our principal reference on majorization. Suppose $x = (x_1,x_2,\ldots,x_d)^t$ and $y = (y_1,y_2,\ldots,y_d)^t$ are real d-dimensional vectors; here $x = (x_1,x_2,\ldots,x_d)^t$ denotes the transpose of the row vector $(x_1,x_2,\ldots,x_d)$. Then x is majorized by y (equivalently y majorizes x), which is written $x \prec y$, if for each k in the range $1,\ldots,d$, $\sum_{i=1}^{k} x_i^{\downarrow} \leqslant \sum_{i=1}^{k} y_i^{\downarrow}$ with the equality holding when $k = d$, and where $x_i^{\downarrow}$ indicates that elements are to be taken in descending order; so, for example, $x_1^{\downarrow}$ is the largest element in $(x_1,\ldots,x_d)$. The majorization relation is a partial order on real vectors, with $x \prec y$ and $y \prec x$ if and only if $x^{\downarrow} = y^{\downarrow}$.

In the following, we introduce the concepts of incoherent states and incoherent operations taken from [13]. Let $\mathcal{H}$ be a finite-dimensional Hilbert space with $\dim(\mathcal{H}) = d$. Fixing a particular basis $\{|i\rangle\}_{i=1}^{d}$, we call all density operators (quantum states) that are diagonal in this basis incoherent. This set of quantum states is labeled by $\mathcal{I}$; all density operators $\rho \in \mathcal{I}$ are of the form

$$\rho = \sum_{i=1}^{d} \lambda_i |i\rangle\langle i|. \tag{1}$$

Quantum operations are specified by a finite set of Kraus operators $\{K_n\}$ satisfying $\sum_n K_n^{\dagger}K_n = I$, where I is the identity operator on $\mathcal{H}$. Quantum operations are incoherent if they fulfill $K_n\rho K_n^{\dagger}/\mathrm{Tr}(K_n\rho K_n^{\dagger}) \in \mathcal{I}$ for all $\rho \in \mathcal{I}$ and for all n.

Results. To state our central result linking coherence manipulation with majorization, we need some notation. Suppose $|\psi\rangle = \sum_{i=1}^{d}\psi_i|i\rangle$ and $|\phi\rangle = \sum_{i=1}^{d}\phi_i|i\rangle$ are any pure states. $|\psi\rangle \xrightarrow{\text{ico}} |\phi\rangle$ (read "$|\psi\rangle$ transforms incoherently to $|\phi\rangle$") indicates that $|\psi\rangle\langle\psi|$ transforms to $|\phi\rangle\langle\phi|$ by incoherent operations. Then we have the following theorem:

Theorem 1. $|\psi\rangle$ transforms to $|\phi\rangle$ using incoherent operations if and only if $(|\psi_1|^2,\ldots,|\psi_d|^2)^t$ is majorized by $(|\phi_1|^2,\ldots,|\phi_d|^2)^t$. More succinctly,

$$\begin{aligned}&|\psi\rangle \xrightarrow{\text{ico}} |\phi\rangle \quad \text{if and only if}\\ &(|\psi_1|^2,\ldots,|\psi_d|^2)^t \prec (|\phi_1|^2,\ldots,|\phi_d|^2)^t.\end{aligned} \tag{2}$$

One direct consequence of Theorem 1 is that there exist pairs $|\psi\rangle$ and $|\phi\rangle$ with neither $|\psi\rangle \xrightarrow{\text{ico}} |\phi\rangle$ nor $|\phi\rangle \xrightarrow{\text{ico}} |\psi\rangle$. For example, when $d = 3$,

$$|\psi\rangle = \sqrt{0.4}|1\rangle + \sqrt{0.3}|2\rangle + \sqrt{0.3}|3\rangle, \tag{3}$$

$$|\phi\rangle = \sqrt{0.5}|1\rangle + \sqrt{0.1}|2\rangle + \sqrt{0.4}|3\rangle. \tag{4}$$

*shuanpingdu@yahoo.com
†Corresponding author: baizhaofang@xmu.edu.cn
‡guoyu3@aliyun.com

These provide an example of essentially different types of coherence, from the point of view of incoherent operations. We will say that $|\psi\rangle$ and $|\phi\rangle$ are incomparable in coherence. In addition, for any two pure states $|\psi\rangle$ and $|\phi\rangle$, $|\psi\rangle$ and $|\phi\rangle$ can be incomparable with respect to incoherence under a change of basis. This may seem odd at first, but it turns out that coherence is a basis-dependent phenomenon.

For entanglement transformations, a major interest has been catalysis. This enables the conversion between two initially inconvertible entangled states assisted by a borrowed entangled state, which is recovered at the end of the process [10,15–19]. For two states $|\psi\rangle$ and $|\phi\rangle$ which are incomparable in coherence, if $|\psi\rangle|\delta\rangle \xrightarrow{\text{ico}} |\phi\rangle|\delta\rangle$, we say $|\psi\rangle$ is transformed into $|\phi\rangle$ under coherence-assisted incoherent operations, and $|\delta\rangle$ is called a coherent catalyst. This state acts much like a catalyst in a chemical reaction: its presence allows a previously forbidden transformation to be realized, and since it is not consumed it can be reused. Here we use the phrase "coherence assisted" because $|\delta\rangle$ must be coherent. Combining Theorem 1 and proofs of Lemmas 1, 2, and 3 in [10], we immediately have the following interesting results:

(i) No incoherent transformation can be catalyzed by a maximally coherent state $|\psi_d\rangle = \sum_{k=1}^{d} \frac{1}{\sqrt{d}}|k\rangle$. This shows a surprising property of coherent catalysts: they must be partially coherent. If the catalyst has not enough coherence, then $|\psi\rangle$ cannot be transformed into $|\phi\rangle$ with certainty, but if it has too much then the result is same.

(ii) Two states are interconvertible (i.e., both $|\psi\rangle \to |\phi\rangle$ and $|\phi\rangle \to |\psi\rangle$) under coherence-assisted incoherent operations if and only if they are equivalent up to a permutation of diagonal unitary transformations. One consequence of this result is that, if a transition that is forbidden under incoherent operation can be catalyzed (i.e., $|\psi\rangle \nrightarrow |\phi\rangle$ under incoherent operation but $|\psi\rangle|\delta\rangle \to |\phi\rangle|\delta\rangle$ for some $|\delta\rangle$), then the reverse transition (from $|\phi\rangle \to |\psi\rangle$) cannot be catalyzed. In particular, only transitions between incomparable states may be catalyzed.

(iii) $|\psi\rangle \to |\phi\rangle$ under coherence-assisted incoherent operation only if both $|\psi_1| \leqslant |\phi_1|$ and $|\psi_d| \geqslant |\phi_d|$.

Theorem 1 provides a necessary condition for coherence measures. From [13], coherence measures should satisfy monotonicity under incoherent operations, i.e., $\mathcal{C}(\Phi(\rho)) \leqslant \mathcal{C}(\rho)$ for any incoherent operation Φ and state ρ. Let $|\psi\rangle = \psi_1|1\rangle + \cdots + \psi_d|d\rangle$, $|\phi\rangle = \phi_1|1\rangle + \cdots + \phi_d|d\rangle$, with $(|\psi_1|^2,\ldots,|\psi_d|^2)^t \prec (|\phi_1|^2,\ldots,|\phi_d|^2)^t$. By Theorem 1, we have $\mathcal{C}(|\phi\rangle\langle\phi|) \leqslant \mathcal{C}(|\psi\rangle\langle\psi|)$. This necessary condition of the coherence measure implies that Result 1 in [20] is not true. That is, the Wigner-Yanase-Dyson skew information

$$\mathcal{C}(\rho,K) = -\tfrac{1}{2}\mathrm{Tr}([\sqrt{\rho},K]^2) \tag{5}$$

is not a good coherence measure since it violates this necessary condition. Assume $d = 3$; let

$$\begin{aligned} K &= |1\rangle\langle 1| + 10|2\rangle\langle 2| + 5|3\rangle\langle 3|,\\ |\psi\rangle &= \frac{1}{\sqrt{3}}|1\rangle + \frac{1}{\sqrt{3}}|2\rangle + \frac{1}{\sqrt{3}}|3\rangle,\\ |\phi\rangle &= \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|2\rangle. \end{aligned} \tag{6}$$

It is easy to check that $(\frac{1}{3},\frac{1}{3},\frac{1}{3})^t \prec (\frac{1}{2},\frac{1}{2},0)^t$ and

$$\mathcal{C}(|\phi\rangle\langle\phi|,K) = \tfrac{81}{4} > \mathcal{C}(|\psi\rangle\langle\psi|,K) = \tfrac{122}{9}. \tag{7}$$

The following construction of coherent measures originates from Theorem 1. For an arbitrary pure state $|\psi\rangle = \sum_{i=1}^{d}\psi_i|i\rangle$, we define $C_l(|\psi\rangle\langle\psi|) = \sum_{i=l}^{d}|\psi_i|^{2\downarrow}$ $(l = 2,3,\ldots,d)$; here $(|\psi_1|^{2\downarrow},|\psi_2|^{2\downarrow},\ldots,|\psi_d|^{2\downarrow})^t$ is the vector obtained by rearranging the coordinates of $(|\psi_1|^2,|\psi_2|^2,\ldots,|\psi_d|^2)^t$ in decreasing order, and extending it over the whole set of density matrices as $C_l(\rho) = \min_{p_j,\rho_j}\sum_j p_j C_l(\rho_j)$, where the minimization is to be performed over all the pure-state ensembles of ρ, i.e., $\rho = \sum_j p_j\rho_j$. In [21], we show that C_l are coherence measures.

Theorem 1 also paves the way for the following question: suppose there is a pure coherent state $|\psi\rangle = \sum_{i=1}^{d}\psi_i|i\rangle$ which we would like to convert into another pure coherent state $|\phi\rangle = \sum_{i=1}^{d}\phi_i|i\rangle$ by incoherent operations. What is the greatest probability of success in such a conversion? In [21], we give the explicit formula for the greatest probability $P(|\psi\rangle \xrightarrow{\text{ico}} |\phi\rangle)$. A parallel result in entanglement theory is the optimal local conversion strategy between any two pure entangled states of a bipartite system [8].

Proof. Now we do some preparatory work to prove Theorem 1 by collecting some useful facts:

(i) For real vectors x,y, $x \prec y$ if and only if $x = Ay$ for some doubly stochastic matrix. Recall that a $d\times d$ matrix $A = (a_{ij})$ is called doubly stochastic if $a_{ij} \geqslant 0$ and $\sum_{i=1}^{d}a_{ij} = \sum_{j=1}^{d}a_{ij} = 1$.

(ii) For every doubly stochastic matrix A, it is a matrix that may be written as a product of at most $d-1$ T transforms. A T transform, by definition, acts as the identity on all but two matrix components. On those two components, it has the form

$$T = \begin{pmatrix} t & 1-t \\ 1-t & t \end{pmatrix}, \tag{8}$$

where $0 \leqslant t \leqslant 1$. In terms of transformation, $T(x_1,x_2,\ldots,x_d)^t = (x_1,\ldots,x_{i-1},tx_i + (1-t)x_j,x_{i+1},\ldots x_{j-1},(1-t)x_i + tx_j,x_{j+1},\ldots,x_d)^t$ for some indices i,j and $0 \leqslant t \leqslant 1$.

(iii) Let π be a permutation of $\{1,2,\ldots,d\}$ and P_π be the permutation matrix corresponding to π that is obtained by permuting the rows of a $d\times d$ identity matrix according to π. A permutation matrix has exactly one entry 1 in each row and each column and 0 elsewhere.

(iv) For the quantum operation $\Phi(\cdot) = \sum_n K_n K_n^\dagger$, it is easy to see that Φ is incoherent if and only if every column of K_n in the fixed basis $\{|i\rangle\}_{i=1}^{d}$ has at most one nonzero entry.

Now, we are in a position to give the proof of Theorem 1.

Proof. First, we can suppose that all $\psi_k,\phi_k (k = 1,2,\ldots,d)$ are non-negative and sorted in descending order. Indeed, in the general case, let $\psi_k = |\psi_k|e^{i\alpha_k}$, $\phi_k = |\phi_k|e^{i\beta_k}$ and $|\psi_{\pi(1)}| \geqslant |\psi_{\pi(2)}| \geqslant \cdots \geqslant |\psi_{\pi(d)}|$, $|\phi_{\sigma(1)}| \geqslant |\phi_{\sigma(2)}| \geqslant \cdots \geqslant |\phi_{\sigma(d)}|$, where π,σ are two permutations of $\{1,2,\ldots,d\}$. One can define $U = P_\pi\mathrm{diag}(e^{-i\alpha_1},e^{-i\alpha_2},\ldots,e^{-i\alpha_d})$ and $V = P_\sigma\mathrm{diag}(e^{-i\beta_1},e^{-i\beta_2},\ldots,e^{-i\beta_d})$; here P_π and P_σ are permutation matrices corresponding to π and σ, respectively. Note that $U|\psi\rangle \xrightarrow{\text{ico}} V|\phi\rangle \Leftrightarrow |\psi\rangle \xrightarrow{\text{ico}} |\phi\rangle$; we can replace $|\psi\rangle$ and $|\phi\rangle$ by $U|\psi\rangle$ and $V|\phi\rangle$.

Now, we prove the "if" part. Assume that $(|\psi_1|^2,\ldots,|\psi_d|^2)^t \prec (|\phi_1|^2,\ldots,|\phi_d|^2)^t$. We will apply the inductive method.

Assume $\dim H = 2$. If $\psi_2 = 0$, from the majorization, it follows that $\phi_2 = 0$. That is, $|\psi\rangle = |\phi\rangle = |1\rangle$. Then the identity operation is the one desired. Now we may suppose $\psi_2 \neq 0$. Let $A = \begin{pmatrix} a & 1-a \\ 1-a & a \end{pmatrix}$ $(0 \leqslant a \leqslant 1)$ be a doubly stochastic matrix such that

$$\begin{pmatrix}\psi_1^2\\ \psi_2^2\end{pmatrix} = \begin{pmatrix} a & 1-a \\ 1-a & a\end{pmatrix}\begin{pmatrix}\phi_1^2\\ \phi_2^2\end{pmatrix}. \tag{9}$$

Define

$$K_1 = \begin{pmatrix}\sqrt{a}\frac{\phi_1}{\psi_1} & 0\\ 0 & \sqrt{a}\frac{\phi_2}{\psi_2}\end{pmatrix}, \tag{10}$$

$$K_2 = \begin{pmatrix} 0 & \sqrt{1-a}\frac{\phi_1}{\psi_2}\\ \sqrt{1-a}\frac{\phi_2}{\psi_1} & 0\end{pmatrix}. \tag{11}$$

One can check that the incoherent operation whose Kraus operators are K_1 and K_2 is the one desired.

Assume that the result holds true for $\dim H \leqslant d-1$; we will prove that the result holds true for $\dim H = d$ and divide the proof into two cases.

Case 1. There is a k $(1 < k < d)$ such that $\psi_k \neq 0$ and $\psi_{k+1} = \cdots = \psi_d = 0$. From the majorization, it follows that $\phi_{k+1} = \cdots = \phi_d = 0$. The k-level vector $(|\psi_1|^2,\ldots,|\psi_k|^2)^t$ is majorized by $(|\phi_1|^2,\ldots,|\phi_k|^2)^t$. From the inductive assumption, there is an incoherent operation $\widetilde{\Phi}$ on M_k [the set of all $(k\times k)$-level matrices], with the specified Kraus operators $\widetilde{K_n}$ $(n = 1,2,\ldots,N)$ such that $\sum_{i=1}^k \psi_i|i\rangle \xrightarrow{\widetilde{\Phi}} \sum_{i=1}^k \phi_i|i\rangle$. Define $K_n = \widetilde{K_n} \oplus \frac{1}{\sqrt{N}} I_{d-k}$; then $\Phi(\cdot) = \sum_{i=1}^N K_n K_n^\dagger$ is an incoherent operation which transforms $|\psi\rangle\langle\psi|$ to $|\phi\rangle\langle\phi|$.

Case 2. $\psi_d \neq 0$. Let A be a doubly stochastic matrix with $(|\psi_1|^2,\ldots,|\psi_d|^2)^t = A(|\phi_1|^2,\ldots,|\phi_d|^2)^t$. Note that the composition of incoherent operations is also incoherent; by the fact (ii), A can be reduced to a T transform for some indices i,j and $0 \leqslant t \leqslant 1$. Let $\pi = (1,2,\ldots,i-1,j,i+1,\ldots,j-1,i,j+1,\ldots,d)$ be a permutation of $\{1,2,\ldots,d\}$, and

$$K_1 = \sqrt{t}\,\mathrm{diag}\left(\frac{\phi_1}{\psi_1},\ldots,\frac{\phi_d}{\psi_d}\right), \tag{12}$$

$$K_2 = \sqrt{1-t}\,\mathrm{diag}\left(\frac{\phi_1}{\psi_1},\ldots,\frac{\phi_{i-1}}{\psi_{i-1}},\frac{\phi_i}{\psi_j},\frac{\phi_{i+1}}{\psi_{i+1}},\ldots,\frac{\phi_{j-1}}{\psi_{j-1}},\frac{\phi_j}{\psi_i},\frac{\phi_{j+1}}{\psi_{j+1}},\ldots,\frac{\phi_d}{\psi_d}\right)P_\pi. \tag{13}$$

Then

$$K_1^\dagger K_1 = t\,\mathrm{diag}\left(\frac{\phi_1^2}{\psi_1^2},\ldots,\frac{\phi_d^2}{\psi_d^2}\right), \tag{14}$$

$$K_2^\dagger K_2 = (1-t)\,\mathrm{diag}\left(\frac{\phi_1^2}{\psi_1^2},\ldots,\frac{\phi_{i-1}^2}{\psi_{i-1}^2},\frac{\phi_j^2}{\psi_i^2},\frac{\phi_{i+1}^2}{\psi_{i+1}^2},\ldots,\frac{\phi_{j-1}^2}{\psi_{j-1}^2},\frac{\phi_i^2}{\psi_j^2},\frac{\phi_{j+1}^2}{\psi_{j+1}^2},\ldots,\frac{\phi_d^2}{\psi_d^2}\right). \tag{15}$$

From $(|\psi_1|^2,\ldots,|\psi_d|^2)^t = A(|\phi_1|^2,\ldots,|\phi_d|^2)^t$, it follows that $K_1^\dagger K_1 + K_2^\dagger K_2 = I$. Furthermore, it is easy to check that $\Phi(\cdot) = \sum_{n=1}^2 K_n K_n^\dagger$ transforms $|\psi\rangle\langle\psi|$ to $|\phi\rangle\langle\phi|$. Note that each column of K_n $(n = 1,2)$ has at most one nonzero entry, so Φ is incoherent. This finishes the proof of the "if" part.

To prove the converse, we consider only the three-dimensional case; other cases can be treated similarly. Now, we suppose that $\dim H = 3$ and there is an incoherent operation Φ that transforms $|\psi\rangle\langle\psi|$ to $|\phi\rangle\langle\phi|$. Let

$$\Phi(|\psi\rangle\langle\psi|) = \sum_n K_n|\psi\rangle\langle\psi|K_n^\dagger = |\phi\rangle\langle\phi|. \tag{16}$$

Hence there exist complex numbers α_n such that $K_n|\psi\rangle = \alpha_n|\phi\rangle$. Let $k_j^{(n)}$ $(j = 1,2,3)$ be the nonzero element of K_n in the jth column (if there is no nonzero element in the jth column, then $k_j^{(n)} = 0$). Suppose $k_j^{(n)}$ is located in the $i(j)$th row. Let

$$\delta_{s,t} = \begin{cases}1, & s = t\\ 0, & s\neq t\end{cases}.$$

Then there is a permutation π_n such that

$$K_n = P_{\pi_n}\begin{pmatrix} k_1^{(n)} & \delta_{1,i(2)}k_2^{(n)} & \delta_{1,i(3)}k_3^{(n)}\\ 0 & \delta_{2,i(2)}k_2^{(n)} & \delta_{2,i(3)}k_3^{(n)}\\ 0 & 0 & \delta_{3,i(3)}k_3^{(n)}\end{pmatrix}. \tag{17}$$

From $\sum_n K_n^\dagger K_n = I$, we get that

$$\begin{aligned} &\sum_n |k_j^{(n)}|^2 = 1, \quad (j = 1,2,3),\\ &\sum_n \overline{k_1^{(n)}}\delta_{1,i(2)}k_2^{(n)} = 0,\\ &\sum_n \overline{k_1^{(n)}}\delta_{1,i(3)}k_3^{(n)} = 0,\\ &\sum_n (\delta_{1,i(2)}\delta_{1,i(3)} + \delta_{2,i(2)}\delta_{2,i(3)})\overline{k_2^{(n)}}k_3^{(n)} = 0.\end{aligned} \tag{18}$$

By a direct computation, one can get

$$K_n|\psi\rangle = P_{\pi_n}\begin{pmatrix} k_1^{(n)}\psi_1 + \delta_{1,i(2)}k_2^{(n)}\psi_2 + \delta_{1,i(3)}k_3^{(n)}\psi_3 \\ \delta_{2,i(2)}k_2^{(n)}\psi_2 + \delta_{2,i(3)}k_3^{(n)}\psi_3 \\ \delta_{3,i(3)}k_3^{(n)}\psi_3 \end{pmatrix}, \tag{19}$$

and so

$$\begin{aligned} k_1^{(n)}\psi_1 + \delta_{1,i(2)}k_2^{(n)}\psi_2 + \delta_{1,i(3)}k_3^{(n)}\psi_3 &= \alpha_n\phi_{\pi_n^{-1}(1)}, \\ \delta_{2,i(2)}k_2^{(n)}\psi_2 + \delta_{2,i(3)}k_3^{(n)}\psi_3 &= \alpha_n\phi_{\pi_n^{-1}(2)}, \\ \delta_{3,i(3)}k_3^{(n)}\psi_3 &= \alpha_n\phi_{\pi_n^{-1}(3)}. \end{aligned} \tag{20}$$

Applying $\sum_n |\cdot|^2$ to the above equations, we have

$$\begin{aligned} \psi_1^2 + \delta_{1,i(2)}\psi_2^2 + \delta_{1,i(3)}\psi_3^2 &= \sum_n |\alpha_n|^2\phi^2_{\pi_n^{-1}(1)}, \\ \delta_{2,i(2)}\psi_2^2 + \delta_{2,i(3)}\psi_3^2 &= \sum_n |\alpha_n|^2\phi^2_{\pi_n^{-1}(2)}, \\ \delta_{3,i(3)}\psi_3^2 &= \sum_n |\alpha_n|^2\phi^2_{\pi_n^{-1}(3)}. \end{aligned} \tag{21}$$

Note that, for $s = 1,2,3$,

$$\sum_n |\alpha_n|^2\phi^2_{\pi_n^{-1}(s)} = \sum_{n,\pi_n^{-1}(s)=1} |\alpha_n|^2\phi_1^2 + \sum_{n,\pi_n^{-1}(s)=2} |\alpha_n|^2\phi_2^2 + \sum_{n,\pi_n^{-1}(s)=3} |\alpha_n|^2\phi_3^2. \tag{22}$$

Let $d_{ij} = \sum_{n,\pi_n^{-1}(i)=j} |\alpha_n|^2$, $1 \leqslant i, j \leqslant 3$; then the matrix $D = (d_{ij})$ is a doubly stochastic matrix, since $\sum_n |\alpha_n|^2 = 1$. Furthermore,

$$D(\phi_1^2,\phi_2^2,\phi_3^2)^t = (\psi_1^2 + \delta_{1,i(2)}\psi_2^2 + \delta_{1,i(3)}\psi_3^2, \delta_{2,i(2)}\psi_2^2 + \delta_{2,i(3)}\psi_3^2, \delta_{3,i(3)}\psi_3^2)^t. \tag{23}$$

This implies that

$$(\psi_1^2 + \delta_{1,i(2)}\psi_2^2 + \delta_{1,i(3)}\psi_3^2, \delta_{2,i(2)}\psi_2^2 + \delta_{2,i(3)}\psi_3^2, \delta_{3,i(3)}\psi_3^2)^t \prec (\phi_1^2,\phi_2^2,\phi_3^2)^t. \tag{24}$$

It is easy to check that

$$(\psi_1^2,\psi_2^2,\psi_3^2)^t \prec (\psi_1^2 + \delta_{1,i(2)}\psi_2^2 + \delta_{1,i(3)}\psi_3^2, \delta_{2,i(2)}\psi_2^2 + \delta_{2,i(3)}\psi_3^2, \delta_{3,i(3)}\psi_3^2)^t. \tag{25}$$

Therefore $(\psi_1^2,\psi_2^2,\psi_3^2)^t \prec (\phi_1^2,\phi_2^2,\phi_3^2)^t$.

Outlook. Our results raise many interesting questions. It would be of great interest to determine when a mixed state ρ can be transformed to a mixed state σ by incoherent operations. We get the result that if σ is incoherent then there exists an incoherent operation Φ such that $\Phi(\rho) = \sigma$ for any state ρ. We show this by explicitly constructing an incoherent operation that achieves the transformation in the Appendix. What are sufficient conditions for the existence of catalysts? Finally, all of the considerations above implicitly assumed a finite-dimensional setting, but this is neither necessary nor desirable as there are very relevant physical situations that require infinite-dimensional systems for their description. Most notable are the quantum states of light, that is, quantum optics, whose bosonic character requires infinite-dimensional systems, harmonic oscillators, for their description. Hence, coherence manipulation and existence of catalysts in infinite-dimensional systems are needed. Mirroring analogous developments in entanglement manipulation [22], we expect that the manipulation of coherence in infinite-dimensional systems can be built.

Conclusions. In this paper, we give a complete characterization of coherence manipulation for pure states in terms of majorization. This result offers an affirmative answer to the open question of whether, given two states ρ and σ, ρ can be transformed into σ or vice versa under incoherent operations [13]. The proof of the result also provides an effective constructive method to find the incoherent operation transforming $|\psi\rangle$ to $|\phi\rangle$, whenever $(|\psi_1|^2,\dots,|\psi_d|^2)^t \prec (|\phi_1|^2,\dots,|\phi_d|^2)^t$. The majorization approach used here is similar to that used to establish the ordering of entangled states, which led to advancement in the field of quantum computation. Based on Theorem 1, some interesting properties of coherent catalysts were discovered.

Acknowledgments. The authors thank the referees for their valuable comments which improved the presentation of this paper. This work was completed while the authors were visiting the Institute of Quantum Computation of the University of Waterloo under the support of the China Scholarship Council. We thank Professor David W. Kribs and Professor Bei Zeng for their hospitality. This work is partially supported by the Natural Science Foundation of China (Grants No. 11001230, No. 11301312, and No. 11171249), the Natural Science Foundation of Fujian (Grants No. 2013J01022 and No. 2014J01024), the Natural Science Foundation of Shanxi (Grants No. 2013021001-1 and No. 2012011001-2), and the Research Start-up Fund for Doctors of Shanxi Datong University under Grant No. 2011-B-01.

APPENDIX: TRANSITION OF MIXED STATES

We will show that if the output mixed state σ is incoherent, i.e., $\sigma \in \mathcal{I}$, then for any quantum state ρ there exists an incoherent operation Φ such that $\Phi(\rho) = \sigma$. We do this by an explicit construction of an incoherent operation. Define the incoherent operation

$$\Phi_1(\rho) := \sum_{i=1}^{d} |i\rangle\langle i|\rho|i\rangle\langle i|.$$

The effect of this operation is to remove all off-diagonal elements of $\lambda_{i,j}|i\rangle\langle j|$ $(i \neq j)$ from $\rho = \sum_{i,j=1}^d \lambda_{i,j}|i\rangle\langle j|$, leaving the diagonal elements $\lambda_{i,i}|i\rangle\langle i|$ intact. Denote by $\{\lambda_i = \langle i|\rho|i\rangle\}_{i=1}^d$ and $\{\mu_i\}_{i=1}^d$ the eigenvalues of $\Phi_1(\rho)$ and σ, respectively. Let

$$A_1 = \sqrt{\mu_1}|1\rangle\langle 1| + \sqrt{\mu_2}|2\rangle\langle 2| + \cdots + \sqrt{\mu_d}|d\rangle\langle d|,$$
$$A_2 = \sqrt{\mu_2}|1\rangle\langle 2| + \sqrt{\mu_3}|2\rangle\langle 3| + \cdots + \sqrt{\mu_d}|d-1\rangle\langle d| + \sqrt{\mu_1}|d\rangle\langle 1|,$$
$$\vdots,$$
$$A_i = \sqrt{\mu_i}|1\rangle\langle i| + \cdots + \sqrt{\mu_{m_{s+i-1}}}|s\rangle\langle m_{s+i-1}| + \cdots + \sqrt{\mu_{m_{d+i-1}}}|d\rangle\langle \mu_{m_{d+i-1}}|,$$
$$\vdots,$$
$$A_d = \sqrt{\mu_d}|1\rangle\langle d| + \sqrt{\mu_1}|2\rangle\langle 1| + \cdots + \sqrt{\mu_{d-1}}|d\rangle\langle d-1|.$$

Here $m_x = x - [\frac{x-1}{d}]d$. It is easy to check that $\sum_{i=1}^{d} A_i A_i^\dagger = I$. By a direct computation, one can get $\Phi_2(\Phi_1(\rho)) = \sum_{i=1}^{d} A_i^\dagger \Phi_1(\rho) A_i = \sigma$. Let $\Phi = \Phi_2 \circ \Phi_1$; then Φ is an incoherent operation satisfying $\Phi(\rho) = \sigma$.

[1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).

[2] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, Phys. Rev. A **54**, 3824 (1996).

[3] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, Phys. Rev. A **53**, 2046 (1996).

[4] C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. A. Smolin, and W. K. Wootters, Phys. Rev. Lett. **76**, 722 (1996).

[5] V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight, Phys. Rev. Lett. **78**, 2275 (1997).

[6] V. Vedral and M. B. Plenio, Phys. Rev. A **57**, 1619 (1998).

[7] M. A. Nielsen, Phys. Rev. Lett. **83**, 436 (1999).

[8] G. Vidal, Phys. Rev. Lett. **83**, 1046 (1999).

[9] D. Jonathan and M. B. Plenio, Phys. Rev. Lett. **83**, 1455 (1999).

[10] D. Jonathan and M. B. Plenio, Phys. Rev. Lett. **83**, 3566 (1999).

[11] L. Hardy, Phys. Rev. A **60**, 1912 (1999).

[12] M. A. Nielsen and G. Vidal, Quantum Inf. Comput. **1**, 76 (2001).

[13] T. Baumgratz, M. Cramer, and M. B. Plenio, Phys. Rev. Lett. **113**, 140401 (2014).

[14] R. Bhatia, *Matrix Analysis* (Springer-Verlag, New York, 1997).

[15] I. Ghiu, M. Bourennane, and A. Karlsson, Phys. Lett. A **287**, 12 (2001).

[16] W. Song, Y. Huang, N. L. Liu, and Z. B. Chen, J. Phys. A: Math. Gen. **40**, 785 (2007).

[17] I. Chattopadhyay and D. Sarkar, Phys. Rev. A **77**, 050305(R) (2008).

[18] G. Bowen and N. Datta, IEEE Trans. Inf. Theory **54**, 3677 (2008).

[19] J. Åberg, Phys. Rev. Lett. **113**, 150402 (2014).

[20] D. Girolami, Phys. Rev. Lett. **113**, 170401 (2014).

[21] S.-P. Du, Z.-F. Bai, and X.-F. Qi, arXiv:1504.02862v1.

[22] M. Owari, S. L. Braunstein, K. Nemoto, and M. Murao, Quantum Inf. Comput. **8**, 30 (2008).