Direct measurement of general quantum states using strong measurement

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The direct state measurement (DSM) based on the weak measurement has the advantage of simplicity, versatility, and directness. However, the weak measurement will introduce an unavoidable error in the reconstructed quantum state. We modify the DSM by replacing the weak coupling between the system and the pointer by a strong one, and present two procedures for measuring quantum states, one of which can give the wave function or the density matrix directly. We can also measure the Dirac distribution of a discrete system directly. Furthermore, we propose quantum circuits for realizing these procedures, and the main body of the circuits consists of Toffoli gates. By numerical simulation, we find that our scheme can eliminate the biased error effectively.

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I. INTRODUCTION

The quantum state is described by the wave function for pure states and the density operator for both pure and mixed states. If we know the wave function or the density matrix of a quantum system, then measurement outcomes can be predicted. Conversely, the quantum state information can also be extracted with appropriate measurements. The experiment method to reconstruct the wave function or the density matrix of an unknown quantum state by measurement is called quantum state tomography (QST). The ability of high-fidelity QST is demanded in many areas of technology such as quantum computing, quantum cryptography, and quantum communications. In the past decades, different QST schemes have been presented [1–5]. The standard QST techniques require designing processes to measure a complete set of noncommutating observables of the system and then determine the quantum state that is most compatible with the measurement results. The requirement may be very demanding for the systems with a large number of degrees of freedom. The direct state measurement (DSM) using weak measurement proposed by Lundeen *et al.* [6,7] takes a different approach, in which a value proportional to the amplitude of the wave function can be measured directly. The DSM using weak measurement is very simple in the experiment realization: it only consist of a weak coupling of the system with an external pointer, a postselection of the final state of the system, and a projective measurement of two complementary observables of the pointer. However, The essential weak coupling in the experiment makes the DSM a biased procedure, which introduces an unavoidable error in the reconstructed quantum state [8].

Alternatively, Di Lorenzo [9,10] demonstrates that an exact quantum state tomography can be accomplished efficiently by making a sequential measurement of two pointers. The procedure relies on the quasicharacteristic function, i.e., the Fourier transform of the Wigner function.

In this paper, we make an improvement over the state reconstruction protocol in [7]. We focus on the quantum state measurement in discrete systems (spins or qubits), and we use one qubit as the ancillary pointer. The pointer in the DSM can be continuous [11,12] or discrete [7,8], and a qubit pointer is as efficient as an infinite-dimensional continuous-variable pointer system [8]. We modify the DSM scheme by replacing the weak coupling between the system and the pointer by a strong one, and this can also eliminate the biased error as in [9,10]. The main differences of our schemes with that of [9,10] are as follows. (i) We use only one pointer, and this pointer is simply a qubit. (ii) Our schemes directly measure the wave function or the Dirac distribution [13], while [9,10] measure the Fourier transform of the Wigner function. We present two measurement procedures for reconstructing the state of a quantum system. In experiments, the coupling can be implemented by n-qubit Toffoli gates and local rotations on individual qubits.

II. WEAK VALUE AND DIRECT STATE MEASUREMENT

We first briefly introduce the weak values and direct state measurement. The formalism of weak values was introduced by Aharonov and co-workers [14]. For an observable $\hat{A}$, the expression

$$\langle A\rangle_W = \frac{\langle\psi_f|\hat{A}|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle} \tag{1}$$

is called the weak value of observable $\hat{A}$ for a quantum system preselected in state $|\psi_i\rangle$ and postselected in state $|\psi_f\rangle$. Weak values characterize the relative correction to a detection probability $|\langle\psi_f|\psi_i\rangle|^2$ due to a small intermediate perturbation $\hat{U}(\epsilon) = \exp(-i\epsilon\hat{A})$ that results in a modified detection probability $|\langle\psi_f|\hat{U}(\epsilon)\psi_i\rangle|^2$ [15]. The weak value can be determined by weak measurements [7,14,16]. In the standard von Neumann model, an ancillary pointer is introduced to interact with the system upon which we want

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to measure $\hat{A}$. The interaction Hamiltonian is

$$H_{\text{int}} = g\hat{A}\hat{p}, \tag{2}$$

where $\hat{p}$ is the momentum operator of the pointer. Assuming that initially the system is prepared in the state $|\psi_i\rangle$ and the pointer is in the state $|\phi\rangle$, then after interaction time t, the system and pointer will be in state $\exp(-igt\hat{A}\hat{p})|\psi_i\rangle\,|\phi\rangle$. By postselecting the system in the state $|\psi_f\rangle$, the pointer state (subnormalized) will be

$$|\phi_f\rangle = \langle\psi_f|\exp(-igt\hat{A}\hat{p})|\psi_i\rangle\,|\phi\rangle\,. \tag{3}$$

If the interaction is weak enough such that the first-order approximation is satisfactory, we have

$$\begin{aligned}|\phi_f\rangle &\simeq \langle\psi_f|I - igt\hat{A}\hat{p}|\psi_i\rangle\,|\phi\rangle\\ &= \langle\psi_f|\psi_i\rangle(I - igt\langle A\rangle_W\hat{p})\,|\phi\rangle\\ &\simeq \langle\psi_f|\psi_i\rangle\exp(-igt\langle A\rangle_W\hat{p})\,|\phi\rangle\,,\end{aligned} \tag{4}$$

where I is the identity operator. The subsequent measurement results on the pointer will depend on $\langle A\rangle_W$, and we can measure $\langle A\rangle_W$ in this way. The procedure is similar for discrete pointer [7,8].

One of the applications of weak values is that they can be used to directly determine a quantum state. Considering an N-dimensional discrete Hilbert space. We choose an orthonormal basis $\{|a_k\rangle(k = 0,1,\ldots,N-1)\}$ in which wave function will be measured. There exists a complementary basis $\{|c_\alpha\rangle(\alpha = 0,1,\ldots,N-1)\}$ such that $\langle a_k|c_\alpha\rangle = \omega^{\alpha k}/\sqrt{N}$, where $\omega = e^{i2\pi/N}$. $\{|c_\alpha\rangle\}$ is the Fourier transform of $\{|a_k\rangle\}$; $|c_\alpha\rangle = 1/\sqrt{N}\sum_{k=0}^{k=N-1}\exp(i2\pi\alpha k/N)|a_k\rangle$. $\{|a_k\rangle\}$ and $\{|c_\alpha\rangle\}$ are the so-called mutually unbiased bases (MUB) [17]. For pure state, we just need state $|c_0\rangle$ such that $\langle c_0|a_k\rangle = 1/\sqrt{N}$ for all k. The wave function $|\psi\rangle$ is expanded in the basis $\{|a_k\rangle\}$ as

$$|\psi\rangle = \sum_k\langle a_k|\psi\rangle\,|a_k\rangle\,. \tag{5}$$

By multiplying a constant $\nu = \langle c_0|a_k\rangle\,/\,\langle c_0|\psi\rangle = [\sqrt{N}\,\langle c_0|\psi\rangle]^{-1}$ (which is independent of k) on both sides, we obtain

$$\nu\,|\psi\rangle = \sum_k\langle\Pi_{a_k}\rangle_W\,|a_k\rangle\,, \tag{6}$$

where $\langle\Pi_{a_k}\rangle_W = \frac{\langle c_0|\Pi_{a_k}|\psi\rangle}{\langle c_0|\psi\rangle}$ is just the weak value of the project operator $\Pi_{a_k} = |a_k\rangle\,\langle a_k|$. So each complex amplitude of the pure state $|\psi\rangle$ is proportional to the weak value $\langle\Pi_a\rangle_W$. By stepping through all the projector $\Pi_{a_k} = |a_k\rangle\,\langle a_k|$ in a series of weak measurement experiments to measure $\langle\Pi_{a_k}\rangle_W$, one can directly measure $|\psi\rangle$ in the basis $\{|a_k\rangle\}$.

III. STANDARD MEASUREMENT SCHEME

In the above weak value measurement procedures, the interaction must be weak enough to fulfill the linear relationship in Eq. (4), i.e., the high-order corrections can be neglected; this will introduce an unavoidable error in the reconstructed state [8]. Here we propose a modified scheme to eliminate this error. We use the fact that the weak value of a projector can be obtained by standard measurement.

A. Pure state measurement scheme

First we consider the situation of pure states. The bullet point algorithm of the procedure is as follows.

(a) Implement the unitary coupling $U_k = \exp(-i\pi/2\,|a_k\rangle\langle a_k|\otimes\sigma_y)$ between the system and the pointer qubit.

(b) Project the system in the basis $\{|c_k\rangle\}$ complementary to $\{|a_k\rangle\}$. If the result state is $|c_0\rangle$, then measure the expectation value $\langle\sigma_x\rangle$ ($\langle\sigma_y\rangle$, $\langle\sigma_z\rangle$) of the pointer qubit; otherwise, discard the result.

(c) Calculate the weak value $\langle\Pi_{a_k}\rangle_W$ from the data collected in step (b).

(d) Repeat (a), (b), and (c) for other k till we get all weak values $\langle\Pi_{a_k}\rangle_W$ $(k = 1,2,\ldots,N)$; then from Eq. (6), we get the wave function.

Now we discuss the procedure in more details. First, let the system interact with the pointer qubit though a unitary coupling $U_k = \exp(-i\theta\Pi_{a_k}\otimes\sigma_y)$ (here σ_y is the Pauli operator of the pointer qubit). It should be pointed out that for a general operator, to isolate the weak value, the coupling strength θ must be small enough to make Eq. (4) hold. However, for the projection operator Π_{a_k}, this constraint can be released and here we take $\theta = \pi/2$. Assuming that initially the system is prepared in the state $|\psi_i\rangle$ and the pointer qubit is in the state $|0\rangle$, then after the interaction the state of the joint system will be

$$U_k|\psi_i\rangle|0\rangle = \big(I-\Pi_{a_k}\big)|\psi_i\rangle|0\rangle + \Pi_{a_k}|\psi_i\rangle\exp\left(-i\frac{\pi}{2}\sigma_y\right)|0\rangle. \tag{7}$$

When a postselection $|c_0\rangle$ is performed on the system, the pointer state (unnormalized) is collapsed to

$$|\phi_f\rangle = \langle c_0|\psi_i\rangle\big(\big(1-\langle\Pi_{a_k}\rangle_W\big)|0\rangle + \langle\Pi_{a_k}\rangle_W|1\rangle\big). \tag{8}$$

To obtain the weak value $\langle\Pi_{a_k}\rangle_W$, we need to measure the expectation values of the Pauli operators σ_x, σ_y, and σ_z of the pointer qubit. Simple calculation gives the following expression:

$$\langle\Pi_{a_k}\rangle_W = \left(\frac{1}{2} - \frac{\langle\sigma_z\rangle_k}{2(1+\langle\sigma_x\rangle_k)}\right) + i\frac{\langle\sigma_y\rangle_k}{2(\langle\sigma_x\rangle_k+1)}. \tag{9}$$

Again by stepping through all Π_{a_k} we can obtain all amplitudes of a pure state $|\psi\rangle$ in the basis $\{|a_k\rangle\}$.

B. General state measurement scheme

Next we show how to determine the density matrix ρ for a general quantum state of the system. The bullet point algorithm of the procedures is as follows.

(a) Implement the unitary coupling $U_k = \exp(-i\pi/2\,|a_k\rangle\langle a_k|\otimes\sigma_y)$ between the system and the pointer qubit.

(b) Project the system in the basis $\{|c_k\rangle\}$ complementary to $\{|a_k\rangle\}$. If the result state is $|c_\alpha\rangle$, measure the expectation value of σ_x (σ_y, σ_z) of the pointer qubit, and value is denoted by $\langle\sigma_x\rangle_{\alpha,k}$ ($\langle\sigma_y\rangle_{\alpha,k}$, $\langle\sigma_z\rangle_{\alpha,k}$).

(c) Repeat (a) and (b) for all other k and α.

(d) Calculate density matrix of the unknown state from the data collected in step (c).

As above, we also couple the system to the pointer initialized in the state $|0\rangle$ with the unitary operation U_k. Then we make a projective measurement on the system in the complementary basis $\{|c_\alpha\rangle\}$. If the result state of the system is $|c_\alpha\rangle$, then the state of the pointer qubit will be

$$\varrho(\alpha,k) = \begin{pmatrix} \sum'_{lm} \rho_{ml}\omega^{\alpha(l-m)} & \sum'_l \rho_{lk}\omega^{\alpha(k-l)} \\ \sum'_l \rho_{kl}\omega^{\alpha(l-k)} & \rho_{kk} \end{pmatrix} \Big/ R(\alpha,k), \tag{10}$$

where $\rho_{ml} = \langle a_m|\rho|a_l\rangle$ is the matrix element in the basis $\{|a_k\rangle\}$. $R(\alpha,k) = \sum'_{lm} \rho_{ml}\omega^{\alpha(l-m)} + \rho_{kk}$ is the normalized factor and the prime means that $l \neq k$, $m \neq k$ in the summations. The expectation values of σ_x, σ_y, and σ_z of the pointer are

$$\langle\sigma_x\rangle_{\alpha,k} = \frac{2\,\mathrm{Re}\big(\sum'_l \rho_{kl}\omega^{\alpha(l-k)}\big)}{R(\alpha,k)}, \tag{11}$$

$$\langle\sigma_y\rangle_{\alpha,k} = \frac{2\,\mathrm{Im}\big(\sum'_l \rho_{kl}\omega^{\alpha(l-k)}\big)}{R(\alpha,k)}, \tag{12}$$

$$\langle\sigma_z\rangle_{\alpha,k} = 1 - \frac{2\rho_{k,k}}{R(\alpha,k)}, \tag{13}$$

where the subscript α,k means that the expectation values are measured under the situation of the unitary coupling U_k and the system state $|c_\alpha\rangle$ after measurement. From the above three equations, we have

$$\sum_l \rho_{kl}\omega^{\alpha(l-k)} = \frac{R(\alpha,k)}{2}(\langle\sigma_x\rangle_{\alpha,k} + i\,\langle\sigma_y\rangle_{\alpha,k} - \langle\sigma_z\rangle_{\alpha,k} + 1). \tag{14}$$

The density matrix ρ can be recovered by inverting the above equation,

$$\rho_{kq} = \frac{1}{2N}\sum_\alpha \omega^{\alpha(k-q)}R(\alpha,k)(\langle\sigma_x\rangle_{\alpha,k} + i\,\langle\sigma_y\rangle_{\alpha,k} - \langle\sigma_z\rangle_{\alpha,k} + 1). \tag{15}$$

The value of $R(\alpha,k)$ can be determined from Eq. (11) and Eq. (13), and the result is

$$R(\alpha,k) = \frac{2}{(1 + \langle\sigma_x\rangle_{\alpha,k})\sum_m \frac{1-\langle\sigma_z\rangle_{\alpha,m}}{1+\langle\sigma_x\rangle_{\alpha,m}}}. \tag{16}$$

In writing the equation we have used the fact that $R(\alpha,k) + 2\,\mathrm{Re}(\sum'_l \rho_{kl}\omega^{\alpha(l-k)}) = \sum_{l,m}\rho_{ml}\omega^{\alpha(l-m)}$ is independent of k and $\sum_k \rho_{kk} = 1$. Repeating the experiment to collect the data of $\langle\sigma_x\rangle_{\alpha,k}$, $\langle\sigma_y\rangle_{\alpha,k}$, and $\langle\sigma_z\rangle_{\alpha,k}$ for all α and k, we can get the whole density matrix from Eq. (15) and Eq. (16).

C. Quantum circuit and numerical simulation

The quantum circuit for realizing the above procedure is depicted in Fig. 1, which presents an incredible experimental simplicity of the DSM scheme. In the circuit, we choose the computational basis as $\{|c_\alpha\rangle\}$, that is $|c_0\rangle = |0,0,\ldots,0\rangle$, $|c_{N-1}\rangle = |1,1,\ldots,1\rangle$, and so on. The complementary basis states $|a_k\rangle$, which are the Fourier transform of $|c_\alpha\rangle$, are product states of the individual qubit [18]. By local rotations each $|a_k\rangle$ can be transformed to $|1,1,\ldots,1\rangle$. Thus the unitary coupling U_k is transformed to $\exp(-i\pi/2\,|1,1,\ldots,1\rangle\,\langle 1,1,\ldots,1| \otimes \sigma_y)$, the Toffoli gate. Therefore, U_k can be implemented by an $(n+1)$-qubit Toffoli gate (n is the number of the qubits in the system, $n = \log_2 N$) and local rotations. The n-qubit Toffoli gate plays a key role in many quantum algorithms. The common approach to implement it is to decompose the gate into two-qubit gates and one-qubit rotations [19,20]. For example, a quadratic-size, linear-depth quantum circuit for Toffoli gates is designed in [20], from which we need $2n^2 + O(n)$ two-qubit gates and $O(n)$ one-qubit rotations to perform U_k. For different k, we just need to change the local rotations before and after the Toffoli gate. Because the postselection states $|c_\alpha\rangle$ are the computational basis states, in the experiment we just need to make a projective measurement on each qubit of the system along σ_z.

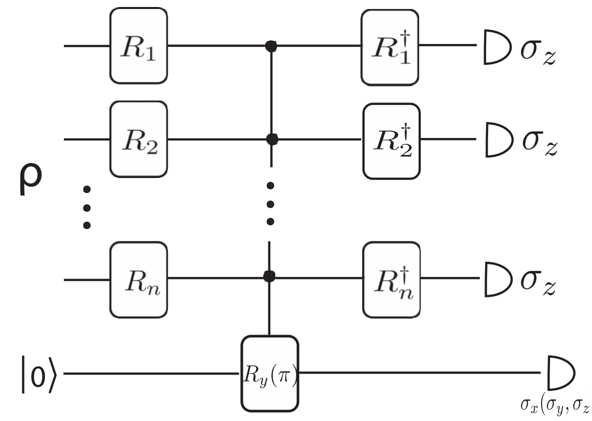

FIG. 1. Implementation of the coupling between the system initialized in the state ρ (or the pure state $|\psi\rangle$) and the pointer initialized in the state $|0\rangle$, followed by measurements on system and then on pointer. We choose $|c_\alpha\rangle$ the usual computational basis; its complementary basis $|a_k\rangle$ is a product state, which can be transformed to $|c_{N-1}\rangle = |11\ldots1\rangle$ by local rotations. So the unitary coupling U_k consists of local rotations R_m ($R_m^\dagger$) and an $(n+1)$-qubit Toffoli gate. A z direction projective measurement is performed on each qubit of the system. For pure states, only $|c_0\rangle$ is postselected; others will be discarded. For general ρ, postselection is not needed; all the results provide data that is employed in the reconstruction. On pointer qubit, we need to measure the expectations of $\sigma_x,\sigma_y,\sigma_z$, respectively.

In [8], a comparison between the DSM and the standard QST is carried out through accurate Monte Carlo techniques. In the numerical simulation, only the statistical errors are considered: in experiments one must estimate the expectation values from a finite number N_c of copies of the system. The trace distance $D(\rho_t,\rho_r) = \mathrm{Tr}(|\rho_t - \rho_r|)/2$ between the true state ρ_t and reconstructed one ρ_r is used to characterize the experiment errors. They find that because of the bias, the trace distance will saturate to a nonzero value, with no long decrease when increasing the number of copies; thus an error is introduced. In our measurement scheme, there is no approximation, so there is no bias and relevant errors. Through the same method as in [8], we also present a comparison between our generalized DSM (gDSM) and the standard QST. The simulation results are shown in Fig. 2. We can find that the trace distance of DSM will not saturate to a nonzero value now; it will continue to decrease when the number of copies of the system increases as in the case of the QST. So this gDSM has eliminated the biased error effectively.

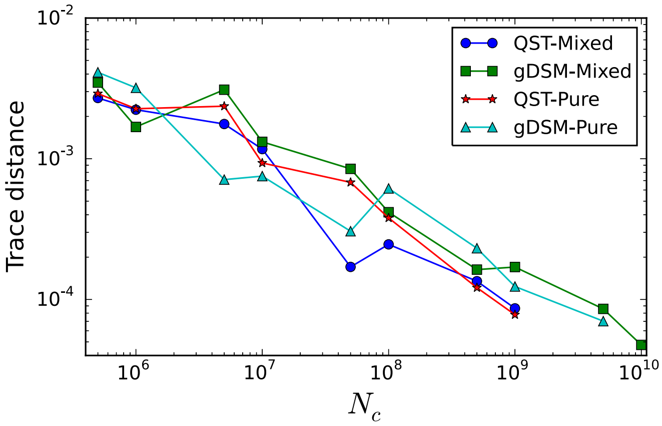

FIG. 2. (Color online) Comparison between the generalized DSM (gDSM) and the standard tomography (QST) for both pure and mixed states of one qubit. The trace distance is plotted as a function of the total number of copies N_c of the system employed in the reconstruction. The pure state is $\rho = \frac{I+1/\sqrt{3}\sigma_x+1/\sqrt{3}\sigma_y+1/\sqrt{3}\sigma_z}{2}$ and the mixed state is $\rho = \frac{I+0.9/\sqrt{3}\sigma_x+0.9/\sqrt{3}\sigma_y+0.9/\sqrt{3}\sigma_z}{2}$. The details of the gDSM are described in the text. The QST is just to measure the expectation of σ_x,σ_y,σ_z of the unknown state, respectively. Because of the removing of the bias, the trace distance of the gDSM continues to decrease when N_c is increasing, as of the QST. For smaller Nc the trace distance has large statistical fluctuation.

The DSM (include the generalized scheme here) seems to be less efficient than the standard QST because of the postselection. In some cases, this will not be a trouble. For example, reconstruction of a pure state needs measure $3N$ observables for N-dimensional Hilbert space, while the QST needs much more depending on detail schemes. For an ensemble system such as a nuclear magnetic resonance (NMR) [18] system, the DSM will be more handy, because an observable such as σ_x can be measured using only one copy.

IV. DIRECT MEASUREMENT OF THE DIRAC DISTRIBUTION AND THE DENSITY MATRIX

The above scheme is not direct for a density matrix, which is determined after sweeping all the $|a_k\rangle$ and $|c_\alpha\rangle$. If we only want to know partial information of the density matrix, this method is tedious and not necessary. We now propose a direct measurement scheme for the density matrix, in which the elements of the matrix can be directly determined one by one. The scheme can also measure the Dirac distribution directly. The quantum circuit of the experiment procedure is depicted in Fig. 3. The pointer qubit is initialized in the state $|+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$, and the system is in an unknown state ρ. The density matrix will be reconstructed in the computational basis $\{|c_\alpha\rangle\}$. The bullet point algorithm of the procedure is as follows.

(a) Implement the controlled-U_1 gate and controlled-U_1' gate on $|+\rangle\langle+|\otimes\rho$ separately, where $U_1 = \exp(-i\pi\,|c_\alpha\rangle\langle c_\alpha|)$ and $U_1' = \exp(-i\pi\,|a_k\rangle\langle a_k|)$. Then measure the expectation value $\langle\sigma_x\rangle$ of the pointer qubit.

(b) Implement the controlled-U_1 gate then the controlled-U_2 gate, where $U_1 = \exp(-i\pi|c_\alpha\rangle\langle c_\alpha|)$ and $U_2 = \exp(-i\pi|a_k\rangle\langle a_k|)$. Then measure the expectation values $\langle\sigma_x\rangle$ and $\langle\sigma_y\rangle$ of the pointer qubit.

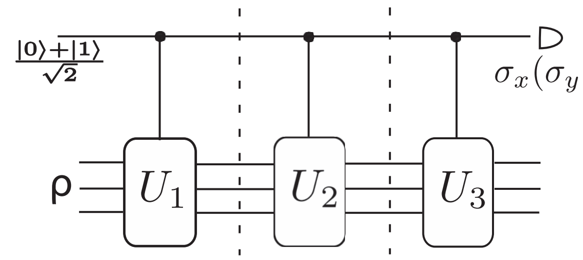

FIG. 3. Circuit for implementing a direct measurement of the Dirac distribution and the density matrix. The top qubit is the ancillary qubit initialized in state $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$, the bottom lines denote the system with an unknown state ρ. One to three controlled-U gates may be required depending on different purposes. If the expectations of σ_x and σ_y of the pointer qubit are measured after the first controlled-U gate, the population of the state ρ will be obtained. If the same is performed after the second controlled-U gate, the Dirac distribution of the state ρ will be obtained. By measuring the expectation of σ_x and σ_y after the third controlled-U gate, we can obtain the elements of the density matrix ρ.

(c) Implement the controlled-U_1 gate, controlled-U_2 gate, and controlled-U_3 gate sequentially, where $U_1 = \exp(-i\pi|c_\alpha\rangle\langle c_\alpha|)$, $U_2 = \exp(-i\pi|a_0\rangle\langle a_0|)$, and $U_3 = \exp(-i\pi|c_\beta\rangle\langle c_\beta|)$. Then measure the expectation values $\langle\sigma_x\rangle$ and $\langle\sigma_y\rangle$ of the pointer qubit.

(d) Calculate the diagonal element of the density matrix from the data collected in (a) and the nondiagonal element from the data collected in (a)–(c). Calculate the Dirac distribution from the data collected in (a) and (b).

Now we have more discussions about the procedure. Depending on different purposes the procedure consists of one to three controlled-U gates, with the control on the pointer qubit and with unitary operation U acting on the system. Measurements only performed on the pointer qubit. After the action of a controlled-U, the expectations of σ_x and σ_y of the pointer are the real and imaginary part of $\mathrm{Tr}(U\rho)$, respectively [21]. That is

$$\langle\sigma_x\rangle + i\langle\sigma_y\rangle = \mathrm{Tr}(U\rho). \tag{17}$$

The first controlled-U operation is chosen as a controlled-PHASE flip gate, $U_1 = \exp(-i\pi\,|c_\alpha\rangle\langle c_\alpha|) = I - 2\,|c_\alpha\rangle\langle c_\alpha|$ and $\mathrm{Tr}(U_1\rho) = 1 - 2\rho_{\alpha,\alpha}$. From Eq. (17), we have $\rho_{\alpha\alpha} = (1 - \langle\sigma_x\rangle)/2$. If the expectation of σ_x of the pointer is measured, the population of the state $|c_\alpha\rangle$ can be obtained. The whole diagonal of the density matrix can be obtained by scanning $|c_\alpha\rangle$.

In the second controlled-U operation, $U_2 = \exp(-i\pi\,|a_k\rangle\langle a_k|) = I - 2\,|a_k\rangle\langle a_k|$, where basis $\{|a_k\rangle\}$ is the Fourier transform of $\{|c_\alpha\rangle\}$. If the measurements of the expectation values of σ_x and σ_y on the pointer qubit are performed now, they will be the real and imaginary part of $\mathrm{Tr}(U_2U_1\rho) = 1 - 2\rho_{c_\alpha,c_\alpha} - 2\rho_{a_k,a_k} + 4\langle c_\alpha|\rho|a_k\rangle\langle a_k|c_\alpha\rangle$, respectively. We thus directly obtained $S_\rho(\alpha,k) = \langle c_\alpha|\rho|a_k\rangle\langle a_k|c_\alpha\rangle = \mathrm{Tr}[\rho S_{\alpha,k}]$ (here $S_{\alpha,k} = |a_k\rangle\langle a_k|c_\alpha\rangle\langle c_\alpha|$), which is just the Dirac distribution [13] in the discrete Hilbert space [7,22,23].

The Dirac distribution is a quasiprobability distribution like the Wigner function, Husimi Q function, and Glauber-Sudarshan P distribution. It is an underused but elegant way to describe a general quantum state and it is also useful for visualizing discrete systems [24]. The connection between Dirac distribution, joint probabilities, and the weak value was explored in [25]. The Dirac distribution is related to the density operator by a discrete Fourier transform, $\rho_{\alpha\beta} = \sum_{k=0}^{k=N-1} S_\rho(\alpha,k)e^{i2\pi k(\alpha-\beta)/N}$. Furthermore, the distribution has the classical-like feature that transforms according to Bayes' law [23,25]. Recent works have measured the Dirac distribution of the pure state in one qubit system [24], and of the quantum state corresponding to the transverse position of a photon for mixed and pure states by weak measurement [7,23]. Here we propose a standard direct measurement scheme of the Dirac distribution in a discrete system for a general state, and it is convenient for experiment realization.

To determine an element of the density matrix from the Dirac distribution, one needs to scan $|a_k\rangle$. Although it is much better than the first scheme, it is still tedious and not directly. A third controlled-U operation, $U_3 = \exp(-i\pi|c_\beta\rangle\langle c_\beta|) = I - 2|c_\beta\rangle\langle c_\beta|$, will make the direct measurement possible. Now $U_2 = \exp(-i\pi|a_0\rangle\langle a_0|)$ is fixed, where $\langle a_0|c_\alpha\rangle = 1/\sqrt{N}$ for all $|c_\alpha\rangle$. By measuring the expectation values of σ_x and σ_y on the pointer qubit, we can get $\langle c_\alpha|\rho|c_\beta\rangle/N = \rho_{\alpha,\beta}/N$. Scanning α and β will make one obtain the entire density matrix. The measured value changes as $1/N$. For large N, it will need high-fidelity gate operations and measurements. We can choose different intermediate unitary operations U_2 for different elements. For example, for $\rho_{\alpha,\beta}$, we can take $U_2 = \exp(-i\pi|\psi\rangle\langle\psi|)$, where $|\psi\rangle = \frac{|c_\alpha\rangle+|c_\beta\rangle}{\sqrt{2}}$.

The controlled phase flip gate in the circuit can be transformed to an $(n+1)$-qubit Toffoli gate by local rotations. Thus it needs $4n^2 + O(n)$ two-qubit gates for directly determining the Dirac distribution and $6n^2 + O(n)$ for the density matrix. If U_2 is not fixed, one needs more two-qubit gates for reconstructing the density matrix.

V. CONCLUSION

Focusing on the discrete system, the weak value of a projector can be determined by standard measurement. We have generalized the direct state measurement scheme by replacing the weak coupling between the system and the pointer qubit by a strong one. We present two measurement procedures for reconstructing the state of a quantum system, one of which can give the wave function or the density matrix directly. By numerical simulation, we find that this generalized scheme can effectively eliminate the biased error introduced by the weak measurement based DSM. We can also measure the Dirac distribution of a discrete system in pure or mixed state by standard measurement. We have also presented the concrete quantum circuits for realizing the procedures experimentally.

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