Nonlinear waves in two-component Bose-Einstein condensates: Manakov system and Kowalevski equations

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Traveling waves in two-component Bose-Einstein condensates whose dynamics is described by the Manakov limit of the Gross-Pitaevskii equations are considered in a general situation with relative motion of the components when their chemical potentials are not equal to each other. It is shown that in this case the solution is reduced to the form known in the "Kowalevski top" theory of motion. Typical situations are illustrated by the particular cases when the general solution can be represented in terms of elliptic functions and their limits. Depending on the parameters of the wave, both density waves (with in-phase motions of the components) and polarization waves (with counterphase motions) are considered.

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I. INTRODUCTION

Realization of Bose-Einstein condensates (BECs) of atoms which can occupy several quantum states at extremely low temperatures has drawn interest to nonlinear dynamics of such multicomponent systems (see, e.g., the review article [1]). In particular, if the components are miscible, then in such a condensate two types of linear waves can propagate—usual sound waves (that is, density waves) with in-phase oscillations of the components and so-called *polarization waves* with counterphase oscillations. Correspondingly, generally speaking, there exist two Mach cones and two channels of Cherenkov radiation which leads to considerable changes in the character of excitations in the system compared with one-component situation. The same holds true for the nonlinear excitations—solitons and breathers. For example, oblique solitons can be generated by the flow of two-component BECs past a nonpolarized obstacle which repels both components [2] and oblique breathers are generated in the case of polarized obstacles which repel one component and attract the other one [3].

The dynamics of two-component BECs becomes even richer if one component moves with respect to another. As was found experimentally in [4–6], the relative motion of components leads to generation of nonlinear periodic waves of polarization. In these experiments the components correspond to the quantum states $|1,-1\rangle$ and $|2,-2\rangle$ of the hyperfine structure of ^{87}Rb atoms with very close values of their interatomic scattering lengths. Hence their quasi-one-dimensional dynamics in elongated cigar-shaped traps can be described with high accuracy by the Gross-Pitaevskii equations of the Manakov type [7]. In the standard nondimensional units these equations can be written in the form

$$i\psi_{k,t} + \tfrac{1}{2}\psi_{k,xx} - (|\psi_1|^2 + |\psi_2|^2)\psi_k = 0, \qquad k = 1,2, \quad (1)$$

where ψ_k denotes the wave function of the kth component, x is the coordinate along the trap, and t is the time variable. Such multicomponent (vector) nonlinear Schrödinger equations appeared first in nonlinear optics [8] and they have been studied intensely in this physical context, where it is natural to suppose that the wave numbers (which are analogs of velocities of the BEC components) of both components are equal to each other (see, e.g., [9]), but the interaction constants are different. Thus, so far the situation with equal nonlinearity constants and nonvanishing relative velocity of the components was studied very little. An important particular case of such type of solutions is the so-called *dark-bright soliton* with vanishing of one of the background densities far enough from the soliton location. This means that the component with nonvanishing background density forms a "trap" for another component localized inside such a trap (see, e.g., [10,11]). Although this solution of the Manakov system describes an important type of nonlinear excitation in two-component BECs, it cannot explain dense lattices of dark-bright solitons observed in recent experiments [4–6]. An attempt of such an explanation was done in Refs. [12–15] where particular cases of nonlinear waves with relative motions of the components were studied. Indeed, solutions in the form of counterphase oscillations (polarization waves) were found in these papers; however, they were limited to BECs with equal chemical potentials and this condition is quite restrictive for adequate description of experimental observations.

In this paper we shall consider the general situation with nonequal chemical potentials for the case of one-phase traveling waves. It will be shown that in this case the Manakov system can be reduced to the equations studied first by Kowalevski in her theory of rotation of the so-called *Kowalevski top* [16,17]. (Similar reduction was performed for the Manakov system with attractive (focusing) nonlinear interaction and without relative motion of the components in Refs. [18–20].)

The physical conditions that the densities of the components must be positive and nonsingular impose heavy restrictions on the admissible solutions of the Kowalevski equations. The typical situations will be illustrated by several particular cases. In particular, it will be shown that the solutions studied previously in [12–15] can be obtained as a special limiting case of the general solution of the Kowalevski equations. Another particular case of the dark-bright solitons is also obtained as a result of simple degeneration of the Kowalevski equations. The so-called *Appelrot class* of solutions of the Kowalevski equations, studied previously in the context of rotations of the Kowalevski top, leads now in the context of two-component condensate flows to the nonlinear density

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waves with very specific dispersion law conditioned by the relative motion of the components. In the soliton limit this periodic solution reduces to the dark-dark solitons which generalize the known Manakov soliton solution to the situation with relative motion of the components. The general nonlinear wave in the two-component BEC is illustrated by the so-called Legendre-Jacobi case when the hyperelliptic integrals are reduced to the elliptic ones. The physical implications of the found solutions are discussed in the conclusion of the paper.

II. EQUATIONS OF MOTION AND KOWALEVSKI VARIABLES

It is convenient to transform the Manakov system (1) to the hydrodynamic-like form by means of the Madelung transformation

$$\psi_k = \sqrt{\rho_k}\exp\left(i\int^x u_k dx - i\mu_k t\right), \quad k = 1,2, \tag{2}$$

where $\mu_{1,2}$ are constant chemical potentials. In a standard way we arrive at the system

$$\begin{aligned} &\rho_{k,t} + (\rho_k u_k)_x = 0,\\ &u_{k,t} + \left(\frac{1}{2}u_k^2 + \rho_1 + \rho_2 + \frac{\rho_{k,x}^2}{8\rho_k^2} - \frac{\rho_{k,xx}}{4\rho_k}\right)_x = 0 \end{aligned} \tag{3}$$

with real variables. Here $\rho_{1,2}$ denote the densities of the components, and $u_{1,2}$ denote their flow velocities.

A traveling wave is described by one-phase solution of Eqs. (3) with all variables depending on $\xi = x - Vt$ only,

$$\rho_k = \rho_k(\xi), \quad u_k = u_k(\xi), \quad \xi = x - Vt, \tag{4}$$

where V is a constant velocity of the wave. Substitution of this ansatz into Eqs. (3) and subsequent integration yields expressions for u_k in terms of ρ_k,

$$u_k(\xi) = V + \frac{\alpha_k}{\rho_k(\xi)}, \quad k = 1,2, \tag{5}$$

and the equation for $\rho_k(\xi)$,

$$\frac{\rho_{k,\xi}^2}{8\rho_k^2} - \frac{\rho_{k,\xi\xi}}{4\rho_k} + \frac{\alpha_k^2}{2\rho_k^2} + \rho_1 + \rho_2 = \beta_k, \tag{6}$$

where α_k, β_k are the integration constants. As follows from these relations, the constants α_k, β_k, $k = 1, 2$, are real. In the case of a uniform flow with constant $\rho_k = \rho_{k0}$ and $u_k = u_{k0}$ we have $\beta_k = \rho_{10} + \rho_{20} + \alpha_k^2/(2\rho_{k0}^2)$. Hence, the parameters β_k are related with the chemical potentials $\mu_k = u_k^2/2 + \rho_{10} + \rho_{20}$ by the formulae

$$\mu_k = \frac{1}{2}V^2 + \frac{V\alpha_k}{\rho_{k0}} + \beta_k, \quad k = 1,2. \tag{7}$$

The solutions studied in Refs. [12–15] were limited to the case $\beta_1 = \beta_2$ which meant that in the uniform case the chemical potentials are equal to each other: $\mu_1 = \mu_2$. Here we will consider the general case including situations with $\beta_1 \neq \beta_2$.

The Manakov system (1) is completely integrable [7] and its solutions can be found by the inverse scattering transform method. Therefore the dynamical system (6) obtained from (1) by reduction to the class of the traveling-wave solutions must be also completely integrable and can be solved by reduction of the solution to quadratures. Such a possibility was indicated in Refs. [21,22] and the method based on the so-called Lax representation of the corresponding integrable dynamical system was used in Refs. [19,20] for studying the traveling-wave solutions of the Manakov system in the case of the attractive (focusing) interaction. In Ref. [18] the same problem was discussed with the use of the alternative Stäckel method of separation of variables in the Hamilton-Jacobi equation related with the corresponding nonlinear system. In Appendix A of this paper we develop a similar approach for the case of repulsive interaction between the BEC components with account of relative motion of the components. The resulting Kowalevski-type equations are studied below for derivation of the physically interesting exact solution of the Manakov system.

As shown in Appendix A, if $\beta_1 \neq \beta_2$, introduction of new variables q_1, q_2 according to

$$\rho_1 = \frac{(q_1+\beta)(q_2+\beta)}{2\beta}, \quad \rho_2 = -\frac{(q_1-\beta)(q_2-\beta)}{2\beta}, \tag{8}$$

where we have denoted $\beta \equiv \beta_1 - \beta_2$ (the limit $\beta \to 0$ will be considered below) reduces the system (6) to the Kowalevski form

$$\begin{aligned} &\frac{dq_1}{\sqrt{\mathcal{R}(q_1)}} + \frac{dq_2}{\sqrt{\mathcal{R}(q_2)}} = 0,\\ &\frac{q_1 dq_1}{\sqrt{\mathcal{R}(q_1)}} + \frac{q_2 dq_2}{\sqrt{\mathcal{R}(q_2)}} = \pm 2\, d\xi, \end{aligned} \tag{9}$$

where

$$\begin{aligned} \mathcal{R}(q) = {}& q^5 - (\beta_1+\beta_2)q^4 - 2(\beta^2 - h)q^3\\ &-\left[(\alpha_1+\alpha_2)^2 - 2(\beta_1+\beta_2)\beta^2 + k\right]q^2\\ &+\beta\left[\beta^3 - 2h\beta + 2\left(\alpha_1^2 - \alpha_2^2\right)\right]q\\ &-\beta^2[(\beta_1+\beta_2)\beta^2 - k + (\alpha_1-\alpha_2)^2] \end{aligned} \tag{10}$$

is a 5th-degree polynomial within q and h and k are the values of two integrals of motion of the system (6). The system (9) can be also written in the form

$$\frac{dq_1}{d\xi} = \frac{2\sqrt{\mathcal{R}(q_1)}}{q_1 - q_2}, \quad \frac{dq_2}{d\xi} = -\frac{2\sqrt{\mathcal{R}(q_2)}}{q_1 - q_2}, \tag{11}$$

which can be more convenient in some situations.

The systems (9) or (11) can be solved formally in terms of Riemann θ functions (more precisely, in terms of Göpel and Rosenhein hyperelliptic functions; modern exposition of this method can be found, e.g., in [23]) but such a form of the general solution is mathematically involved and hardly can produce essential understanding of physical behavior of waves in a two-component BEC. Therefore we shall confine ourselves here to the most important particular solutions which provide useful information about such typical nonlinear excitations in BECs as (quasi)periodic waves and solitons.

III. NONLINEAR WAVES IN A TWO-COMPONENT BEC

The physical variables ρ_1 and ρ_2 (i.e., densities of BEC components) must be positive and this condition imposes important restrictions on the variables q_1 and q_2 which obey the systems (9) or (11). Supposing for definiteness that $\beta > 0$,

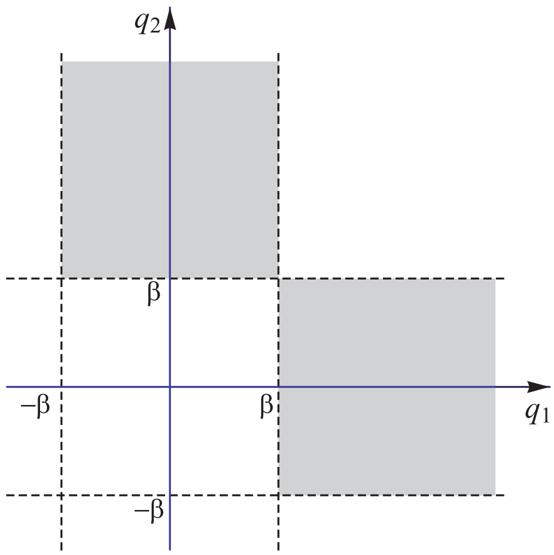

FIG. 1. (Color online) Regions of variations of the parameters q_1 and q_2 [see Eq. (12)] corresponding to the conditions of positivity of the densities ρ_1 and ρ_2 defined by Eqs. (8).

it is easy to find that q_1 and q_2 can vary in the intervals (see Fig. 1)

$$\begin{aligned} -\beta \leqslant q_1 \leqslant \beta, \quad q_2 \geqslant \beta, \\ \text{or} \quad q_1 \geqslant \beta, \quad -\beta \leqslant q_2 \leqslant \beta. \end{aligned} \tag{12}$$

Formal integration of the system (9) yields

$$\begin{aligned} \int_{q_{10}}^{q} \frac{dq}{\sqrt{\mathcal{R}(q)}} + \int_{q_{20}}^{q_2} \frac{dq}{\sqrt{\mathcal{R}(q)}} = 0, \\ \int_{q_{10}}^{q} \frac{q\,dq}{\sqrt{\mathcal{R}(q)}} + \int_{q_{20}}^{q_2} \frac{q\,dq}{\sqrt{\mathcal{R}(q)}} = \pm 2\xi, \end{aligned} \tag{13}$$

where q_{10} and q_{20} are integration constants equal to the values of q_1 and q_2 at $\xi = 0$, respectively. Every solution of the system (9) is parametrized by five zeros ν_i, $i = 1,\dots,5$, of the polynomial (10),

$$\mathcal{R}(q) = \prod_{i=1}^{5} (q - \nu_i), \tag{14}$$

and, depending on their values, we obtain different classes of solutions. Since the solution is symmetric with respect to transposition of q_1 and q_2, for definiteness we assume that they change in the intervals (the zeros ν_i are numerated in the order of their values)

$$\nu_1 \leqslant q_1 \leqslant \nu_2, \quad \nu_3 \leqslant q_2 \leqslant \nu_4, \tag{15}$$

where the polynomial $\mathcal{R}(q)$ is positive.

A. Limit $\beta \to 0$

If $\beta \equiv \beta_1 - \beta_2 \to 0$, then we must have $\nu_1 \to 0$ and $\nu_2 \to 0$ to satisfy the conditions (12). At the same time, to get in this singular limit from Eqs. (8) finite values of ρ_1 and ρ_2, we have to define new variables and parameters as

$$q_1 = \beta \widetilde{q}_1, \quad \nu_1 = \beta \widetilde{\nu}_1, \quad \nu_2 = \beta \widetilde{\nu}_2 \tag{16}$$

so that Eqs. (8) reduce to

$$\rho_1 = \tfrac{1}{2} q_2 (1 + \widetilde{q}_1), \quad \rho_2 = \tfrac{1}{2} q_2 (1 - \widetilde{q}_1). \tag{17}$$

The Kowalevski equations (11) are then transformed to

$$\begin{aligned} \frac{d\widetilde{q}_1}{d\xi} &= \frac{2\sqrt{\nu_3 \nu_4 \nu_5}}{q_2} \sqrt{(\widetilde{q}_1 - \widetilde{\nu}_1)(\widetilde{\nu}_2 - \widetilde{q}_2)}, \\ \frac{dq_2}{d\xi} &= -2\sqrt{(q_2 - \nu_3)(\nu_4 - q_2)(\nu_5 - q_2)}. \end{aligned} \tag{18}$$

We notice that the expression $\rho \equiv \rho_1 + \rho_2 = q_1 + q_2$ reduces to $\rho = q_2$ in this limit. Introducing also $\widetilde{q}_1 = \cos\theta$, $\widetilde{\nu}_1 = \cos\theta_1$, $\widetilde{\nu}_2 = \cos\theta_2$, $\nu_3 = r_1$, $\nu_4 = r_2$, $\nu_5 = r_3$, we arrive at the equations

$$\begin{aligned} \frac{d\cos\theta}{d\xi} &= -\frac{2\sqrt{r_1 r_2 r_2}}{\rho} \sqrt{(\cos\theta - \cos\theta_1)(\cos\theta_2 - \cos\theta)}, \\ \frac{d\rho}{d\xi} &= -2\sqrt{(\rho - r_1)(r_2 - \rho)(r_3 - \rho)}, \end{aligned} \tag{19}$$

identical to the equation obtained for this special case in Refs. [12–15].

It is worth noticing that obtained in Appendix A the integrals of motion (A6) and (A7) of the dynamical system (6) can be cast after introduction of these new variables ρ and θ to the form

$$\begin{aligned} H = \frac{\rho_\xi^2}{8\rho} - \frac{\rho^2}{2} + \tilde{\beta}\rho \\ + \frac{\rho^2 \sin^2\theta \cdot \theta_\xi^2 + 8\left[\alpha_1^2 + \alpha_2^2 - \left(\alpha_1^2 - \alpha_2^2\right)\cos\theta\right]}{8\rho \sin^2\theta} = h, \end{aligned} \tag{20}$$

$$\begin{aligned} K = \frac{\rho^2 \sin^2\theta \cdot \theta_\xi^2 + 8\left[\alpha_1^2 + \alpha_2^2 - \left(\alpha_1^2 - \alpha_2^2\right)\cos\theta\right]}{4\sin^2\theta} \\ - (\alpha_1 + \alpha_2)^2 = k. \end{aligned} \tag{21}$$

The angle θ can be excluded from Eq. (20) with the use of Eq. (21) which gives the equation for a single variable ρ,

$$\rho_\xi^2 = 4\rho^3 - 8\tilde{\beta}\rho^2 + 8h\rho - 4[k + (\alpha_1 + \alpha_2)^2], \tag{22}$$

which is another form of the second equation (19). Its solution can be expressed in terms of elliptic functions. When ρ is known, then $\theta = \theta(\xi)$ can be obtained by integration of Eq. (21) or

$$\theta_\xi^2 = \frac{8}{\rho^2}\left[\frac{k + (\alpha_1 + \alpha_2)^2}{2} - \frac{\alpha_1^2 + \alpha_2^2 - \left(\alpha_1^2 - \alpha_2^2\right)\cos\theta}{\sin^2\theta}\right], \tag{23}$$

which also coincides up to the notation with the first equation (19). When their solutions are found then the component densities are given by (17) transformed to

$$\rho_1(\xi) = \rho(\xi)\cos^2\frac{\theta(\xi)}{2}, \quad \rho_2(\xi) = \rho(\xi)\sin^2\frac{\theta(\xi)}{2}. \tag{24}$$

Dependence of θ on ξ leads to oscillations of the component densities even for the case of constant total density ρ. Variation of ρ with ξ means "modulation" of the component oscillations.

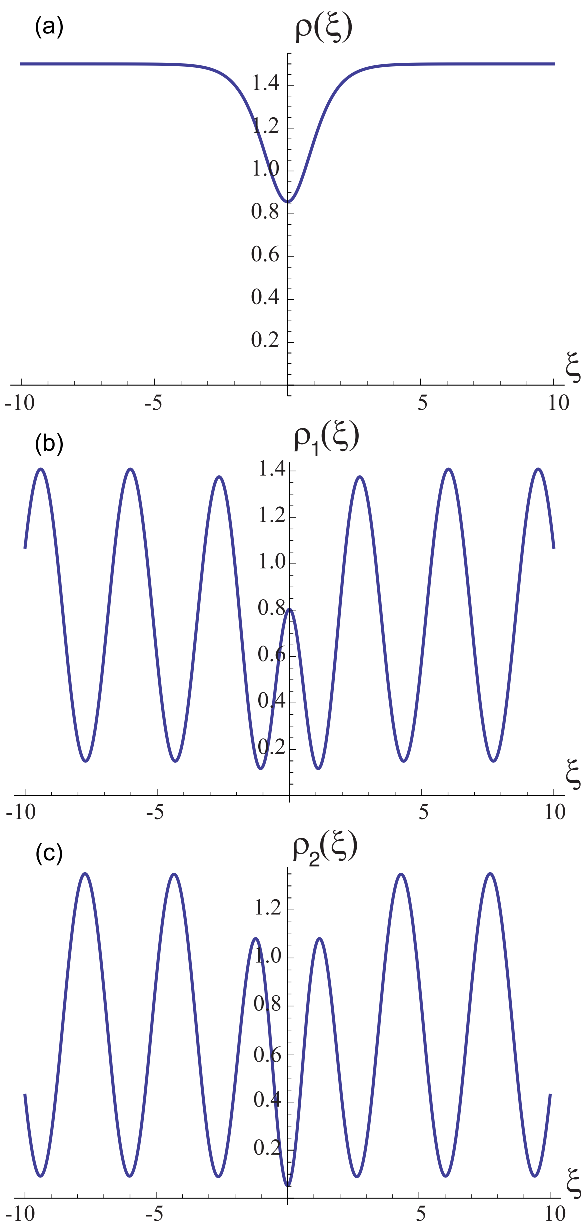

FIG. 2. (Color online) Dependence of the total density $\rho(\xi)$ (a) and the component densities $\rho_1(\xi)$ (b) and $\rho_2(\xi)$ (c) on ξ in a "quasisoliton." Parameters are equal to $V = 0.5$, $r_1 = 0.857$, $r_2 = r_3 = 1.0$, $\theta = 2.5$, $\theta_2 = 0.5$.

Most spectacularly such behavior is expressed in the soliton limit $r_2 = r_3$ when the total density profile has the form of the soliton. In the well-known dark-dark soliton solutions the component density profiles repeat the total density profile up to constant factors independent of ξ. This situation corresponds to a particular case $\theta = \theta_1 = \theta_2$ when the solution of Eq. (23) is trivial. Explicit formulas for the general case $\theta_1 \neq \theta_2$ were found in Refs. [13,14] and we shall not reproduce them here. Instead, to illustrate such a behavior of the two-component BEC systems, we show typical plots for this "quasisoliton" solution in Fig. 2.

More details about these solutions can be found in Refs. [12–15].

B. Appelrot class of solutions

The systems (9) and (11) were applied for the first time to a real mechanical problem by Kowalevski in her theory of rotation of the so-called Kowalevski top [16]. After that some particular especially remarkable motions of this top were discussed by other authors, in particular, by Appelrot and Delone (see, e.g., [17]). Here we shall apply their method to the special case of nonlinear motion of a two-component BEC which we shall also call the *Appelrot case*.

Let us suppose that the polynomial $\mathcal{R}(q)$ has a double root $q = \bar{\nu}$, the other roots we denote as $\nu_1 \leqslant \nu_2 \leqslant \nu_3$; that is, we have

$$\mathcal{R}(q) = (q-\bar{\nu})^2(q-\nu_1)(q-\nu_2)(q-\nu_3) \equiv (q-\bar{\nu})^2\mathcal{R}_1(q), \tag{25}$$

where $\mathcal{R}_1(q)$ is the 3rd-degree polynomial with the roots ν_1, ν_2, ν_3. In this case it is convenient to use the system (11) written now in the form

$$\begin{aligned} 2(q_1-\bar{\nu})\sqrt{\mathcal{R}_1(q_1)} &= (q_1-q_2)\frac{dq_1}{d\xi}, \\ 2(q_2-\bar{\nu})\sqrt{\mathcal{R}_1(q_2)} &= -(q_1-q_2)\frac{dq_2}{d\xi}. \end{aligned} \tag{26}$$

It is easy to see that this system is satisfied if

$$\begin{aligned} q_1 = \bar{\nu}, \quad \frac{dq_2}{d\xi} &= 2\sqrt{\mathcal{R}_1(q_2)} \\ \text{or} \quad q_2 = \bar{\nu}, \quad \frac{dq_1}{d\xi} &= 2\sqrt{\mathcal{R}_1(q_1)}. \end{aligned} \tag{27}$$

Both solutions lead to the same physical solution due to symmetry of Eqs. (8) with respect to transposition of q_1 and q_2. For definiteness we shall take the second solution in (27). Then the variable q_1 oscillates in the interval $\nu_1 \leqslant q_1 \leqslant \nu_2$, where $\mathcal{R}_1(q_1) \geqslant 0$ and, hence, for $\beta > 0$, we have according to (12) two choices for the parameters $\bar{\nu}, \nu_1, \nu_2$,

$$\begin{aligned} \beta \leqslant \nu_1 < \nu_2, &\quad -\beta \leqslant \bar{\nu} \leqslant \beta, \\ -\beta \leqslant \nu_1 < \nu_2 \leqslant \beta, &\quad \bar{\nu} \geqslant \beta. \end{aligned} \tag{28}$$

As we shall see, the second choice cannot give the soliton solution, so we shall consider the first one.

In the standard way we obtain the solution expressed in terms of the Jacobi elliptic sn function,

$$q_1(\xi) = \nu_1 + (\nu_2-\nu_1)\mathrm{sn}^2(\sqrt{\nu_3-\nu_1}\,(\xi-\xi_0),m), \tag{29}$$

where

$$m = \frac{\nu_2-\nu_1}{\nu_3-\nu_1}, \tag{30}$$

and, to simplify the notation, from now on we shall put the integration constant $\xi_0 = 0$. Then the component densities are given by

$$\begin{aligned} \rho_1 &= \frac{\bar{\nu}+\beta}{2\beta}[\beta+\nu_1+(\nu_2-\nu_1)\mathrm{sn}^2(\sqrt{\nu_3-\nu_1}\,\xi,m)], \\ \rho_2 &= \frac{\bar{\nu}-\beta}{2\beta}[\beta-\nu_1-(\nu_2-\nu_1)\mathrm{sn}^2(\sqrt{\nu_3-\nu_1}\,\xi,m)], \end{aligned} \tag{31}$$

and their substitution into (5) yields the flow velocities. These formulas represent the periodic nonlinear wave which can be called the *density wave*, since the densities oscillate in phase and in the small-amplitude limit this wave reduces to the sound wave, which describes oscillations of the total density $\rho = \rho_1 + \rho_2$.

Let us consider the soliton limit when $\nu_3 \to \nu_2$ ($m \to 1$):

$$\begin{aligned} \rho_1 &= \frac{\beta + \bar{\nu}}{2\beta}\left(\beta + \nu_2 - \frac{\nu_2 - \nu_1}{\cosh^2(\sqrt{\nu_2 - \nu_1}\xi)}\right), \\ \rho_2 &= \frac{\beta - \bar{\nu}}{2\beta}\left(\nu_2 - \beta - \frac{\nu_2 - \nu_1}{\cosh^2(\sqrt{\nu_2 - \nu_1}\xi)}\right), \end{aligned} \tag{32}$$

where $\beta < \nu_1 < \nu_2$ and $-\beta < \bar{\nu} < \beta$. These parameters can be expressed in terms of the constant densities at $|\xi| \to \infty$,

$$\rho_{10} = \frac{1}{2\beta}(\bar{\nu} + \beta)(\beta + \nu_2), \quad \rho_{20} = \frac{1}{2\beta}(\beta - \bar{\nu})(\nu_2 - \beta). \tag{33}$$

Solving this system with respect to ν_2 and $\bar{\nu}$ gives

$$\begin{aligned} \nu_2 &= \tfrac{1}{2}(\rho_{10} + \rho_{20} + \sqrt{(\rho_{10} - \rho_{20} - 2\beta)^2 + 4\rho_{10}\rho_{20}}), \\ \bar{\nu} &= \tfrac{1}{2}(\rho_{10} + \rho_{20} - \sqrt{(\rho_{10} - \rho_{20} - 2\beta)^2 + 4\rho_{10}\rho_{20}}). \end{aligned} \tag{34}$$

The parameter β can be also expressed in terms of ρ_{10}, ρ_{20} at $|\xi| \to \infty$. From (5) and (6) we get

$$\alpha_1 = \rho_{10}(u_{10} - V), \quad \alpha_2 = \rho_{20}(u_{20} - V) \tag{35}$$

and

$$\begin{aligned} \beta_1 &= \tfrac{1}{2}(u_{10} - V)^2 + \rho_{10} + \rho_{20}, \\ \beta_2 &= \tfrac{1}{2}(u_{20} - V)^2 + \rho_{10} + \rho_{20}, \end{aligned} \tag{36}$$

hence

$$\beta = \tfrac{1}{2}[(u_{10} - V)^2 - (u_{20} - V)^2]. \tag{37}$$

To determine the last unknown parameter ν_1, we remark that the Viète formula for the polynomial (10) in our case gives $\beta_1 + \beta_2 = \nu_1 + 2(\nu_2 + \bar{\nu})$ and, consequently,

$$\nu_1 = \tfrac{1}{2}[(u_{10} - V)^2 + (u_{20} - V)^2]. \tag{38}$$

The solution (32) exists if $\nu_2 > \nu_1$ and this condition gives restrictions for the soliton velocity,

$$\frac{\rho_{10}}{(V - u_{10})^2} + \frac{\rho_{20}}{(V - u_{20})^2} > 1. \tag{39}$$

Note that for the second choice in (28) the condition $\nu_2 > \nu_1$ cannot be fulfilled. In the limiting case of a one-component quiescent condensate ($\rho_{20} = 0, u_{10} = 0$) the condition (39) reduces to the well-known fact that the soliton velocity is smaller than the sound velocity, $V < c_s = \sqrt{\rho_{10}}$.

As is clear from Eqs. (32), this solution describes a dark-dark soliton with density dips in both components. However, due to the relative flow of components, in this solution the component density profiles are not proportional to each other, in contrast to the well-known soliton solution propagating along the quiescent condensate. This situation is illustrated in Fig. 3 where it is shown that the density curves can intersect each other for a certain choice of the parameters of the flow at infinity. Thus, the relative velocity introduces the asymmetry into behavior of the components.

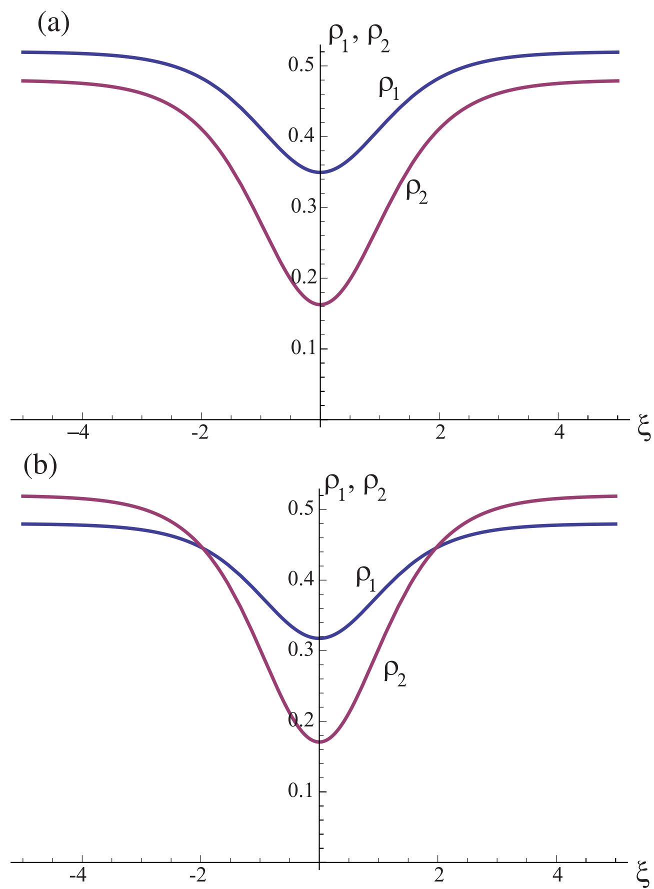

FIG. 3. (Color online) Plots of the densities of the components ρ_1 and ρ_2 as functions of ξ [see Eqs. (32)]. In all plots $u_{10} = 0$, $u_{20} = 0.5$, and $V = 1.0$. In panel (a) $\rho_{10} = 0.52$, $\rho_{20} = 0.48$, and in panel (b) $\rho_{10} = 0.48$, $\rho_{20} = 0.52$.

Dependence of the inverse width $\kappa = \sqrt{\nu_2 - \nu_1}$ of the soliton on its velocity V is given by

$$\begin{aligned} \kappa = \frac{1}{\sqrt{2}}\{&\rho_{10} + \rho_{20} - (u_{10} - V)^2 - (u_{20} - V)^2 \\ &+ \sqrt{[\rho_{10} - \rho_{20} - (u_{10} - V)^2 + (u_{20} - V)^2]^2 + 4\rho_{10}\rho_{20}}\}^{1/2}. \end{aligned} \tag{40}$$

This expression can be also obtained by linearization of Eq. (6) with respect to small deviations ρ_k' around asymptotic densities ($\rho_k = \rho_{k0} + \rho_k'$) and seeking the solution of the linearized equations in the form $\rho_k' \propto \exp(-\kappa|\xi|)$. This calculation shows that the Appelrot class of solutions yields in the corresponding limit *all* soliton solutions with exponentially decaying tails around *nonzero* background densities $\rho_{k0} \neq 0$.

Dependence (40) is illustrated in Fig. 4 for different values of the relative velocity of the BEC components. The remarkable feature is that this dependence can be nonmonotonic and for large enough values of the relative velocity the region of possible values of the velocity V splits into two separated regions in sharp contrast with the one-component situation. The appearance of two regions of velocity can be illustrated graphically in the following way. We introduce for convenience the variables $X = V - u_{10}$, $Y = V - u_{20}$; then the boundary

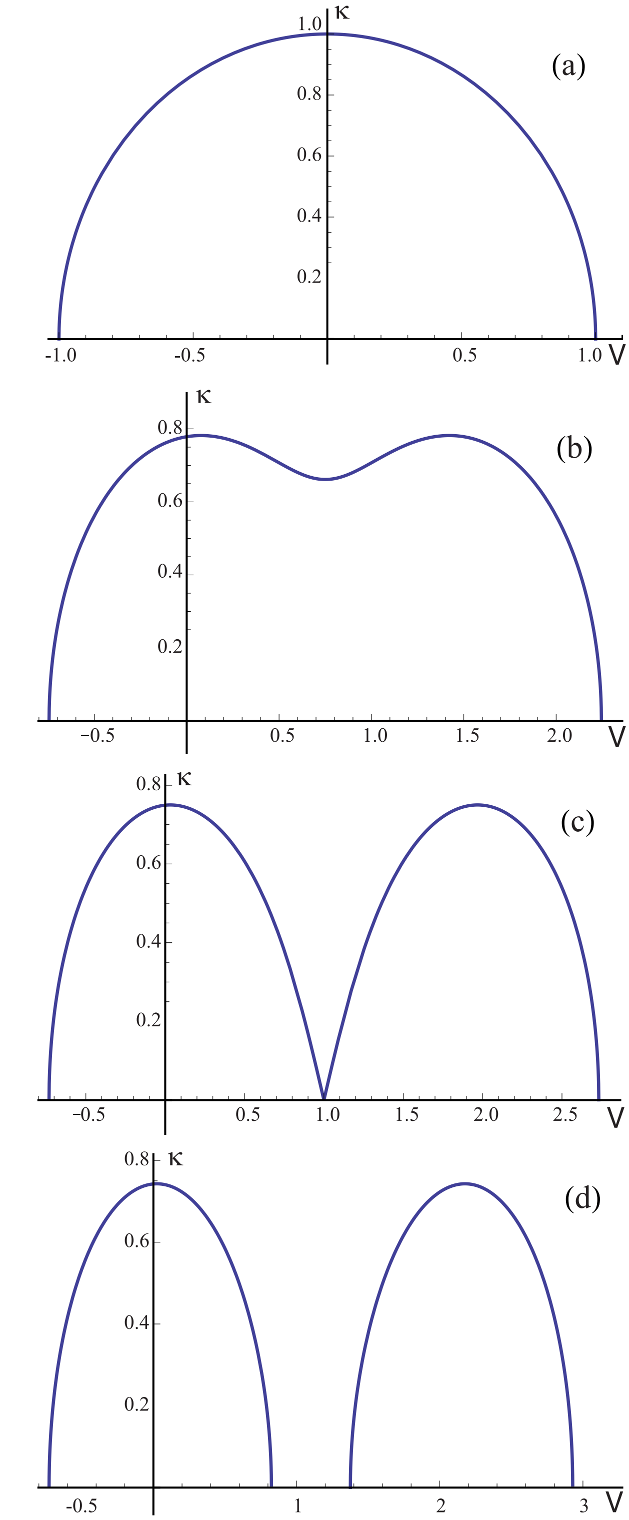

FIG. 4. (Color online) Dependence of the inverse width κ of a dark-dark soliton on its velocity V for different values of the relative velocity between condensates. In all plots $\rho_{10} = \rho_{20} = 0.5$, $u_{10} = 0$, and V is measured with respect to quiescent condensate: (a) $u_{20} = 0$, this is the case of well-known dark-dark solitons propagating through two still condensates; (b) plot corresponding to $u_{20} = 1.5$ illustrates a nonmonotonic dependence; (c) at $u_{20} = 2.0$ the region of possible velocities splits with formation of two disconnected regions; (d) plot for $u_{20} = 2.2$ illustrates appearance of two disconnected regions of possible soliton velocities V.

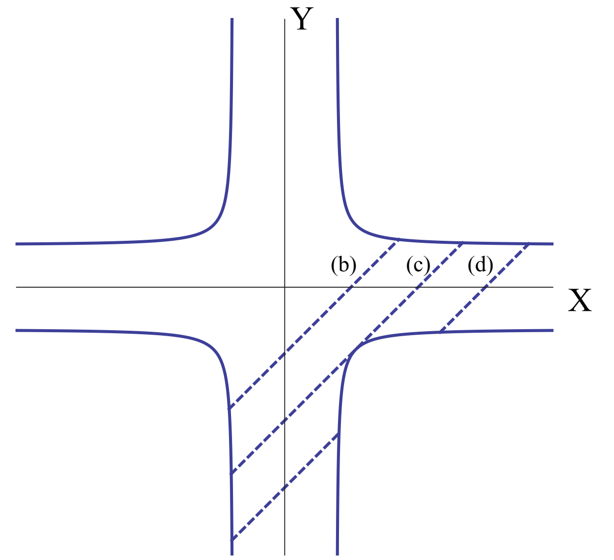

FIG. 5. (Color online) Plot of the curve (41) (solid line) and of the straight lines $X - Y = U_0$ (dashed lines): (b) corresponds to a single region of possible values of V [see Fig. 2(b)]; (c) corresponds to the relative velocity at which the region splits into two regions [see Fig. 2(c)]; (d) corresponds to two separated regions [see Fig. 2(d)].

of the region (39) is given by the equation

$$X^2Y^2 - \rho_{10}Y^2 - \rho_{20}X^2 = 0. \tag{41}$$

Its plot is shown in Fig. 5 by a solid line and the admissible values of V are located inside this line (that is, in the area including the origin of the coordinate system). If we fix the value of the relative velocity $U_0 = u_{20} - u_{10} \equiv X - Y$, then the possible values of V correspond to points of the straight line located between its intersections with the curve (41). Consequently, the splitting of the region of possible values of V correspond to such U_0 that the straight line $X - Y = U_0$ touches the curve (41) at the point where $dY/dX = 1$ or

$$XY^2 + X^2Y - \rho_{10}Y - \rho_{20}X = 0 \tag{42}$$

(see Fig. 5). The system (41) and (42) can be easily solved to give

$$\begin{aligned} X &= V - u_{10} = \pm\rho_{10}^{1/3}\sqrt{\rho_{10}^{1/3} + \rho_{20}^{1/3}}, \\ Y &= V - u_{20} = \mp\rho_{20}^{1/3}\sqrt{\rho_{10}^{1/3} + \rho_{20}^{1/3}}, \end{aligned} \tag{43}$$

and hence the critical value of the relative velocity is given by

$$U_0 = \left(\rho_{10}^{1/3} + \rho_{20}^{1/3}\right)^{3/2}. \tag{44}$$

C. Dark-bright soliton solution

If one of the background densities vanishes (say, $\rho_{20} = 0$), then the so-called dark-bright soliton solutions of the Manakov system are obtained (see, e.g., [10,11]). Here we show that this type of solutions is a specialization of general solutions of the Kowalevski equations when the polynomial $\mathcal{R}(q)$ has two double zeros. In this case the condition (15) is fulfilled if one of the double zeroes coincides with β. Thus, we assume that

$$-\beta \leqslant \nu_1 \leqslant q_1 \leqslant \beta, \quad \beta \leqslant q_2 \leqslant \bar{\nu} = \nu_4 = \nu_5. \tag{45}$$

Then the system (11) reduces to

$$\begin{aligned} 2(\beta - q_1)(\bar{\nu} - q_1)\sqrt{q_1 - \nu_1} &= (q_1 - q_2)\frac{dq_1}{d\xi}, \\ 2(q_2 - \beta)(\bar{\nu} - q_2)\sqrt{q_2 - \nu_1} &= -(q_1 - q_2)\frac{dq_2}{d\xi}. \end{aligned} \tag{46}$$

As in the preceding subsection, we see that the second equation is satisfied identically by $q_2 = \bar{\nu}$ and the first equation $dq_1/d\xi = -2(\beta - q_1)\sqrt{q_1 - \nu_1}$ can be easily integrated to give $q_1 = \beta - (\beta - \nu_1)\cosh^{-2}(\sqrt{\beta - \nu_1}\,\xi)$. As a result we obtain the densities

$$\begin{aligned} \rho_1 &= (\bar{\nu} + \beta)\left(1 - \frac{(\beta - \nu_1)/(2\beta)}{\cosh^2(\sqrt{\beta - \nu_1}\,\xi)}\right), \\ \rho_2 &= (\bar{\nu} - \beta)\frac{(\beta - \nu_1)/(2\beta)}{\cosh^2(\sqrt{\beta - \nu_1}\,\xi)}, \end{aligned} \tag{47}$$

which obviously correspond to the dark-bright soliton: the density ρ_1 has a dip at $\xi = 0$ and approaches to the background density $\rho_0 = \bar{\nu} + \beta$ as $|\xi| \to \infty$ whereas ρ_2 has a hump at $\xi = 0$ and vanishes as $|\xi| \to \infty$.

Let us relate the parameters of formulas (47) with standard physical parameters of the soliton solution. To this end we define the inverse half-width κ of the soliton by the equation $\kappa = \sqrt{\beta - \nu_1}$ and introduce the ratio of the component densities at the center of the soliton $\gamma = [\rho_0 - \rho_1(0)]/\rho_2(0) = (\bar{\nu} + \beta)/(\bar{\nu} - \beta)$. Besides that we assume that there is no flow of the first component at infinity: $u_{10} = 0$. Then from Eqs. (5) and (6) we find $\alpha_1 = -V\rho_0, \alpha_2 = 0, \beta_1 = \rho_0 + V^2/4$, $\beta_2 = \rho_0 - \kappa^2/2$ and hence

$$\beta = \frac{V^2}{4} + \frac{\kappa^2}{2} = \rho_0\frac{\gamma - 1}{2\gamma}, \quad \nu_1 = \beta - \kappa^2, \quad \bar{\nu} = \rho_0\frac{\gamma + 1}{2\gamma}. \tag{48}$$

The dependence of the soliton's inverse width on its velocity is given by the formula

$$\kappa(V) = \sqrt{\frac{\gamma - 1}{\gamma}\rho_0 - \frac{V^2}{2}}. \tag{49}$$

These formulas are equivalent to those found in Ref. [10] and a typical plot of the bright-dark soliton solution is illustrated in Fig. 6.

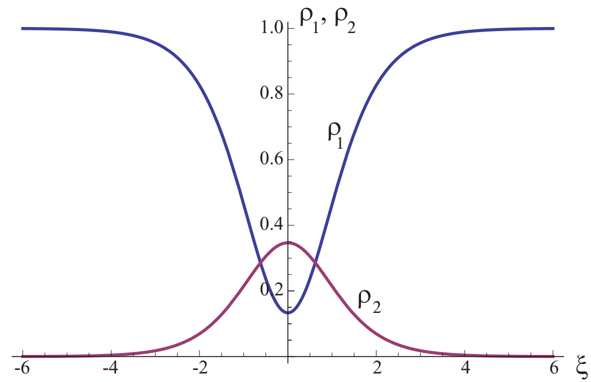

FIG. 6. (Color online) Plots of the densities of the components ρ_1 and ρ_2 as functions of ξ [see Eqs. (47)]. The parameters are equal to $\rho_0 = 1.0$, $\gamma = 2.5$, $V = 0.4$.

D. Legendre-Jacobi class of solutions

The general one-phase traveling waves described by the Manakov system can be illustrated by an easy numerical solution of the Kowalevski equations (11). On the other hand, the analytical solution of these equations can be expressed in terms of Riemann θ functions by the methods used already by Kowalevski (see [16,17]) and developed further in the algebraic-geometric approach to integrable equations (see, e.g., [23]). This method was applied to the one-phase solutions of the focusing Manakov system in Refs. [19,20]. However, the resulting expressions are quite inconvenient for practical use. Therefore we shall confine ourselves here to a particular case, when the solution can be reduced to the much better known special functions (elliptic integrals) which permit one to understand the characteristic features of the solution in a much simpler way. Here we shall consider such a situation first noticed by Legendre [24] and generalized by Jacobi [25].

Let the zeros of the polynomial $\mathcal{R}(q)$ be given by

$$\begin{aligned} &\nu_1 = c - 1/b, \quad \nu_2 = c - 1/a, \quad \nu_3 = c, \\ &\nu_4 = c + 1, \quad \nu_5 = c + 1/ab, \end{aligned} \tag{50}$$

where $0 < b \leqslant a \leqslant 1$ and the parameter c satisfies the conditions

$$\max\{\beta, 1/b - \beta\} < c < \beta + 1/a, \tag{51}$$

so that q_1 and q_2 oscillate within the intervals

$$-\beta < \nu_1 \leqslant q_1 \leqslant \nu_2 < \beta, \quad \beta < \nu_3 \leqslant q_2 \leqslant \nu_4. \tag{52}$$

Let us assume for definiteness that at $\xi = 0$ we have $q_1(0) = c - 1/a$ and $q_2 = c$ (other choices of the initial conditions can be considered in a similar way). Then, introducing the variables

$$z_{1,2} = q_{1,2} - c, \tag{53}$$

we represent the solution (13) in the form

$$\begin{aligned} \int_{-1/a}^{z_1} \frac{dz}{\sqrt{\widetilde{\mathcal{R}}(z)}} + \int_0^{z_2} \frac{dz}{\sqrt{\widetilde{\mathcal{R}}(z)}} &= 0, \\ \int_{-1/a}^{z_1} \frac{z\,dz}{\sqrt{\widetilde{\mathcal{R}}(z)}} + \int_0^{z_2} \frac{z\,dz}{\sqrt{\widetilde{\mathcal{R}}(z)}} &= \pm 2\xi, \end{aligned} \tag{54}$$

where

$$\widetilde{\mathcal{R}}(z) = z(1 - z)(1 - abz)(1 + az)(1 + bz)/(ab)^2. \tag{55}$$

As Jacobi showed [25], the integrals here can be expressed in terms of incomplete elliptic integrals of the first kind. Since Jacobi did not provide the details of his method, this calculation is discussed briefly in Appendix B. As a result, we obtain a particular solution of Eqs. (11) in the form

$$\begin{aligned} &F(\varphi_{1a},k_1) + F(\varphi_{1b},k_2) - F(\varphi_2,k_1) - F(\varphi_2,k_2) = 0, \\ &F(\varphi_{1a},k_1) - F(\varphi_{1b},k_2) - F(\varphi_2,k_1) + F(\varphi_2,k_2) \\ &\quad = \pm 2\sqrt{\frac{(1+a)(1+b)}{ab}}\,\xi, \end{aligned} \tag{56}$$

where

$$\varphi_{1a} = \begin{cases} \pi - \arcsin\sqrt{\frac{(1+az_1)(1+bz_1)}{z_1(\sqrt{a}-\sqrt{b})^2}}, & -\frac{1}{b} \leqslant z_1 \leqslant -\frac{1}{\sqrt{ab}}, \\ \arcsin\sqrt{\frac{(1+az_1)(1+bz_1)}{z_1(\sqrt{a}-\sqrt{b})^2}}, & -\frac{1}{\sqrt{ab}} \leqslant z_1 \leqslant -\frac{1}{a}; \end{cases} \tag{57}$$

$$\varphi_{1b} = \arcsin\sqrt{\frac{(1+az_1)(1+bz_1)}{z_1(\sqrt{a}+\sqrt{b})^2}}, \quad -\frac{1}{b} \leqslant z_1 \leqslant -\frac{1}{a}; \tag{58}$$

$$\varphi_2 = \arcsin\sqrt{\frac{(1+a)(1+b)z_2}{(1+az_2)(1+bz_2)}}, \quad 0 \leqslant z_2 \leqslant 1. \tag{59}$$

These equations determine implicitly the dependence of z_1 and z_2, and, hence, of q_1 and q_2, on ξ in the interval of ξ until the first turning point is met ($z_1 = -1/b$ or $z_2 = 1$). After that the sign before the corresponding square root in the Kowalevski equations (11) must be changed and the replacement in the solution (54)

$$\int_{-1/a}^{z_1} \frac{dz}{\sqrt{\tilde{\mathcal{R}}(z)}} \to \int_{-1/a}^{-1/b} \frac{dz}{\sqrt{\tilde{\mathcal{R}}(z)}} - \int_{-1/b}^{z_1} \frac{dz}{\sqrt{\tilde{\mathcal{R}}(z)}}$$

or

$$\int_{0}^{z_2} \frac{dz}{\sqrt{\tilde{\mathcal{R}}(z)}} \to \int_{0}^{1} \frac{dz}{\sqrt{\tilde{\mathcal{R}}(z)}} - \int_{1}^{z_2} \frac{dz}{\sqrt{\tilde{\mathcal{R}}(z)}}$$

must be done with similar changes in the expressions (56). Making such changes at every successive turning point, we find the solution in any necessary interval of ξ. Substitution of the resulting $q_1 = z_1 + c$ and $q_2 = z_2 + c$ into Eqs. (8) yields the dependence of densities ρ_1 and ρ_2 on ξ. Typical resulting plots are shown in Fig. 7.

As one can see, in the general solution the periodicity of the wave in space and time is lost and the wave pattern demonstrates quite complicated behavior as a function of ξ. This is a quasiperiodic solution of the dynamical system (6).

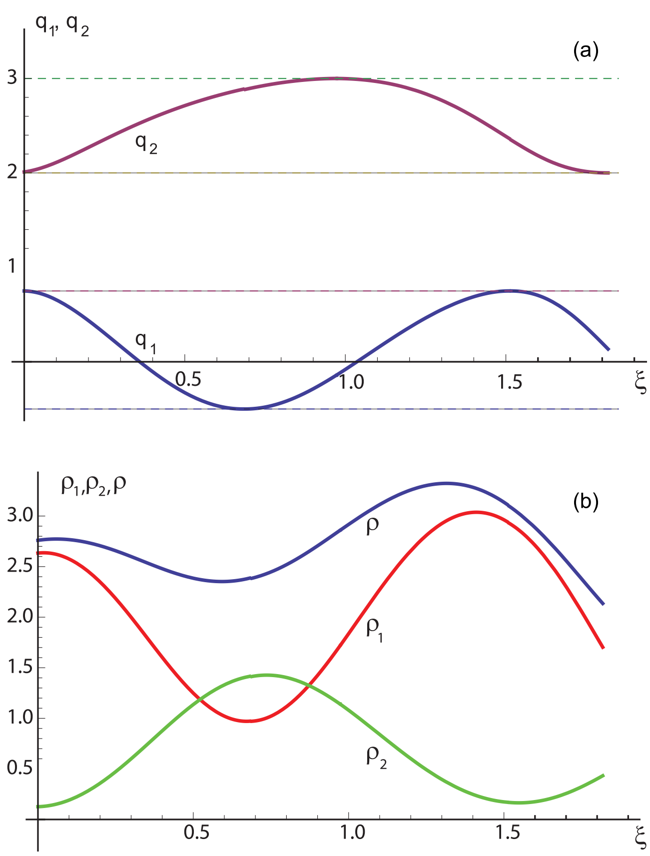

FIG. 7. (Color online) Plots for the solution (56) of the Kowalevski equations for the values of the parameters $a = 0.8$, $b = 0.4$, $c = 2$ which correspond to $\nu_1 = -0.5$, $\nu_2 = 0.75$, $\nu_3 = 2.0$, $\nu_3 = 3.0$, $\nu_5 = 5.125$. The initial conditions are given by $q_1(0) = \lambda_2$, $q_2(0) = \lambda_3$. (a) Plots of q_1 and q_2 for the interval of ξ corresponding to full cycle of q_2 variable; dashed lines indicate the values of ν_i, $i = 1,2,3,4$; (b) plots of the components densities ρ_1 and ρ_2 and of the total density $\rho = \rho_1 + \rho_2$.

IV. CONCLUSION

In this paper we have found the one-phase traveling wave solution of the Manakov system which describes evolution of two-component BEC. It is shown that in this case the Manakov system reduces to the equations which Kowalevski derived in her study of rotation of a heavy top in the completely integrable case discovered by her. We show that the previously found solutions of the Manakov system appear in this scheme as particular cases. Besides that, solutions are found which were either missed in previous analysis or cannot be obtained by more elementary methods when parameters of the solutions are chosen in such a way that the evolution equations are greatly simplified. In particular, we have found a dark-dark soliton solution for a two-component BEC with the nonzero relative motion of the components. This solution has very unusual dependence of the inverse width on the soliton's velocity. In principle, this can lead to forms of dispersive shock waves evolved from initial steplike distributions of the components densities or velocities.

For applications of the developed theory to the description of the polarization wave patterns observed in the experiments [4–6], the Whitham modulation theory [26,27] for these waves has to be developed. Some particular situations have already been studied in Refs. [28] (genus-zero case) and [15] (genus-one case for the limit $\beta = 0$). The results of the present paper demonstrate that the general one-phase solution is described by the 5th-degree polynomial $\mathcal{R}(q)$ whose zeros as well as the wave velocity must be related in the framework of the finite-gap integration method with the modulation parameters appearing in the Whitham theory of modulations of nonlinear waves. Thus, the results obtained here provide the necessary step to development of the modulation theory which can be applied to description of dispersive polarization shock waves observed experimentally.

ACKNOWLEDGMENTS

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APPENDIX A

It is convenient to cast the system (3) to a mathematically simpler form by means of the complex substitution $v_k = u_k - i\rho_{k,x}/(2\rho_k)$:

$$\rho_{k,t} + \left(\rho_k v_k + \frac{i}{2}\rho_{k,x}\right)_x = 0,$$
$$v_{k,t} + \left(\frac{1}{2}v_k^2 + \rho_1 + \rho_2 - \frac{i}{2}v_{k,x}\right)_x = 0. \tag{A1}$$

Let us introduce also the imaginary "time" variable $\tau = -2i\xi$. Then after obvious integrations we get

$$\frac{d\rho_k}{d\tau} = \alpha_k - \rho_k w_k, \quad \frac{dw_k}{d\tau} = \frac{1}{2}w_k^2 + \rho_1 + \rho_2 - \beta_k, \tag{A2}$$

and $w_k = v_k - V$.

Our approach is based on the fact that the system (A2) is Hamiltonian with the Hamiltonian

$$H = -\tfrac{1}{2}\left(\rho_1 w_1^2 + \rho_2 w_2^2\right) - \tfrac{1}{2}(\rho_1 + \rho_2)^2 + \alpha_1 w_1 + \alpha_2 w_2 + \beta_1\rho_1 + \beta_2\rho_2 \tag{A3}$$

and Poisson brackets

$$\{\rho_i,\rho_j\} = \{w_i,w_j\} = 0, \quad \{w_i,\rho_j\} = \delta_{ij}. \tag{A4}$$

The corresponding equations of motion

$$\frac{d\rho_k}{d\tau} = \frac{\partial H}{\partial w_k}, \quad \frac{dw_k}{d\tau} = -\frac{\partial H}{\partial \rho_k} \tag{A5}$$

possess the integral of energy

$$H(\rho_k,w_k) = h = \text{const}. \tag{A6}$$

For complete integrability of this system with two degrees of freedom we need, according to the Liouville-Arnold theorem (see, e.g., [29]), one more integral. One can check that there is such an integral quadratic in momenta w_k:

$$K = -\rho_1\rho_2(w_1 - w_2)^2 + 2(w_1 - w_2)(\alpha_1\rho_2 - \alpha_2\rho_1) - (\beta_1 - \beta_2)\left[\rho_1 w_1^2 - \rho_2 w_2^2 + \rho_1^2 - \rho_2^2 - 2(\alpha_1 w_1 - \alpha_2 w_2) - 2(\beta_1\rho_1 - \beta_2\rho_2)\right]. \tag{A7}$$

Thus, integration of the system (A5) can be reduced to quadratures.

If $\beta_1 \neq \beta_2$, then we can make a canonical transformation

$$\rho_1 = \frac{(q_1+\beta)(q_2+\beta)}{2\beta}, \quad \rho_2 = -\frac{(q_1-\beta)(q_2-\beta)}{2\beta},$$
$$w_1 = \frac{(q_1-\beta)p_1 - (q_2-\beta)p_2}{q_1 - q_2},$$
$$w_2 = \frac{(q_1+\beta)p_1 - (q_2+\beta)p_2}{q_1 - q_2}, \tag{A8}$$

where $\beta \equiv \beta_1 - \beta_2$. As we shall see, the dynamics is separable in these new variables q_i, p_i, $i = 1,2$. The Poisson brackets preserve their canonical form

$$\{q_i,q_j\} = \{p_i,p_j\} = 0, \quad \{p_i,q_j\} = \delta_{ij}, \tag{A9}$$

and the Hamiltonian becomes

$$H = \frac{\left(q_1^2 - \beta^2\right)p_1^2 - 2[(\alpha_1+\alpha_2)q_1 - (\alpha_1-\alpha_2)\beta]p_1}{2(q_2 - q_1)} + \frac{\left(q_2^2 - \beta^2\right)p_2^2 - 2[(\alpha_1+\alpha_2)q_2 - (\alpha_1-\alpha_2)\beta]p_2}{2(q_1 - q_2)} - \frac{1}{2}\left[q_1^2 + q_2^2 + q_1q_2 - (\beta_1+\beta_2)(q_1+q_2) - \beta^2\right]. \tag{A10}$$

The equations of motion are given by

$$\frac{dq_1}{d\tau} = \frac{\partial H}{\partial p_1} = \frac{\left(q_1^2-\beta^2\right)p_1 + (\alpha_1-\alpha_2)\beta - (\alpha_1+\alpha_2)q_1}{q_1 - q_2},$$
$$\frac{dp_1}{d\tau} = -\frac{\partial H}{\partial q_1} = \frac{(p_1-p_2)[(\alpha_1-\alpha_2)\beta - (\alpha_1+\alpha_2)q_2]}{(q_1-q_2)^2} - \frac{\left(q_1^2+\beta^2\right)p_1^2 - \left(q_2^2-\beta^2\right)p_2^2 + 2q_1q_2p_1^2}{2(q_1-q_2)^2} + \frac{1}{2}(\beta_1+\beta_2-2q_1-q_2), \tag{A11}$$

and similar equations can be written for q_2 and p_2. They have two integrals of motion—the energy $H(q_1,p_1,q_2,p_2) = h = \text{const}$ and

$$K(q_1,p_1,q_2,p_2) = \left\{2(\alpha_1+\alpha_2)(p_1-p_2)q_1q_2 - \left(p_1^2q_1 - p_2^2q_2\right)q_1q_2 + \beta^2\left(p_1^2q_2 - p_2^2q_1\right) - 2\beta(\alpha_1-\alpha_2)(p_1q_2 - p_2q_1)\right\}/(q_1-q_2) + (\beta_1+\beta_2-q_1-q_2)q_1q_2 = k = \text{const}. \tag{A12}$$

As was mentioned above, integration of a Hamiltonian system with two degrees of freedom and two integrals of motion can be reduced to quadratures. Actual integration can be performed in our case as follows. Eliminating variables q_2, p_2 from the integrals $H(q,p) = h$ and $K(q,p) = k$, we obtain

$$\Phi = \left(q_1^2-\beta^2\right)\left(\beta_1+\beta_2-q_1-p_1^2\right) + 2p_1[(\alpha_1+\alpha_2)q_1 - \beta(\alpha_1-\alpha_2)] - 2hq_1 + k = 0, \tag{A13}$$

which demonstrates the mentioned above separation of variables. Taking into account Eqs. (A11), we get

$$\frac{\partial\Phi}{\partial p_1} = 2\left(q_1^2-\beta^2\right)\left(\beta_1+\beta_2-q_1-p_1^2\right) - 2[\beta(\alpha_1-\alpha_2) - (\alpha_1+\alpha_2)q_1] = -2(q_1-q_2)\frac{dq_1}{d\tau}. \tag{A14}$$

We solve Eq. (A13) with respect to p_1 and substitute the result into (A14). After this and similar manipulations with the variables q_2, p_2 we obtain the system

$$\pm\sqrt{-\mathcal{R}(q_1)} = -(q_1-q_2)\frac{dq_1}{d\tau},$$
$$\pm\sqrt{-\mathcal{R}(q_2)} = (q_1-q_2)\frac{dq_2}{d\tau}, \tag{A15}$$

where

$$
\begin{aligned}
\mathcal{R}(q) &= q^5 - (\beta_1+\beta_2)q^4 - 2(\beta^2-h)q^3 \\
&\quad - \left[(\alpha_1+\alpha_2)^2 - 2(\beta_1+\beta_2)\beta^2 + k\right]q^2 \\
&\quad + \beta\left[\beta^3 - 2h\beta + 2\left(\alpha_1^2-\alpha_2^2\right)\right]q \\
&\quad - \beta^2[(\beta_1+\beta_2)\beta^2 - k + (\alpha_1-\alpha_2)^2] \qquad \text{(A16)}
\end{aligned}
$$

is a 5th-degree polynomial with respect to q. Then after simple manipulations we arrive at the system

$$
\begin{aligned}
&\frac{dq_1}{\sqrt{\mathcal{R}(q_1)}} + \frac{dq_2}{\sqrt{\mathcal{R}(q_2)}} = 0,\\
&\frac{q_1 dq_1}{\sqrt{\mathcal{R}(q_1)}} + \frac{q_2 dq_2}{\sqrt{\mathcal{R}(q_2)}} = \pm 2\, d\xi,
\end{aligned} \qquad \text{(A17)}
$$

where we have returned to the real variable $\xi = i\tau/2$. Sometimes it is convenient to rewrite this system in the Kowalevski form

$$
\frac{dq_1}{d\xi} = \frac{2\sqrt{\mathcal{R}(q_1)}}{q_1-q_2}, \quad \frac{dq_2}{d\xi} = -\frac{2\sqrt{\mathcal{R}(q_2)}}{q_1-q_2}. \qquad \text{(A18)}
$$

This approach can be generalized on the multicomponent Manakov system with an arbitrary number of components.

APPENDIX B

We have to calculate the integrals

$$
\begin{aligned}
&I_1 = \int_0^z \frac{dz}{\sqrt{\tilde{\mathcal{R}}(z)}}, \quad I_1' = \int_0^z \frac{z dz}{\sqrt{\tilde{\mathcal{R}}(z)}},\\
&I_2 = \int_{-1/a}^z \frac{dz}{\sqrt{\tilde{\mathcal{R}}(z)}}, \quad I_2' = \int_{-1/a}^z \frac{z dz}{\sqrt{\tilde{\mathcal{R}}(z)}}.
\end{aligned} \qquad \text{(B1)}
$$

The integrals I_1 and I_1' are calculated with the use of the substitution

$$
1 + abz^2 = uz \qquad \text{(B2)}
$$

or

$$
\sqrt{z} = \left(\sqrt{u+2\sqrt{ab}} \pm \sqrt{u-2\sqrt{ab}}\right)/(2\sqrt{ab}), \qquad \text{(B3)}
$$

where u is a new integration variable. It is easy to see that the function (B3) with the lower sign maps the interval $1+ab \leqslant u < \infty$ on $0 \leqslant z \leqslant 1$ and with the upper sign maps the same interval on $1/(ab) \leqslant z < \infty$. Then substitution (B3) with the lower sign into I_1 gives after simple manipulations

$$
\begin{aligned}
I_1 &= \frac{ab}{2}\int_{u(z)}^{\infty} \frac{du}{\sqrt{(u+a+b)(u+2\sqrt{ab})(u-1-ab)}}\\
&\quad + \frac{ab}{2}\int_{u(z)}^{\infty} \frac{du}{\sqrt{(u+a+b)(u-2\sqrt{ab})(u-1-ab)}},
\end{aligned} \qquad \text{(B4)}
$$

where

$$
u(z) = (1+abz^2)/z. \qquad \text{(B5)}
$$

Elliptic integrals in (B4) are transformed to the standard form by the substitution

$$
u = \frac{(1+a)(1+b)}{\sin^2\varphi} - a - b. \qquad \text{(B6)}
$$

As a result we obtain

$$
I_1 = \frac{ab}{\sqrt{(1+a)(1+b)}}\left\{F(\varphi,k_1) + F(\varphi,k_2)\right\}, \qquad \text{(B7)}
$$

where $F(\varphi,k)$ denotes the elliptic integral of the first kind,

$$
k_1 = \frac{\sqrt{a}-\sqrt{b}}{\sqrt{(1+a)(1+b)}}, \quad k_2 = \frac{\sqrt{a}+\sqrt{b}}{\sqrt{(1+a)(1+b)}}, \qquad \text{(B8)}
$$

and φ is related with the upper limit of integration z by the formula

$$
\sin^2\varphi = \frac{(1+a)(1+b)z}{(1+az)(1+bz)}. \qquad \text{(B9)}
$$

The integral I_1' is calculated by the same method and the result reads

$$
I_1' = \sqrt{\frac{ab}{(1+a)(1+b)}}\left\{-F(\varphi,k_1) + F(\varphi,k_2)\right\}. \qquad \text{(B10)}
$$

The integrals I_2 and I_2' in (B1) can be calculated with the use of the substitution

$$
u(z) = -(1+abz^2)/z \qquad \text{(B11)}
$$

or

$$
\sqrt{-z} = \left(\sqrt{u+2\sqrt{ab}} \pm \sqrt{u-2\sqrt{ab}}\right)/(2\sqrt{ab}), \qquad \text{(B12)}
$$

which map the interval $2\sqrt{ab} \leqslant u \leqslant a+b$ on the intervals $-1/b \leqslant z \leqslant -1/\sqrt{ab}$ and $-1/\sqrt{ab} \leqslant z \leqslant -1/a$, correspondingly for upper and lower signs. This transforms I_2 to

$$
\begin{aligned}
I_2 &= \frac{ab}{2}\int_{a+b}^{u(z)} \frac{du}{\sqrt{(u-a-b)(u+2\sqrt{ab})(u+1+ab)}}\\
&\quad + \frac{ab}{2}\int_{a+b}^{u(z)} \frac{du}{\sqrt{(u-a-b)(u-2\sqrt{ab})(u+1+ab)}}.
\end{aligned} \qquad \text{(B13)}
$$

These integrals are reduced to standard form of elliptic integrals by substitutions

$$
u = a + b - (a + b \pm 2\sqrt{ab})\sin^2\varphi. \qquad \text{(B14)}
$$

As a result we obtain

$$
I_2 = -\frac{ab}{\sqrt{(1+a)(1+b)}}\left\{F(\varphi_a,k_1) + F(\varphi_b,k_2)\right\}, \qquad \text{(B15)}
$$

where

$$
\varphi_a = \begin{cases} \pi - \arcsin\sqrt{\frac{(1+az)(1+bz)}{z(\sqrt{a}-\sqrt{b})^2}}, & -\frac{1}{b} \leqslant z \leqslant -\frac{1}{\sqrt{ab}},\\ \arcsin\sqrt{\frac{(1+az)(1+bz)}{z(\sqrt{a}-\sqrt{b})^2}}, & -\frac{1}{\sqrt{ab}} \leqslant z \leqslant -\frac{1}{a}; \end{cases} \qquad \text{(B16)}
$$

and

$$\varphi_b = \arcsin \sqrt{\frac{(1+az)(1+bz)}{z(\sqrt{a}+\sqrt{b})^2}}, \quad -\frac{1}{b} \leqslant z \leqslant -\frac{1}{a}. \quad \text{(B17)}$$

Similar calculation yields

$$I_2' = \sqrt{\frac{ab}{(1+a)(1+b)}} \{F(\varphi_a,k_1) - F(\varphi_b,k_2)\}. \quad \text{(B18)}$$

[1] M. Abad and A. Recati, Eur. Phys. J. D **67**, 148 (2013).
[2] Yu. G. Gladush, A. M. Kamchatnov, Z. Shi, P. G. Kevrekidis, D. J. Frantzeskakis, and B. A. Malomed, Phys. Rev. A **79**, 033623 (2009).
[3] A. M. Kamchatnov and Y. V. Kartashov, Phys. Rev. Lett. **111**, 140402 (2013).
[4] C. Hamner, J. J. Chang, P. Engels, and M. A. Hoefer, Phys. Rev. Lett. **106**, 065302 (2011).
[5] M. A. Hoefer, J. J. Chang, C. Hamner, and P. Engels, Phys. Rev. A **84**, 041605(R) (2011).
[6] C. Hamner, Y. Zhang, J. J. Chang, C. Zhang, and P. Engels, Phys. Rev. Lett. **111**, 264101 (2013).
[7] S. V. Manakov, Zh. Eksp. Teor. Fiz. **65**, 505 (1973) [Sov. Phys. JETP **38**, 248 (1974)].
[8] A. L. Berkhoer and V. E. Zakharov, Zh. Eksp. Teor. Fiz. **58**, 903 (1970) [Sov. Phys. JETP **31**, 486 (1970)].
[9] Yu. S. Kivshar and G. P. Agrawal, *Optical Solitons: From Fibers to Photonic Crystals* (Academic, San Diego, 2003).
[10] A. P. Sheppard and Y. S. Kivshar, Phys. Rev. E **55**, 4773 (1997).
[11] J. Smyrnakis, M. Magiropoulos, G. M. Kavoulakis, and A. D. Jackson, Phys. Rev. A **81**, 063601 (2010).
[12] O. C. Wright, Physica D **264**, 1 (2013).
[13] A. M. Kamchatnov, Europhys. Lett. **103**, 60003 (2013).
[14] A. M. Kamchatnov, Zh. Eksp. Teor. Fiz. **145**, 719 (2014) [JETP **118**, 630 (2014)].
[15] A. M. Kamchatnov, J. Phys. A: Math. Theor. **47**, 145203 (2014).
[16] S. Kowalevski, Acta Mathematica **12**, 177 (1889).
[17] V. V. Golubev, *Lectures on Integration of the Equations of Motion of a Rigid Bode about a Fixed Point* (GITTL, Moscow, 1953) (in Russian) [English transl.: Israel Program Sci. Trans., Jerusalem, and Office of Tech. Services, U.S. Dept. Commerce, Washington, DC (1960)].
[18] C. Polymilis, K. Hizanidis, and D. J. Frantzeskakis, Phys. Rev. E **58**, 1112 (1998).
[19] P. L. Christiansen, J. C. Eilbeck, V. Z. Enolskii, and N. A. Kostov, Proc. R. Soc. London A **456**, 2263 (2000).
[20] J. C. Eilbeck, V. Z. Enolskii, and N. A. Kostov, J. Math. Phys. **41**, 8236 (2000).
[21] D. V. Choodnovsky and G. V. Choodnovsky, Lett. Nuovo Cimento **22**, 47 (1978).
[22] S. Wojciechowski, Phys. Scr. **31**, 433 (1985).
[23] E. D. Belokolos, A. I. Bobenko, V. Z. Enol'skii, A. R. Its, and V. B. Matveev, *Algebro-Geometric Approach to Nonlinear Integrable Equations* (Springer, Berlin, 1994).
[24] A. M. Legendre, *Traité des Fonctions Elliptiques et des Integrales Eulériennes*, t. 3 (Huzard-Courcier, Paris, 1828).
[25] C. G. J. Jacobi, J. reine ang. Mat. **8**, 413 (1832); *Gesammelte Werke*, Bd. 1, S. 373 (Chelsea, New York, 1969).
[26] G. B. Whitham, *Linear and Nonlinear Waves* (Wiley, New York, 1974).
[27] A. M. Kamchatnov, *Nonlinear Periodic Waves and Their Modulations: An Introductory Course* (World Scientific, Singapore, 2000).
[28] O. C. Wright, Physica D **82**, 1 (1995).
[29] V. I. Arnold, *Mathematical Methods of Classical Mechanics* (Springer, Berlin, 1989).