Interference effects in tunneling of Schrödinger cat wave-packet states

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We analyze tunneling of a single particle, whose initial state is given by a superposition of spatially separated wave-packet modes. It is shown that "pile up" of different components in the scatterer may change the tunneling probabilities, making such states a convenient tool for probing the barrier's scattering times. Interference effects arising in resonance tunneling are studied in detail. The analysis allows us to gain further insight into the origin of interference effects in scattering of several identical particles.

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I. INTRODUCTION

Currently, there is considerable interest in the interference effects accompanying scattering of several noninteracting identical particles (see, for example, [1–5]). In the celebrated Hong-Ou-Mandel setup [3,4], effects of this kind are observed if the particles, incident from opposite sides, coincide in the scatterer. Recently it was shown that a different kind of interference effect may arise in the case where identical particles, incident from the same side, and detained in the scatterer, coincide there due to a kind of "pile-up" effect [6,7]. As a result, in resonance tunneling, the transmission probability was shown to oscillate as a function of the temporal delay between the arriving particles, provided two or more metastable states in the barrier can be accessed [6].

In this paper, we discuss a closely related case, where the state of a *single* particle, incident on a barrier, consists of several spatially separated wave-packet modes. Such an exotic "cat" state can be created, for example, by splitting the original wave packet into parts, which experience different time delays before being recombined [8] or, in the case of cold atoms, by using techniques similar to those described in Ref. [9]. We will show that the "pile up" of the modes, caused by a delay in the barrier region, can cause observable changes in the tunneling probability. One purpose of this paper is to analyze the use of such systems as an alternative tool for probing the barrier's scattering times (for a tunneling time review see [10–12]). Its other purpose is to use the analysis in order to gain further insight into the nature of interference effects in scattering of several identical particles.

The rest of the paper is organized as follows. In Sec. I we consider transmission of a multicomponent initial state. In Sec. II we analyze its transmission across a rectangular barrier. In Sec. III we study the case of resonance tunneling, and analyze the interference patterns occurring in the transmission probability. In Sec. IV we discuss the interference mechanism, and its similarity with that in the case of several identical particles. Section V considers transmission of a mixed "cat" state, and Sec. VI contains our conclusions.

II. A MULTICOMPONENT INITIAL STATE

Consider, in one dimension, a particle whose wave function is given by a superposition of N wave packets, $\psi_n(x)$,

$$\Psi_0(x,t) = K^{-1/2}\sum_{n=1}^{N}\psi_n(x,t). \tag{1}$$

Such states can be constructed in different ways. For example, $\psi_n(x,t)$ could be copies of the same wave packet separated in space, or the copies of the same wave packet, created at the same place at different times [13]. In the following we will consider the latter choice, in order to simplify the comparison with the multiparticle case analyzed in Ref. [6]. We, therefore, have

$$\psi_n(x,t) = (2\pi)^{-1/2}\int A_n(p)\exp[ipx - iE(p)(t+t_n)]dp, \tag{2}$$

where $0 = t_1 < t_2 < \cdots < t_N$.

For a particle of mass μ, e.g., for cold atoms [6] or photons in a waveguide [14], the energy is quadratic in the momentum, $E(p) = p^2/2\mu$. For massless particles, e.g., free photons, or electrons in graphene [15], this relation is linear, $E(p) = cp$. The constituent wave packets may, or may not overlap, and for the normalization constant K in Eq. (1) we have

$$K = \sum_{mn}\langle\psi_m|\psi_n\rangle = \int dp A_m^*(p)A_n(p)\exp[iE(p)\tau_{mn}] \equiv \sum_{m,n} I_{mn}, \tag{3}$$

where $\tau_{mn} = t_m - t_n$. The particle is incident on a finite width potential barrier with a transmission amplitude $T(p)$ (see Fig. 1) and, as $t \to \infty$, its transmitted part takes the form,

$$\Psi^T(x,t) \equiv \sum_n \psi_n^T(x,t) = (2\pi K)^{-1/2}\sum_n\int T(p)A_n(p) \times \exp[ipx - iE(p)(t+t_n)]dp. \tag{4}$$

We are interested in times sufficiently large for the particle to have left the barrier region. The tunneling probability P^T is then given by the integral of $|\Psi(x,t)|^2$ from the right edge of the barrier to infinity. Replacing the lower integration limit by $-\infty$, from Eq. (4) we have

$$P^T = \sum_{m,n} T_{mn} \Bigg/ \sum_{m,n} I_{mn}, \tag{5}$$

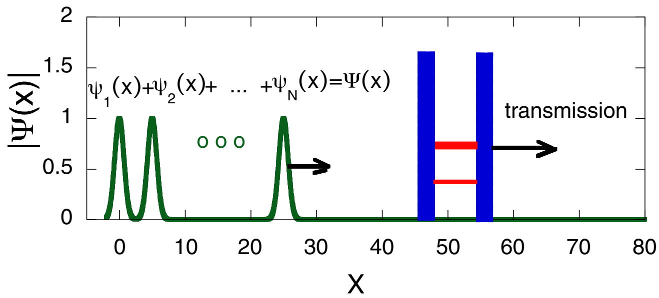

FIG. 1. (Color online) Schematic diagram showing a "cat" state, consisting of N nonoverlapping components, incident on a resonance barrier supporting two metastable states.

where

$$T_{mn}=\langle\psi_m^T|\psi_n^T\rangle=\int dp|T(p)|^2A_m^*(p)A_n(p)\exp[iE(p)\tau_{mn}], \tag{6}$$

is the matrix of the overlaps between the transmitted modes ψ_n^T.

If the delays are large, $|t_m - t_n| \to \infty$, rapid oscillations of the exponentials in Eqs. (3) and (6), cause the off-diagonal elements of the overlap matrix vanish, $I_{mn} = a_n\delta_{mn}$ and $T_{mn} = \int |T(p)^2|A_n(p)|^2dp \times \delta_{mn} \equiv w_n\delta_{mn}$. If so, all components of Ψ_0 are transmitted independently, and we have

$$P^T = \sum_n w_n \equiv P^T_{\text{ind}}. \tag{7}$$

As in Ref. [6], we are interested in the case where $I_{mn} = a_n\delta_{mn}$, and $T_{mn} \neq w_n\delta_{mn}$. This would indicate that the initially nonoverlapping components of Ψ_0 "pile up" in the barrier region, and the interference between them affects the outcome of the tunneling process. A deviation of P^T from P^T_{ind} may, therefore, serve as a crude indicator that a scattered particle spends in the barrier a duration comparable to at least some of $|t_m - t_n|$. Next we apply this test to the case of a rectangular barrier.

III. A RECTANGULAR BARRIER

For a rectangular barrier, $V(x) = V$ for $a \leqslant x \leqslant< b$, and 0 otherwise, the transmission coefficient in the tunneling regime $E(p) < V$ is given by the well-known expression,

$$|T(p)|^2 = 1/\{1 + V^2\sinh^2[q(b-a)]/4V(V-E(p)\}, \tag{8}$$

where $q(p) = [2\mu(V-E)]^{1/2}$ for a massive particle. Henceforth, we will consider Gaussian wave packets with identical momentum distributions, separated by equal time delays,

$$\begin{gathered} A_n(P) = A_m(p) \equiv A(p), \quad t_{n+1} - t_n \equiv \tau, \\ |A(p)|^2 = (2\pi)^{-1/2}\sigma\exp[-(p-p_0)^2\sigma^2/2]. \end{gathered} \tag{9}$$

In the deep tunneling regime, $|T(p)|^2 \sim \exp[-2q(b-a)]$ rapidly grows as E increases, but contains no sharp features. As a result, the momentum distribution of each ψ_n^T is shifted towards higher p's, but not modified sufficiently to prevent integrals in Eq. (6) from being destroyed by the oscillations (see inset in Fig. 2). There is, therefore, no evidence that

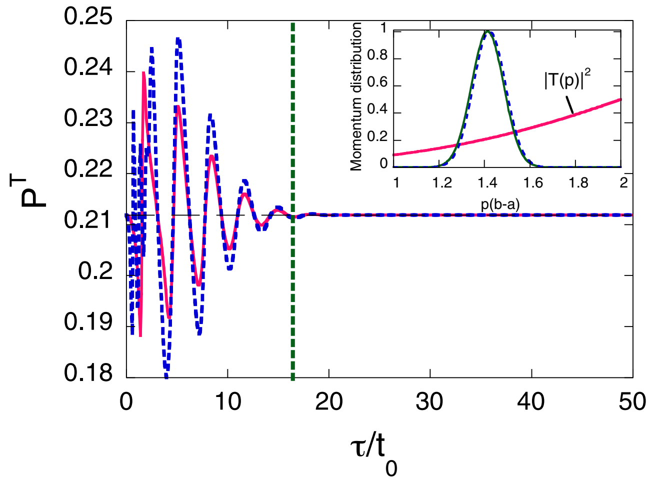

FIG. 2. (Color online) Tunneling probabilities for the rectangular barrier (8) vs τ/t_0, $t_0 \equiv 1/2\mu(b-a)^2$, with $2\mu V(b-a)^2 = 4$. Shown are the cases of $N = 2$ (solid) and $N = 5$ (dashed) Gaussian components (9) with $p_0(b-a) = 1.41$, and $\sigma/(b-a) = 4.47$. To the right the vertical dashed line the overlap between the components, $\sum_{i\neq j}^N |I_{ij}|$, is less than 0.005. Also shown in the inset are $|A(p)|^2$ (solid) and $|T(p)|^2|A(p)|^2$ (dashed), both renormalized to unit heights, as well as $|T(p)|$ (thick solid).

different components of the wave function, which did not overlap initially, may be delayed, and eventually "meet" in the barrier region. The transmission probability for an two- and five-component states is shown in Fig. 2. As expected, as soon as the overlap between different ψ_n vanishes, different modes in Eq. (1) tunnel independently, and we have $P^T = P^T_{\text{ind}}$. The absence of the said pile-up effect is consistent with the original McColl's suggestion that "there is no appreciable delay in the transmission of the packet through the barrier." [16]. It is also consistent with the finding of Ref. [6], where tunneling of two identical particles was studied in a similar context.

IV. RESONANCE TUNNELLING

The situation is different in resonance tunneling across a symmetrical barrier. The transmission coefficient of such a barrier typically exhibits well-separated sharp narrow peaks which, in the Breit-Wigner approximation, have Lorentzian shapes,

$$|T(p)|^2 \approx \sum_j \frac{\Gamma_j^2}{\left(p^2/2\mu - E_j^r\right)^2 + \Gamma_j^2}. \tag{10}$$

Thus, $E_j^r = E(p_j^r)$ gives the position of the jth resonance peak, and Γ_j is its width. Now the transmitted momentum distribution, $|T(p)|^2|A(p)|^2|$, can be made much narrower than the incident one, $|A(p)|^2|$. Approximating both $|A(p)|^2$ and ∂_pE constant for $E(p) \approx E_j^r$, and evaluating the remaining integrals in Eq. (6), then yields

$$T_{mn} = \sum_j C_j\exp(-\Gamma_j|m-n|\tau)\exp\left[iE_j^r(m-n)\tau\right], \tag{11}$$

where $C_j = 2\pi\Gamma_j|A(p_j^r)|^2/\partial_pE(p_j^r)$. Equation (11) shows that $T_{mn}(\tau)$ oscillates with the internal frequencies of the

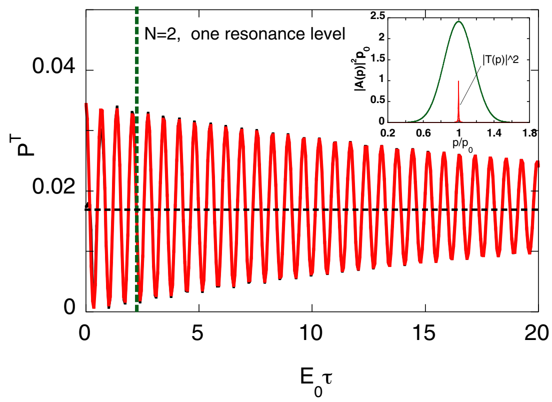

FIG. 3. (Color online) Probability to tunnel with one resonance level accessible to a particle in two-component initial state (9) vs the time lag $\tau = t_2 - t_1$, for $p_0\sigma = 6$, $E_1^r/E_0 = 1$, $\Gamma_1/E_0 = 0.014$, and $E_0 \equiv E(p_0)$. A horizontal dashed line marks P_{ind}^T in Eq. (7); ψ_1 and ψ_2 can be considered nonoverlapping to the right of the vertical dashed line. The inset shows $|A(p)|^2$ and $|T(p)|^2$.

resonance barrier, $\omega_j = E_j^r$, $j = 0,1,2,\ldots$. In particular, for $N = 2$, and just one resonance state at E_1^r, we have the interference correction $\delta P^T(\tau) \equiv P^T(\tau) - P_{\text{ind}}^T(\tau)$ given by ($\tau \equiv \tau_{12}$),

$$\delta P^T(\tau) = \frac{2\mu\pi\Gamma_1}{\partial_p E(p_1^r)} |A(p_1^r)|^2 \exp(-\Gamma_1\tau)\cos(E_1^r\tau). \quad (12)$$

For a narrow resonance, $\delta P^T(\tau)$ in Eq. (12) oscillates with a frequency E_1^r, persists even for the delays at which ψ_1 and ψ_2 no longer overlap, and finally vanishes for τ exceeding $1/\Gamma_1$, as illustrated in Fig. 3(a). This is an agreement with the broadly accepted view that, in resonance tunneling, a particle spends approximately a duration of order of the lifetime of the metastable state supported by the barrier [17] (see also [6] and references therein).

With only two resonances accessible to the incident particle, for $C_1 \approx C_2 = C$, and $|\tau(\Gamma_1 - \Gamma_2)| \ll 1$, we find

$$\delta P^T(\tau) \approx 2C\exp(-\Gamma_1\tau)\cos(\delta\omega\tau)\cos(\overline{\omega}\tau), \quad (13)$$

where $\overline{\omega} = (E_1^r + E_2^r)/2$, and $\delta\omega = (E_2^r - E_1^r)/2$. If two resonance levels are close to each other, $\overline{\omega} \gg \delta\omega$, damped rapid oscillations of $\delta P(\tau)$ are modulated with a much lower frequency $\delta\omega$ (see Fig. 4). We recall that for two identical particles, quantum statistical correction to transmission probability was found to oscillate with the frequency $2\delta\omega$ [6], which suggests a certain similarity between two effects. We will return to discuss this further in Sec. V.

Finally, for an initial state (9), containing N identical modes, we have

$$P^T(\tau) \equiv Nw + \sum_j F_j(\tau, E_j^r, \Gamma_j), \quad (14)$$

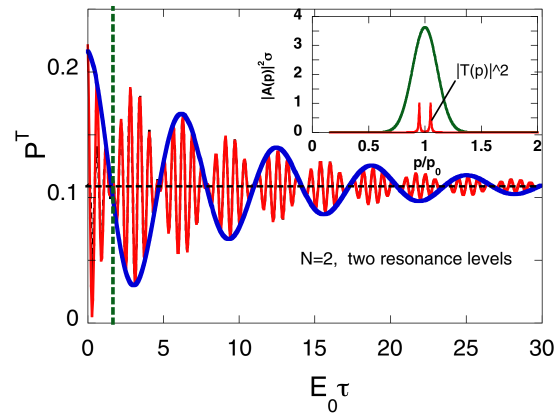

FIG. 4. (Color online) Same as Fig. 3, but for two resonance levels, $p_0\sigma = 9$, $E_1^r/E_0 = 0.9$, $\Gamma_1/E_0 = 0.032$, $E_2^r/E_0 = 1,1$, and $\Gamma_2/E_0 = 0.038$. Also shown by a thick solid line is the envelope $2C\exp(-\Gamma_1\tau)\cos(\delta\omega\tau)$ in Eq. (13).

where the function $F_N(\tau, E^r, \Gamma)$ is given by

$$F_j(\tau, E^r, \Gamma) = C_j \text{Re}\left\{\frac{N}{\exp(-i\mathcal{E}_j^r\tau) - 1} - \frac{\exp[i\mathcal{E}_j^r(N-1)\tau] - \exp(-i\mathcal{E}_j^r\tau)}{[\exp(-i\mathcal{E}_j^r\tau) - 1]^2}\right\}, \quad (15)$$

and we have introduced complex energies $\mathcal{E}_j^r = E_j^r - i\Gamma_j$ to shorten the notations. For $N \gg 1$, it is sufficient to retain only the first term in the curly brackets, which yields

$$F_N(\tau, E^r, \Gamma) = -CN\frac{\cos(E^r\tau) - \exp(-\Gamma\tau)}{\cos(E^r\tau) - \cosh(\Gamma\tau)}. \quad (16)$$

Thus, for $\Gamma\tau \ll 1$, $F_N(\tau, E^r, \Gamma)$ has sharp peaks at $\tau = 2\pi k/E^r$, $k = 1,2,\ldots$, whose heights are proportional to $NE^r/k\Gamma$. When added together, the peaks may give δP^T a highly irregular shape, as shown in Fig. 5.

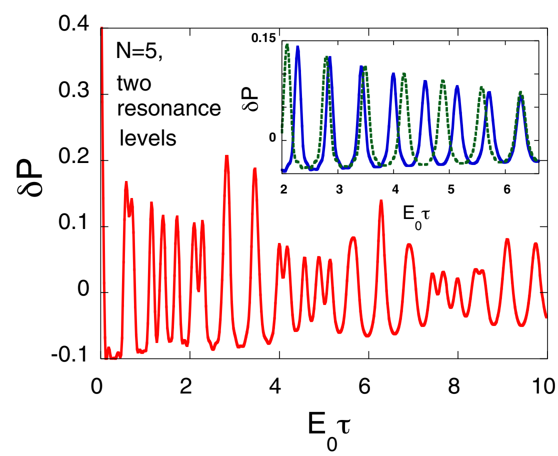

FIG. 5. (Color online) Same as Fig. 4, but for five identical Gaussian components, $N = 5$, and $t_n - t_{n-1} = \tau$. The inset shows the sequences of peaks [cf. Eq. (15)], contributed by each of the two resonances.

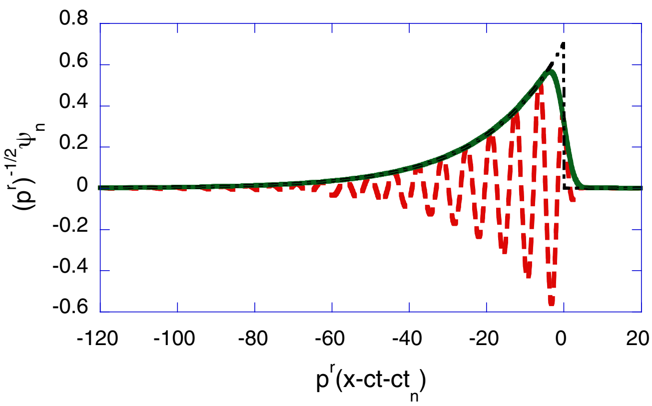

FIG. 6. (Color online) Real part (dashed) and modulus (solid) of ψ_n^T in Eq. (17) transmitted across a barrier supporting a single metastable state, with $p^r\Gamma/c = 0.05$ and $A(p^r)\Gamma/c = 0.11$. Also shown by the dot-dashed line is the modulus of ψ_n^T as given by Eq. (17).

V. THE INTERFERENCE MECHANISM

The task of evaluating $P^T(\tau)$ is particularly straightforward, since we need not follow the evolution of the wave function in the barrier region, and only require the final overlap matrix T_{mn}. By filtering momenta of a ψ_n, a narrow resonance at E_j^r produces a nearly monochromatic transmitted state ψ_{nj}^T with $E \approx E_j^r$. The state is broad in the coordinate space, and the overlaps between ψ_{mj}^T and ψ_{nj}^T, whose relative delay is $(m-n)\tau$, contains a factor $\exp[iE_j^r(m-n)\tau]$. For several well-separated resonances, ψ_{nj}^T and $\psi_{nj'}^T$ with $j \neq j'$ have different energies, and are practically orthogonal, so that transmissions via different metastable states are mutually exclusive events. In this way, summation over all transmitted modes in Eq. (6) produces interference structures shown in Figs. 3–5.

We illustrate this with a simple example by considering nonspreading wave-packet states with a linear dispersion law, $E(p) = cp$. Assuming the Breit-Wigner form for the transmission amplitude, $T(p) = i\Gamma/[(E-E^r)+i\Gamma]$, and putting $A(p) \approx A(p^r)$, we have

$$\psi_n^T(x,t) \approx \frac{2\pi\Gamma A(p^r)}{c}\Phi(x-ct-ct_n), \tag{17}$$

where

$$\Phi(y) \equiv \theta(-y)\exp(ip^r y + \Gamma y/c), \tag{18}$$

and $\theta(y) = 1$ for $y \geqslant 0$ and 0 otherwise. Thus, ψ_n^T in Fig. 6 has a sharp front followed by a long exponential tail, which allows different ψ_n^T to overlap, $\langle\psi_m|\psi_n\rangle \sim \exp[-iE^r(t_m-t_n)]$, even if ψ_n didn't, $I_{mn} = 0$ for $m \neq n$. With two or more resonances involved, ψ_n^T would contain several contributions of the form (17), one for each metastable state.

Our analysis of a *single* particle, prepared in an exotic initial state, helps us to gain an insight into the interference effects accompanying scattering of *several* identical particles. Earlier we considered [6] the case of two fermions or bosons, emitted in the same wave-packet state, with a time delay τ between the emissions. For a barrier with two resonance levels, the two-particle transmission probability $P^T(2,2)$, considered as a function of τ, exhibited oscillations with a frequency $\Delta\omega = E_2^r - E_1^r$. The oscillations disappear if the particles can be distinguished.

We note that in both problems we only require the knowledge of the initial and final overlap matrices, I_{mn}, and T_{mn} [6]. They are, however, used differently. Whereas in the single particle case, considered here, the correction $\delta P^T(\tau) = 2\mathrm{Re}T_{12}$ contains all barrier frequencies E_j^r; the statistical correction to $P^T(2,2)$ depends on $|T_{12}|^2$, and oscillates only with $\Delta\omega = E_2^r - E_1^r$.

What makes the two cases similar is that the particle, or particles, are distributed between wave-packet modes ψ_1 and ψ_2. For a single particle, this is readily seen from Eq. (1). A symmetrized (antisymmetrized) state of two uncorrelated particles, $I_{12} = I_{21} = 0$ is given by $\Psi(x_1,x_2,t) = [\psi_1(x_1,t)\psi_2(x_2,t) \pm \psi_1(x_2,t)\psi_2(x_1,t)]/\sqrt{2}$, which also implies that particle 1 is simultaneously present in both ψ_1 and ψ_2, albeit in a different manner. In both cases, the physical origin of an oscillatory pattern in the transmission probability is the overlap between different ψ_n^T, acquired in the barrier, and the phases carried by different ψ_n, experiencing different time lags. In conclusion, it is worth noticing that our approach relies on the analysis of asymptotic states, thus leaving the details of what actually happens in the scatterer [18] while the particle is inside it beyond the scope of this work.

VI. TRANSMISSION OF A MIXED CAT STATE

Before concluding, we briefly discuss transmission of a mixed cat state with two components, $\psi_1(x,t)$ and $\psi_2(x,t)$, $\langle\psi_m|\psi_m\rangle = \delta_{mn}$, $m,n = 1,2$. The system is prepared in the following way: With a probability $p/2$ it is in one of the states $\psi_1(x,t)$ and $\psi_2(x,t)$, and with a probability $(1-p)$ it is in their coherent superposition, $[\psi_1(x,t)+\psi_2(x,t)]/\sqrt{2}$. The incident density matrix is, therefore, given by

$$\begin{aligned}\rho(x,x') &= [\psi_1(x,t)\psi_1^*(x',t) + \psi_2(x,t)\psi_2^*(x',t)]/2\\ &\quad + (1-p)[\psi_1(x',t)\psi_2^*(x,t) + \psi_2(x,t)\psi_1^*(x',t)]/2,\end{aligned} \tag{19}$$

and for the transmission probability we have

$$P^T = [w_1 + w_2]/2 + (1-p)\mathrm{Re}[T_{12}(\tau)]. \tag{20}$$

Thus, for a pure state, $p = 0$ we recover Eq. (5). As p increases, the last interference term in Eq. (20) becomes smaller, and finally vanishes for the incoherent combination of the two states, $p = 1$, where $P^T = (w_1+w_2)/2$. This simple result is easily extended to the case where the initial state has three, or more, components, $N > 2$.

VII. CONCLUSIONS AND DISCUSSION

In summary, various wave-packet modes of the same one-particle state, well separated (nonoverlapping) initially, may coincide inside a scatterer, provided the particle is detained there for an appreciable period of time. Then the interference, resulting from this "pile-up" effect, may significantly change the tunneling probability P^T. The effect requires that transmitted modes be significantly broadened in the coordinate space, and is absent in tunneling across a rectangular barrier. It is present in resonance tunneling, where P^T, considered as a function of the time between the arrivals of two consecutive modes at the barrier, oscillates with internal frequencies of the barrier. We have shown that the interference patterns predicted for the resonance transmission of several identical particles [6,7], have a similar origin, both resulting from the particle being distributed, in one way or another, between different wave-packet components. The interference patterns are washed out if the initial state is mixed, rather than pure. The proposed type of "interferometry in the time domain" is within the capability of modern experimental techniques [9,19].

ACKNOWLEDGMENTS

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