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Analytical solution for the Lévy-like steady-state distribution of intensities in random lasers

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We derive analytically the Lévy-like steady-state distribution with exponential tempering of emission intensities in random lasers. Our approach is based on the Langevin and associated Fokker-Planck equations describing the dynamics of the amplitudes of the resonance modes in a cavity with a disordered nonlinear dielectric medium. The reported results fully agree with the experimental characterization of the prelasing, Lévy-like, and self-averaged Gaussian lasing regimes in a random laser system as a function of the pump energy and disorder strength, as well as with the recent suggestion of the Lévy exponent α as a universal identifier of the random lasing threshold.

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I. INTRODUCTION

In the last three decades a great diversity of stochastic phenomena have been described in terms of the statistics of Lévy flights and walks, with examples ranging from human mobility [1–3] to animal foraging [4–6], fluid dynamics [7,8], and photons [9], just to name a few. In the photonic world, disordered materials have always been considered detrimental to light propagation. However, advances in basic research have shown that disorder can be exploited to the understanding of light propagation in such media. One of the most studied examples is the Anderson localization of light [10], in analogy with Anderson localization of electrons [11]. Light propagation in turbid media, where disorder is present, has also been well studied with important implications in imaging and information retrieval [12]. It has been proposed [13] and demonstrated [14] that laser emission in random amplifying media, under the proper conditions, can be obtained, and the field of random lasers (RLs) has grown fantastically since the first unambiguous observation of such mesoscopic devices in 1994 [14]. Random lasers differ from conventional lasers by the fact that the optical feedback, which is usually provided by static mirrors, in this case occurs due to the feedback from the scattering particles. Several features are already well understood in RLs, as reviewed in Refs. [15–18]. The scatterers for the RLs can be either dielectric or metallic, which in the latter case can lead to plasmonic enhancement. Both bulk or waveguide geometries, including random fiber lasers [19,20], have been demonstrated. The photon statistics of the RL well above threshold is Poissonian, as for the conventional lasers. Although the RL is cavityless, in the sense of a static mirror cavity, it is not modeless [21]. There are two regimes for the feedback in RLs, known as nonresonant or incoherent feedback and resonant or coherent feedback [22]. The spectral behavior above the threshold is a clear distinct signature between the two regimes, as spikes, which are related to the RL modes, appear in the coherent regime, as opposed to smooth spectrum in the nonresonant regime. This feature has been first demonstrated in Ref. [22], and was also recently corroborated in three-photon pumped ZnO-based RLs in two different designs, namely, a ZnO-on-Si thin film [23] and a submicron scale ZnO powder [24]. Another important characteristic of RLs is the threshold dependence on scatter concentration [25].

Among several cross-disciplinary phenomena studied in RLs, its proposed glassy behavior [26–29] has led to the recent experimental demonstration of replica symmetry-breaking phase transition [30]. In another set of theoretical and experimental work, the RL emission intensity and fluctuations statistical properties have been studied [31–41] and demonstrated to behave as Lévy-like statistics. Intrinsic intensity fluctuations, not arising from the pump intensity fluctuation, were first reported in Ref. [31], while the first theoretical insights were reported in Refs. [32,34,36]. Different statistical regimes of RL fluctuations were clearly identified in [35,39]. More recently, Uppu and Mujumdar [41] proposed the use of the exponent associated with the α-stable Lévy distribution as a universal identifier of the threshold and criticality in RLs by performing experiments with dye-scatterer-based RLs, in which the scatterer was ZnO.

In this paper, we derive analytically the Lévy-like steady-state distribution with exponential tempering of emission intensities in RLs. Our approach is based on the Langevin and associated Fokker-Planck equations describing the dynamics of the amplitudes of the resonance modes in a cavity with a disordered nonlinear dielectric medium. Our theoretical results agree quite well with the behavior reported in Ref. [41].

II. COUPLED LANGEVIN EQUATIONS FOR THE INTENSITY OF THE RESONANCE MODES IN A DISORDERED NONLINEAR MEDIUM

We start by considering a disordered nonlinear dielectric medium in a resonant cavity of a RL. The spatial randomness of the active medium implies a static refractive index with a spatially random profile, $n(\mathbf{r}) = c\sqrt{\mu_0\varepsilon(\mathbf{r})}$. Since the electromagnetic cavity may support a large number N of overlapping resonance modes we write the leading-order electromagnetic

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field solution of the nonlinear Maxwell equations as [26–29]

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\left\{\sum_{n=1}^{N} a_n(t)\mathbf{E}_n(\mathbf{r})\exp(-i\omega_n t)\right\}, \quad (1)$$

$$\mathbf{H}(\mathbf{r},t) = \mathrm{Re}\left\{\sum_{n=1}^{N} a_n(t)\mathbf{H}_n(\mathbf{r})\exp(-i\omega_n t)\right\}, \quad (2)$$

where the real valued field amplitudes $\{\mathbf{E}_n(\mathbf{r}),\mathbf{H}_n(\mathbf{r})\}$ appear as above modified by the complex time-dependent adimensional prefactor $a_n(t)$ due to the nonlinearity of the amplifying medium. We take

$$a_n(t) = A_n(t)\exp[i\varphi_n(t)], \quad (3)$$

with the dynamics of the real amplitude $A_n(t)$ evolving much slower than that of the phase $\varphi_n(t)$. The intensity signal of the RL associated with a given resonance frequency ω_n can be expressed as the time average $I_n = \langle I_n(t)\rangle$, with

$$I_n(t) = c_n A_n^2(t), \quad (4)$$

in which the mode-dependent proportionality constant c_n is fixed through the average flux per area A of the electromagnetic power, $c_n = (2A)^{-1}\int_A dA[\mathbf{E}_n(\mathbf{r})\times\mathbf{H}_n(\mathbf{r})]\cdot\mathbf{n}(\mathbf{r})$. The total intensity is thus $I = \sum_{n=1}^{N} I_n$.

The dynamics of the complex amplitudes $\{a_n(t)\}$ is governed by the system of coupled Langevin equations of motion [27]:

$$\frac{da_n}{dt} = -\frac{1}{2}\sum_{\{p,q,r\}'=1}^{N} g_{npqr}a_q a_r a_p^* + (\gamma_n - \alpha_n)a_n + \eta_n,$$
$$n = 1,2,\ldots,N, \quad (5)$$

where γ_n and α_n denote, respectively, the mode-dependent amplification (gain) and radiation loss coefficient rates, and the complex term $\eta_n(t)$ accounts for the Gaussian (white) optical noise, so that $\langle\eta_n(t)\rangle = \langle\eta_n^*(t)\rangle = 0$ and two-time correlations given by $\langle\eta_n(t)\eta_m(t')\rangle = \langle\eta_n^*(t)\eta_m^*(t')\rangle = 0$ and $\langle\eta_n(t)\eta_m^*(t')\rangle = 2D\delta_{n,m}\delta(t-t')$, with the variance D providing a measure of the noise amplitude related to the heat-bath temperature of the system. The symbol $\{p,q,r\}'$ in Eq. (5) indicates that the sum is restricted to mode combinations such that [42] $\omega_n + \omega_p = \omega_q + \omega_r$.

The fourth-rank complex tensor g_{npqr} in Eq. (5) marks the signature of the nonlinearity of the active medium of volume V through the leading-order nonlinear polarization:

$$g_{npqr} = \frac{1}{2i}\int_V dV \sum_{\{\alpha,\beta,\gamma,\delta\}=x,y,z} \chi^{(3)}_{\alpha\beta\gamma\delta}(\omega_n;\omega_q,\omega_r,-\omega_p,\mathbf{r})$$
$$\times E_n^\alpha(\mathbf{r})E_p^\beta(\mathbf{r})E_q^\gamma(\mathbf{r})E_r^\delta(\mathbf{r}), \quad (6)$$

where the spatial randomness of the disordered medium is also implied in the third-order response susceptibility tensor $\chi^{(3)}_{\alpha\beta\gamma\delta}(\omega_n;\omega_q,\omega_r,-\omega_p,\mathbf{r})$. Indeed, in the absence of disorder the real part of the tensor g_{npqr} assumes a constant value, $g^{\mathrm{R}}_{npqr} = g$, as in the case of standard passively mode-locked laser systems [28,43].

The presence of random disorder in g_{npqr} makes rather difficult the approach to the system (5) of N coupled equations. In this sense, by considering g_{npqr} as equally Gaussian-distributed random variables g and applying replica symmetry-breaking techniques with the phases $\{\varphi_n(t)\}$ as the relevant dynamical variables, a phase diagram was built [26–29] with ferromagnetic- and glassylike phases (depending on the tendency of the disorder to hamper the synchronous oscillation of the modes), as a function of the average g (related to the pumping energy rate and the heat-bath temperature) and variance (disorder strength) of the Gaussian distribution in the strong-coupling regime.

By considering Eqs. (3)–(5) we obtain

$$\frac{1}{c_n}\frac{dI_n}{dt} = -\mathrm{Re}\left\{\sum_{\{p,q,r\}'=1}^{N} g_{npqr}a_n^* a_q a_r a_p^* - 2a_n^*\eta_n\right\}$$
$$+2(\gamma_n - \alpha_n)\frac{I_n}{c_n}. \quad (7)$$

The restricted sum above encompasses three classes of mode combinations [42] $\omega_n = \omega_q$ and $\omega_r = \omega_p$, $\omega_n = \omega_r$, and $\omega_q = \omega_p$, and the remaining possibilities provided that $\omega_n + \omega_p = \omega_q + \omega_r$. In particular, the latter class has been usually disregarded [27,42] so that Eq. (7) becomes

$$\frac{dI_n}{dt} = -g^{\mathrm{R}}_{nnnn}\frac{I_n^2}{c_n} - I_n\sum_{r=1(r\neq n)}^{N}\left(g^{\mathrm{R}}_{nrnr} + g^{\mathrm{R}}_{nrrn}\right)\frac{I_r}{c_r}$$
$$+2c_n\mathrm{Re}\{a_n^*\eta_n\} + 2(\gamma_n - \alpha_n)I_n. \quad (8)$$

By expressing the optical noise as the sum of additive and multiplicative statistically independent stochastic processes [44], so that $\eta_n(t) = \eta_n^{(0)}(t) + a_n(t)\eta_n^{(1)}(t)$, we write

$$\mathrm{Re}\{a_n^*\eta\} = \sqrt{\frac{I_n}{c_n}}\left[\eta_n^{(0)\mathrm{R}}\cos(\varphi_n) + \eta_n^{(0)\mathrm{I}}\sin(\varphi_n)\right] + \frac{I_n}{c_n}\eta_n^{(1)\mathrm{R}}. \quad (9)$$

As the phases $\{\varphi_n(t)\}$ vary much more rapidly [27,42] than the amplitudes $\{A_n(t)\}$, they can be averaged out, leading to the corresponding system of coupled stochastic Langevin equations governing the dynamics of the mode intensities $\{I_n(t)\}$:

$$\frac{dI_n}{dt} = -g^{\mathrm{R}}_{nnnn}\frac{I_n^2}{c_n} - I_n\sum_{r=1(r\neq n)}^{N}\left(g^{\mathrm{R}}_{nrnr} + g^{\mathrm{R}}_{nrrn}\right)\frac{I_r}{c_r}$$
$$+2\eta_n^{(1)\mathrm{R}}I_n + 2(\gamma_n - \alpha_n)I_n,$$
$$n = 1,2,\ldots,N. \quad (10)$$

III. FOKKER-PLANCK EQUATION FOR THE PROBABILITY DISTRIBUTION OF INTENSITIES

The presence of the Gaussian white noise in Eq. (10) allows an exact connection [44] between the system of coupled Langevin equations and the following Fokker-Planck equation for the probability density of intensities $P(\{I_m\},t)$:

$$\frac{\partial P}{\partial t} = -\sum_{n=1}^{N}\frac{\partial}{\partial I_n}[(\mathcal{L}_n + 2QI_n)P] + 2Q\sum_{n=1}^{N}\frac{\partial^2}{\partial I_n^2}\left(I_n^2 P\right), \quad (11)$$

where the parameter Q controls the magnitude of the multiplicative fluctuations through $\langle \eta_n^{(1)\mathrm{R}}(t)\eta_m^{(1)\mathrm{R}}(t')\rangle = Q\delta_{n,m}\delta(t-t')$, and we define

$$\mathcal{L}_n(\{I_m\}) = -g^{\mathrm{R}}_{nnnn}\frac{I_n^2}{c_n} - I_n \sum_{r=1(r\neq n)}^{N} \left(g^{\mathrm{R}}_{nrnr} + g^{\mathrm{R}}_{nrrn}\right)\frac{I_r}{c_r} + 2(\gamma_n - \alpha_n)I_n. \quad (12)$$

With the ansatz $P(\{I_m\},t) = P^{(0)}(\{I_m\})\exp(-\lambda t)$, where $P^{(0)}(\{I_m\}) = P(\{I_m\},t=0)$ represents the initial condition and the boundary condition satisfies $P(\{I_m\}\to\infty,t) = 0$, the solution of Eq. (11) can be expressed in terms of an eigenfunction expansion associated with the (either discrete or continuous) set of eigenvalues $\{-\lambda_i\}$, so that

$$L P_i^{(0)}(\{I_m\}) = -\lambda_i P_i^{(0)}(\{I_m\}), \quad (13)$$

in which the Fokker-Planck operator reads

$$L(\{I_m\}) = -\sum_{n=1}^{N}\frac{\partial}{\partial I_n}(\mathcal{L}_n + 2QI_n) + 2Q\sum_{n=1}^{N}\frac{\partial^2}{\partial I_n^2}I_n^2. \quad (14)$$

In general, exact solutions of Fokker-Planck equations such as Eq. (11), or the associated eigenvalue equations (13), can be hardly ever found [44] mainly if $\mathcal{L}_n(\{I_m\})$ or the Fokker-Planck operator $L(\{I_m\})$ are nonlinear in the intensities $\{I_m\}$, which actually corresponds to the present case. In spite of this, by taking both $\{A_m(t)\}$ and $\{I_m(t)\}$ (with $I_m = c_m A_m^2$) as slowly varying with respect to $\{\varphi_m(t)\}$, in leading order we can approximate $\{I_r\}$ as constants in Eq. (12), so to yield the new set of Langevin and Fokker-Planck equations, respectively,

$$\frac{dI_n}{dt} = d_n I_n - b_n I_n^2 + 2\eta_n^{(1)\mathrm{R}} I_n, \quad (15)$$

and

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial I_n}[(\mathcal{L}_n + 2QI_n)P] + 2Q\frac{\partial^2}{\partial I_n^2}(I_n^2 P), \quad (16)$$

in which

$$\mathcal{L}_n(\{I_m\}) = d_n I_n - b_n I_n^2, \quad (17)$$

and with the new coefficients respectively identified as

$$d_n = -\sum_{r=1(r\neq n)}^{N}\left(g^{\mathrm{R}}_{nrnr} + g^{\mathrm{R}}_{nrrn}\right)\frac{I_r}{c_r} + 2(\gamma_n - \alpha_n) \quad (18)$$

and

$$b_n = \frac{g^{\mathrm{R}}_{nnnn}}{c_n}. \quad (19)$$

The slow dynamics of I_n is also expected to influence the behavior of the distribution P. In this sense the steady-state solution of Eq. (16) can thus be found by straightforward integration:

$$P_{\mathrm{ss}}(I_n) = \frac{\mathcal{A}_n}{I_n^{\mu_n}}\exp\left(-\frac{g^{\mathrm{R}}_{nnnn}}{2c_n Q}I_n\right), \quad (20)$$

where the normalization constant reads

$$\mathcal{A}_n = \frac{1}{\Gamma(d_n/2Q)}\left(\frac{b_n}{2Q}\right)^{d_n/(2Q)}, \quad (21)$$

and with the power-law exponent defined as

$$\mu_n = 1 + \frac{1}{2Q}\sum_{r=1(r\neq n)}^{N}\left(g^{\mathrm{R}}_{nrnr} + g^{\mathrm{R}}_{nrrn}\right)\frac{I_r}{c_r} - \frac{1}{Q}(\gamma_n - \alpha_n). \quad (22)$$

The connection between the quantities displayed in Eq. (22) and the experimental parameters such as the pump energy and the disorder strength can be expressed as follows. First, by writing the displacement and magnetic induction vectors, respectively, as $\mathbf{D} = \varepsilon_0 n^2\mathbf{E} + \mathbf{P}_{\mathrm{NL}}$ and $\mathbf{B} = \mu_0\mathbf{H}$, with the leading-order nonlinear polarization given by

$$\mathbf{P}_{\mathrm{NL}}(\mathbf{r},t) = \mathrm{Re}\left\{\sum_{\{n,p,q,r\}'}\sum_{\alpha,\beta,\gamma,\delta}\mathbf{e}_\alpha\chi^{(3)}_{\alpha\beta\gamma\delta}(\omega_n;\omega_q,\omega_r,-\omega_p,\mathbf{r}) \times E_p^\beta(\mathbf{r})E_q^\gamma(\mathbf{r})E_r^\delta(\mathbf{r})\exp(-i\omega_n t)\right\} \quad (23)$$

and the tensor g_{npqr} defined in Eq. (6), the contribution to the average electromagnetic energy due to the nonlinear character of the active medium reads

$$E_{\mathrm{NL}} = \frac{1}{2}\sum_{n,r}\left(g^{\mathrm{R}}_{nrnr} + g^{\mathrm{R}}_{nrrn}\right)\frac{I_n}{c_n}\frac{I_r}{c_r}. \quad (24)$$

Thus, an increasing pump energy E_{p} induced on the RL by the pumping laser source leads to a larger sum in Eq. (22). On the other hand, the noise amplitude Q is proportional to the heat-bath temperature of the system, and plays an opposite role with respect to the pump energy in Eq. (22). Indeed, it has been experimentally recognized that the increasing of the pump energy or the decreasing of the heat-bath temperature promotes the same qualitative effect in RL systems [29,41,45,46].

Moreover, the tensor elements g^{R}_{nrnr} and g^{R}_{nrrn} also embody the random disorder properties of the active medium. In particular, the strength of disorder can be measured through the variance of the distributions of values of g^{R}_{nrnr} and g^{R}_{nrrn} in Eq. (22). Indeed, in the absence of disorder the dispersion of these quantities is null, so that $g^{\mathrm{R}}_{nrnr} = g^{\mathrm{R}}_{nrrn} = g$, as mentioned above.

IV. LÉVY, TRUNCATED LÉVY, GAUSSIAN STATISTICS, AND RANDOM LASERS

It is currently a well-documented fact [34,36,41] that RLs constituted by a resonant cavity with a disordered nonlinear dielectric medium emit spectra of intensity signals $\{I_n\}$ which fluctuate considerably from shot to shot (in a pulsed laser) or, in general, along a sequence of output measurements. In some circumstances (depending, e.g., on the pumping energy rate and the disorder strength above the random lasing threshold) the statistics of the measured values of I_n for each resonance frequency ω_n have been found to follow Lévy-like distributions [36,40,41].

As mentioned in the Introduction, the range of applications of Lévy-like statistics is far from being restricted to the domain of nonlinear optics and RLs. In this sense consider, for instance, a random variable $u \in (-\infty,\infty)$ whose probability density function (PDF) presents a *diverging* second moment. The generalized central limit theorem states that the sum $x =$

$\sum_{i=1}^{N_s} u_i$ of a number $N_s \gg 1$ of such identically distributed and statistically independent variables is distributed according to the α-stable Lévy PDF $P(x)$, expressed by the Fourier transform of the characteristic function [47,48]:

$$\bar{P}(k) = \exp\{-|ck|^{\alpha}[1 - i\beta\ \text{sgn}(k)\Phi] + ik\nu\}. \tag{25}$$

Above, $\beta \in [-1,1]$, $\nu \in (-\infty,\infty)$ and $c \in (0,\infty)$ denote, respectively, the asymmetry (skewness), location, and scale parameters, and $\Phi = \tan(\pi\alpha/2)$ if $\alpha \neq 1$, whereas $\Phi = -(2/\pi)\ln|k|$ if $\alpha = 1$. The Lévy index $\alpha \in (0,2]$ represents the most important parameter. Indeed, only for three special values of α a closed form expression of $P(x)$ can be found [47,48]: Lévy distribution with $\alpha = 1/2$ and $\beta = 1$, Cauchy distribution for $\alpha = 1$ and $\beta = 0$, and Gaussian distribution for $\alpha = 2$ and any allowed value of β. In addition, the asymptotic large-$|x|$ behavior of $P(x)$ has a power-law heavy tail in the form $P(x) \sim |x|^{-(\alpha+1)}$.

It is also interesting to notice that if the random variable u is power-law (Pareto) distributed, so that

$$P(u) = \frac{A}{|u|^{\mu}}, \quad |u| \geqslant u_0, \tag{26}$$

where the lower cutoff u_0 allows for a finite normalization constant A, then its second moment diverges algebraically [4] for $1 < \mu < 3$ and is finite for $\mu > 3$ (the logarithm divergence of the case $\mu = 3$ must be considered carefully [49] and the case $\mu \leqslant 1$ is non-normalizable). Therefore, according to the reasoning above the statistics of the sum x of N_s of these variables converges, respectively, to the Lévy ($0 < \alpha < 2$, with $\alpha = \mu - 1$) and Brownian regimes ($\alpha = 2$, for $\mu \geqslant 3$). In particular, whenever the central limit theorem holds ($\alpha = 2$) the convergence to the Gaussian PDF $P(x)$ with average $N_s\bar{u}$ and variance $N_s\sigma_u^2$ is quickly achieved [47,48] (in terms of N_s), where $\bar{u}$ and σ_u^2 denote, respectively, the average and variance of the original random variable u.

As infinite intensities I_n in RLs (or, e.g., infinite distances traversed by humans in mobility activity [1–3]) are clearly unaccessible, actual Lévy statistics with diverging variance cannot be strictly associated to most (if not all) realistic phenomena. Instead, *truncated* Lévy PDFs [50] with finite second moment have been largely employed to describe real systems [4]. In this sense a number of possibilities for the truncation of random variables at large values have been designed. The simplest one is just to impose an upper cutoff $u_{\max}$ to the PDF so that $P(u) = 0$ for $u > u_{\max}$. Another form also often used is to temper the power-law Lévy PDF with an exponential decay:

$$P(u) = \frac{\bar{A}}{|u|^{\mu}} \exp(-\gamma|u|), \quad |u| > 0, \tag{27}$$

with the tempering parameter $\gamma > 0$ and normalization constant $\bar{A}$. In contrast with the $\alpha = 2$ Brownian regime discussed above, the convergence of the truncated Lévy PDFs to the Gaussian statistics is ultraslow [50], achieved only for a remarkably large N_s, which increases even further for a larger $u_{\max}$ or smaller γ. As a consequence, the general properties of actual (i.e., nontruncated) Lévy statistics should indeed be retained to a considerable extent in general stochastic phenomena described by truncated Lévy PDFs [51]. In this context the term *Lévy-like* thus sounds more appropriate to characterize this sort of statistical behavior than the term *Lévy* itself.

In the specific context of RLs, Uppu and collaborators have applied [37–41] in a series of recent articles both truncation schemes to power-law Lévy distributions of intensity signals. It is thus instructive to briefly review the arguments [36], based on the the exponential amplification of diffusing spontaneous photons which interact with the gain medium and get multiply scattered, leading to stimulated emission with a power-law PDF of intensities similar to that of Eq. (26).

By denoting L as the first-passage path length of a photon i of wavelength λ exiting the resonant cavity, and since the amplification process scales exponentially with L, then the associated emission intensity signal is $I_i(L,\lambda) = I_0 \exp(L/\ell_g)$, where I_0 is its minimum value as the process starts with a single spontaneous emission photon and $\ell_g(\lambda)$ is the gain length of the active medium, defined as the mean distance traveled for the first amplification of the photon. As the PDF of values of L is Poissonian, the cumulative distribution of the intensity of the exiting photon is $F(I_i) = \int_{\ell_g \ln(I_i/I_0)}^{\infty} \exp(-L/\bar{L})dL/\bar{L}$, where $\bar{L}$ is the mean first-passage length of the exiting photons, which leads [36] to the PDF for the emission signal I_i of a single photon of wavelength λ:

$$P(I_i) = \frac{A}{I_i^{\phi}}, \quad I_i \geqslant I_0, \tag{28}$$

with

$$\phi(\lambda) = 1 + \ell_g/\bar{L}. \tag{29}$$

Therefore, the total emission intensity due to N_{ph} exiting photons with resonance wavelength λ_n is given by the sum of random variables, $I_n = \sum_{i=1}^{N_{ph}} I_i(\lambda_n)$. As discussed above, depending on the value of the power-law exponent ϕ this sum is ultimately attracted by the α-stable Lévy ($1 < \phi < 3$, with $\alpha = \phi - 1$) or Gaussian ($\phi \geqslant 3$, $\alpha = 2$) statistics. However, as also indicated, an infinity value of I_i corresponding to an infinite path length L of the diffusing photon is not physically allowed, and some truncation scheme (either an upper cutoff $I_{\max}$, an exponential tempering, or another) must be imposed to the PDF (28) in order to guarantee a finite variance of the measured values of I_n.

At this point we observe that the comparison between the PDF $P(I_i)$ above and the steady-state distribution of intensity signals $P_{ss}(I_n)$, Eqs. (20)–(22) derived in the preceding section, is particularly enlightening. First, whereas $P(I_i)$ was calculated with a basis on the paths of individual diffusing photons in the active medium, $P_{ss}(I_n)$ took into account from the very beginning the Langevin dynamics and associated Fokker-Planck equation of the complex amplitudes $\{a_n(t)\}$ of the resonance modes, which, as stated in Eqs. (1) and (2), drive the influence of the nonlinearity of the disordered medium on the electromagnetic field. In this sense, it is also worth mentioning that a PDF similar to Eq. (20) for the RL energy field has been also derived in the context of a random walk model and associated Fokker-Planck equation driving the distribution of the positions and energies of the walker. Secondly, the important exponential tempering that removes the unphysical divergence in the second moment of the distribution of I_n values arises naturally in $P_{ss}(I_n)$, whereas it must be externally imposed to the PDF $P(I_i)$ through a suitable truncation factor. In spite of this, it is clear that the diffusion properties of the scattered photons and the dynamics features of the resonance modes are intrinsically interrelated. Their dependence upon the pump energy and disorder strength is analyzed in the next section.

V. DISCUSSION AND CONCLUSIONS

In a recent work, Uppu and Mujumdar have proposed [41] the Lévy exponent α as a universal identifier of the lasing threshold and the distinct statistical regimes characterizing the pulse-to-pulse fluctuations of intensity in RLs. By measuring α as a function of the pump energy E_p and disorder strength for a statistically relevant ($N_s = 2000$) set of emission spectra, they have successfully compared this suggestion with the conventional definitions for the threshold, namely, the probability of coherent random lasing and the intensity enhancement and bandwidth collapse for diffusive random lasing emission. In that work the disorder strength was characterized by the transport mean free path ℓ^* of a photon before it changes direction in the disordered active medium, a parameter which depends on the density of random scatterers and whose measure is accessible by the coherent backscattering technique. In the following we argue that the physical conclusions drawn from our results for the PDF of intensity signals $P_{ss}(I_n)$, Eqs. (20)–(22), are in full agreement with the reported findings by Uppu and Mujumdar [41].

A. Prelasing Gaussian emission

For small pump energies below the lasing threshold, $E_p \leqslant E_{th}$, the emission spectrum of a RL displays no lasing peaks. Indeed, only the standard fluorescence spectrum with weak magnitude of fluctuations is observed in the subthreshold domain. In this regime Uppu and Mujumdar have experimentally measured the value $\alpha = 2$ for the distribution of intensities I_n (see Figs. 2 and 3 in Ref. [41]), a result indicative of Gaussian behavior of its PDF for each resonance mode n.

In terms of our findings expressed by Eqs. (20)–(22), this picture corresponds to the Gaussian regime with $\mu_n \leqslant 0$, in which no Lévy-like power-law decay of the distribution $P_{ss}(I_n)$ can be identified. A quick convergence (in terms of N_s) of the PDF $P_{ss}(I_n)$ with finite second moment to the $\alpha = 2$ Gaussian-attracted distribution of intensities is thus achieved provided that

$$\frac{1}{2}\sum_{r=1(r\neq n)}^{N}\left(g^{R}_{nrnr}+g^{R}_{nrrn}\right)\frac{I_r}{c_r} \leqslant \gamma_n-\alpha_n-Q. \tag{30}$$

Moreover, according to the discussion on Eq. (24) relating the electromagnetic energy in the cavity to the sum above involving the tensor elements g^{R}_{nrnr} and g^{R}_{nrrn}, we observe that this inequality can be actually satisfied for small pump energies, i.e., below the threshold.

B. Lévy-like random lasing

For intermediate pump energies, $E_{th} < E_p \lesssim E_G$, the system enters the coherent lasing regime, with the distribution of intensities presenting non-Gaussian statistics [36,41] characterized by strong fluctuations and the Lévy exponent in the range $0 < \alpha < 2$. The fluctuations are largest and the smallest value of α is reached in the vicinity of the threshold. In this sense, by considering that the ultraslow convergence of truncated Lévy PDFs to the Gaussian behavior occurs only for a remarkably large N_s (see Sec. IV), the distribution $P_{ss}(I_n)$ actually retains the Lévy properties to a considerable extent provided that $0 < \mu_n < 3$, i.e., for

$$\gamma_n-\alpha_n-Q < \frac{1}{2}\sum_{r=1(r\neq n)}^{N}\left(g^{R}_{nrnr}+g^{R}_{nrrn}\right)\frac{I_r}{c_r} < \gamma_n-\alpha_n+2Q. \tag{31}$$

We notice that, since each resonance mode presents its own value of μ_n, a dependence of α and E_{th} on the frequency ω_n is expected, a fact also probed experimentally [41].

As the pump energy increases beyond some typical value E_G, condition (31) is no longer fulfilled and the PDF $P_{ss}(I_n)$ reenters the Gaussian regime since the Lévy-like behavior is not possible for $\mu_n \geqslant 3$ (see below). This result necessarily implies a nonmonotonic behavior of the exponent $\alpha \in (0,2)$ in the Lévy-like RL regime placed between two Gaussian domains with $\alpha = 2$. Indeed, as the power-law exponent μ_n changes signal at the vicinity of the random lasing threshold, the abrupt change from the weak Gaussian to the strong Lévy fluctuation regime is accompanied by a sharp decrease in the Lévy exponent α. From this point, a further enhancement of the pump energy leads to a higher μ_n and a relatively smooth increase in α towards the second Gaussian regime. The above features are clearly seen in Fig. 2 of Ref. [41].

For a given pump energy in the Lévy-like random lasing regime it is also interesting to notice that an increase in the disorder strength, associated with a smaller ℓ^*, promotes a larger width of the distribution of random values of the tensor elements g^{R}_{nrnr} and g^{R}_{nrrn}. This results in a larger μ_n and a weaker Lévy character [larger $\alpha \in (0,2)$] of the intensity fluctuations, as well as in a higher threshold value E_{th} of the pump energy. This reasoning has been also experimentally confirmed [40,41] by varying the density of scatterers in the gain medium of RLs.

Moreover, as also discussed, due to the actual impossibility of infinity intensity the exponentially tempered truncation of the PDF $P_{ss}(I_n)$ will eventually promote a crossover to the Gaussian regime for some extensively large N_s, in agreement with recent numerical results [40] obtained using photon transport Monte Carlo simulations applied to RLs.

C. Gaussian random lasing

As the pump energy exceeds a characteristic value, $E_p \gtrsim E_G$, the system crosses over towards a Gaussian lasing regime for $\mu_n \geqslant 3$, which is in fact very distinct from the subthreshold Gaussian behavior described above for $\mu_n < 0$. Indeed, as the Gaussian character becomes more evident at high pump energies, the narrow lasing peaks and the strong fluctuations of intensity typical of the Lévy-like regime reduce considerably, so that the gain becomes redistributed among the large number of strongly coupled resonance modes in this so-called self-averaged random lasing regime [38,39]. In this sense, Gaussian shot-to-shot fluctuations in the range $\mu_n \geqslant 3$ assure that $P_{ss}(I_n)$ is governed by the central limit theorem in this regime, in which

$$\frac{1}{2}\sum_{r=1(r\neq n)}^{N}\left(g^{R}_{nrnr}+g^{R}_{nrrn}\right)\frac{I_r}{c_r} \geqslant \gamma_n-\alpha_n+2Q. \tag{32}$$

In conclusion, by applying in this work the Langevin dynamics and associated Fokker-Planck equation for the amplitudes of the resonance modes in a RL system, we have analytically derived the steady-state distribution of emission intensities, constituted by a power-law Lévy-like dependence tempered by an exponential truncation. Our results discussed as a function of the pump energy and disorder strength are in agreement with the experimental characterization of the prelasing, Lévy-like, and self-averaged Gaussian lasing regimes, as well as with the recent suggestion by Uppu and Mujumdar [41] of the Lévy exponent α as a universal identifier of the random lasing threshold.

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