Extremal states for photon number and quadratures as gauges for nonclassicality

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Rotated quadratures carry the phase-dependent information of the electromagnetic field, so they are somehow conjugate to the photon number. We analyze this noncanonical pair, finding an exact uncertainty relation, as well as a couple of weaker inequalities obtained by relaxing some restrictions of the problem. We also find the intelligent states saturating that relation and complete their characterization by considering extra constraints on the second-order moments of the variables involved. Using these moments, we construct performance measures tailored to diagnose photon-added and Schrödinger-cat-like states, among others.

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I. INTRODUCTION

Leaving aside interpretational issues [1–3], the quantum state is an essential ingredient of quantum theory: once it is known, the probabilities of the outcomes of any possible measurement may be predicted. Unfortunately, this elusive object cannot be directly determined and must be inferred from a suitable set of measurements, which constitutes the province of quantum tomography [4]. Although this might superficially appear to be a mere statistical estimation, the positivity of the reconstructed state imposes stringent quantum constraints.

Indeed, these constraints endow the set of admissible states with a rich geometry, the boundaries of which somehow establish the realm of the quantum world. Delimiting these borders is, in general, a difficult conundrum. One way to ease these complications is to look at the problem using moments of the relevant quantities, with the hope that only a few of them will be important enough. As a simple yet illustrative example of this, let us examine a single-mode quantum field, which will serve as a thread in this work. The complex amplitude $\hat{a}$ and the photon number $\hat{n} = \hat{a}^\dagger \hat{a}$ for this system must obey $\langle \hat{n} \rangle \geqslant |\langle \hat{a} \rangle|^2$, which can be readily obtained by a simple application of the Cauchy-Schwarz inequality [5]. The extremal states (i.e., those saturating the inequality) turn out to be coherent states. Hence, the difference between the average photon number and the square of the absolute value of the complex amplitude, which must be always positive, can be taken, up to second order, as a reliable indicator of the "quality" of a coherent state.

In classical signal processing, intensity and phase are the basic magnitudes specifying the field. At the quantum level, they translate into photon number and phase. However, the definition of a *bona fide* phase operator is beset by difficulties that have been the object of a heated debate [6–10]. Here, we choose a surrogate approach that considers the phase properly encoded in the field quadratures, as is routinely done in the theory of quantum nondemolition measurements [11–13]. While photon number lies at the heart of the discrete-variable quantum information, quadratures are the primary tool in the continuous-variable domain. Photon number and quadratures bridge these two complementary worlds in a natural way.

Extremal states for these variables were investigated some years ago, fueled by the search for noise minimum states [14,15]. More recently, the quite similar question of the uncertainty relation for the number and the annihilation operator was addressed [16]. Our aim here is to push this research further and explore how these extremal states can be used for the diagnostics of nonclassicality.

The plan of this paper is as follows. In Sec. II we revisit the uncertainty relations for photon number and rotated quadratures, as well as loose approximations thereof. Section III rounds off this discussion by looking at the extra restrictions that quantum theory imposes on the second-order moments of those variables and by looking at the properties of intelligent states, which obey the equality in the previous uncertainty relations. Based on those states, we tailor performance measures especially germane to verify photon-added and Schrödinger-cat-like states, among others. Finally, our conclusions are summarized in Sec. IV.

II. UNCERTAINTY RELATIONS FOR PHOTON NUMBER AND QUADRATURES

A. Tight uncertainty relations

The system we are interested in is a single-mode electromagnetic field, which can be formally deemed a harmonic oscillator. Classically, it is characterized by a complex amplitude that contains information about both the magnitude and the phase of the field. In the quantum formalism, the mode is specified by the action of annihilation ($\hat{a}$) and creation ($\hat{a}^\dagger$) operators satisfying the basic bosonic commutation relation [17]

$$[\hat{a}, \hat{a}^\dagger] = \hat{\mathbb{1}} \,. \tag{2.1}$$

At optical frequencies, the common way of measuring the field is with homodyne detection [18]. The readout in this scheme involves moments of the rotated quadratures

$$\hat{x}_\theta = \frac{1}{\sqrt{2}}(\hat{a}\, e^{+i\theta} + \hat{a}^\dagger\, e^{-i\theta}) \,, \quad \hat{p}_\theta = \frac{1}{\sqrt{2}i}(\hat{a} e^{+i\theta} - \hat{a}^\dagger e^{-i\theta}) \,, \tag{2.2}$$

where θ is the phase of the local oscillator that can be externally varied. The reader should be careful about comparing results on quadrature, as there are a variety of normalizations used in the literature. Notice that $\hat{p}_\theta = -\hat{x}_{\theta+\pi/2}$ and that, for $\theta = 0$, they reduce to the canonical variables $\hat{x}$ and $\hat{p}$. They satisfy the canonical commutation relation (in units $\hbar = 1$ throughout)

$$[\hat{x}_\theta, \hat{p}_\theta] = i\hat{\mathbb{1}} \,. \tag{2.3}$$

Since $\hat{x}_\theta^2 + \hat{p}_\theta^2 = \hat{n} + \hat{\mathbb{1}}/2$, where $\hat{n} = \hat{a}^\dagger \hat{a}$ is the number operator, precise knowledge of the eigenvalue of $\hat{n}$ restricts the possible knowledge about the quadratures. This is quantified by the commutation

$$[\hat{n}, \hat{x}_\theta] = -i\,\hat{p}_\theta \,, \tag{2.4}$$

which, in turn, implies the uncertainty relation

$$V(\hat{n})\; V(\hat{x}_\theta) \geqslant \tfrac{1}{4} |\langle \hat{p}_\theta \rangle|^2 \,. \tag{2.5}$$

Here, $V(\hat{A}) = \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2$ denotes the variance of $\hat{A}$, and the angular brackets $\langle \cdot \rangle$ mean averaging over the state of the system (either pure or mixed).

Equation (2.5) is an exact relation but depends on the local oscillator phase. Differentiation with respect to θ leads to the extremal values $\lambda_\pm$ of $V(\hat{x}_\theta)$; they can be written as [19]

$$\lambda_\pm^2 = C(\hat{a}^\dagger, \hat{a}) \pm |V(\hat{a})| \,, \tag{2.6}$$

where the (symmetrized) covariance is $C(\hat{A}, \hat{B}) = \langle \{\hat{A}, \hat{B}\}/2 \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle$. It is convenient to introduce the quantity λ by

$$\lambda^2(\varphi) = \lambda_+^2 \sin^2 \varphi + \lambda_-^2 \cos^2 \varphi \,, \tag{2.7}$$

with $\varphi = \arg[V(\hat{a})/2] - \arg[\langle \hat{a} \rangle]$, in terms of which Eq. (2.5) takes the form

$$V(\hat{n}) \left[\frac{\lambda_+ \lambda_-}{\lambda(\varphi)} \right]^2 \geqslant |\langle \hat{a} \rangle|^2 \,. \tag{2.8}$$

Apart from the invariant parameters $\lambda_\pm$, this expression also depends on the phase φ. However, this alternative presentation will allow us in the following to devise remarkable simplifications. In addition, it is closely related to the customary ball-and-stick representation of quantum states in phase space [20], where the quadratures $\hat{x}$ and $\hat{p}$ are taken as coordinates. In this picture, sketched in Fig. 1, the stick corresponds to the average value of the field $\langle \hat{a} \rangle$, and the ball corresponds to the fluctuations around the mean value. We display this area as a noise ellipse whose semiaxes are precisely the invariant parameters $\lambda_\pm$. In this way, $\lambda_\pm$, which are eigenvalues of the covariance matrix and related to the universal quantum invariants [21], play a key role in picturing the noise properties of the state [22,23]. The meaning of $\lambda(\varphi)$ can be gathered at once from Fig. 1.

As a final remark, we mention that inequality (2.8) formally makes it possible to introduce a quantity like phase variance $V(\hat{\phi})$, fulfilling a standard uncertainty relation with $V(\hat{n})$, namely,

$$V(\hat{\phi}) = \left[\frac{\lambda_+ \lambda_-}{\lambda(\varphi)} \right]^2 \frac{1}{|\langle \hat{a} \rangle|^2} \,. \tag{2.9}$$

Interestingly enough, an explicit calculation shows that this variance of the putative operator $\hat{\phi}$ tallies with the smallest possible phase resolution in the Shapiro-Wagner phase concept [24] if both quadrature operators are measured simultaneously.

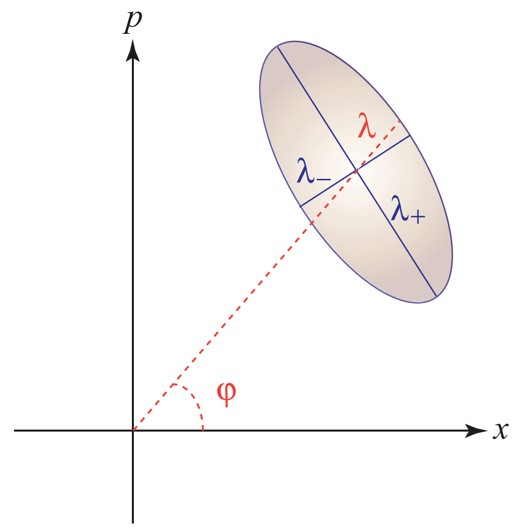

FIG. 1. (Color online) Ball-and-stick diagram in phase space. $\langle a \rangle$ is the complex amplitude of the field, and $\lambda_\pm$ are the semiaxes of the uncertainty ellipse. The variable $\lambda(\varphi)$ has been defined in Eq. (2.7).

B. Relaxing the bounds

The tight uncertainty relation (2.5) and its equivalent (2.8) convey complete information, but they provide phase-dependent bounds. It might be interesting to work out weaker inequalities, which are independent of the orientation of the noise ellipse.

A first option stems from the trivial observation that, according to (2.7), $\inf_\varphi \lambda(\varphi) = \lambda_-$, so (2.8) can be relaxed to

$$V(\hat{n})\, \lambda_+^2 \geqslant |\langle \hat{a} \rangle|^2 \,, \tag{2.10}$$

or, using $C(\hat{a}^\dagger, \hat{a})$,

$$V(\hat{n})\,[C(\hat{a}^\dagger, \hat{a}) + |V(\hat{a})|] \geqslant |\langle \hat{a} \rangle|^2 \,. \tag{2.11}$$

The second possibility comes from the estimate $[\lambda_+ \lambda_- / \lambda(\varphi)]^2 \leqslant \lambda_+^2 + \lambda_-^2$. Now, we can write down

$$V(\hat{n})\,(\lambda_+^2 + \lambda_-^2) \geqslant |\langle \hat{a} \rangle|^2 \,, \tag{2.12}$$

which, using again $C(\hat{a}^\dagger, \hat{a})$, reads as

$$V(\hat{n})\, C(\hat{a}^\dagger, \hat{a}) \geqslant |\langle \hat{a} \rangle|^2 \,. \tag{2.13}$$

This coincides with the expression obtained in Ref. [16]. In spite of its simplicity, this inequality has a drawback: it cannot be exactly saturated (but see the solution worked out in Ref. [25]). This can be confirmed by noticing that Eq. (2.13) is just the sum of

$$V(\hat{n})\; V(\hat{x}) \geqslant \tfrac{1}{4} |\langle \hat{p} \rangle|^2 \,, \quad V(\hat{n})\; V(\hat{p}) \geqslant \tfrac{1}{4} |\langle \hat{x} \rangle|^2 \tag{2.14}$$

since $|\langle \hat{x} \rangle|^2 + |\langle \hat{p} \rangle|^2 = 2|\langle \hat{a} \rangle|^2$. But these two relations cannot be saturated simultaneously [26,27], and as consequence, Eq. (2.13) is not tight.

In Fig. 2 we have plotted both approximate inequalities (2.11) and (2.13) in a three-dimensional space, with the moments $V(\hat{a})$, $C(\hat{a}^\dagger, \hat{a})$, and $V(\hat{n})$ as axes. The region above

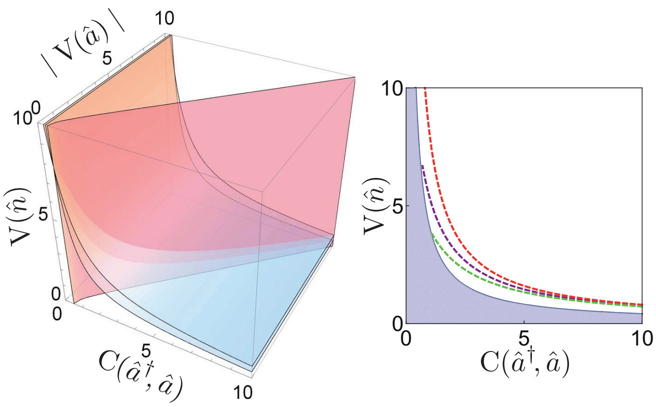

FIG. 2. (Color online) Uncertainty relations (2.8) and (2.11) as a function of the second-order moments of the variables involved. The plane along the diagonal corresponds to the constraint (3.2). In the right panel, we plot sections of the previous figure for several values of $|V(\hat{a})|$ [up to the value permitted by (3.3)]. The solid line represents the bound (2.13), and the shaded region designates the forbidden states.

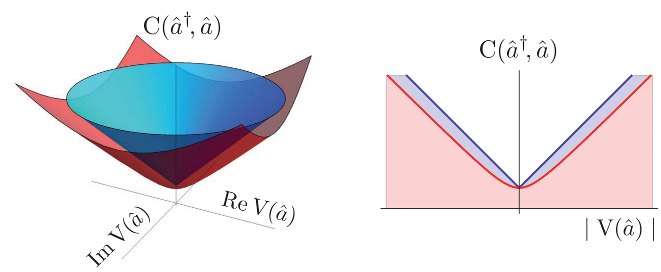

FIG. 3. (Color online) Three-dimensional subspace of all the possible second-order moments for a fixed value of $\langle \hat{a} \rangle$, ranged by the red hyperboloid. The blue cone is the boundary for moments representing squeezed light. On the right, we present a section of that plot; the red (light gray) shaded region represents the forbidden states, while the blue (dark gray) shaded one gives the squeezed states.

these surfaces is the allowed states. We have also plotted several two-dimensional sections for different values of $|V(\hat{a})|$. It is evident that (2.11) is tighter than (2.13), which actually is independent of $|V(\hat{a})|$.

In summary, the exact uncertainty relation (2.8) and its two weaker approximations (2.10) and (2.13) fully specify the complementary nature of photon number and quadratures. They can be regarded as a sensible alternative to the more controversial uncertainty relations for number-phase observables [28,29] and their entropic counterparts [30–32].

III. EXTREMAL STATES

A. Additional restrictions on the moments

The discussion thus far has capitalized on the variances as the proper estimator of quantum uncertainties, as is generally accepted. To have a complete grasp of the problem, we have to assess also the constraints arising in the second-order moments involved in the problem, as they are not independent.

As heralded in the Introduction, an appropriate tool to delimit these moments is the generalized Cauchy-Schwarz inequality, which can be jotted down as [14]

$$|\langle \hat{A}^\dagger \hat{B} \rangle|^2 \leqslant \langle \hat{A}^\dagger \hat{A} \rangle \, \langle \hat{B}^\dagger \hat{B} \rangle \, . \tag{3.1}$$

The equality occurs only for states where $\langle \hat{A}^\dagger \hat{A} \rangle = 0$, $\langle \hat{B}^\dagger \hat{B} \rangle = 0$, or $(\hat{A} - ir\hat{B})\,\hat{\rho} = 0$ for some real scalar r and with $\hat{\rho}$ being the density operator of the state.

A first application of Eq. (3.1), with $\hat{A} = \hat{a}$ and $\hat{B} = \hat{\mathbb{1}}$, gives $\langle \hat{a}^\dagger \hat{a} \rangle \geqslant \langle \hat{a}^\dagger \rangle \langle \hat{a} \rangle$. As a result, from its very definition, the covariance fulfills

$$C(\hat{a}^\dagger, \hat{a}) \geqslant \tfrac{1}{2} \, , \tag{3.2}$$

which is saturated by the coherent states. Repeating the same procedure, but now with $\hat{A}^\dagger = \hat{B} = \hat{a} - \langle \hat{a} \rangle$, we get

$$|V(\hat{a})|^2 \leqslant C(\hat{a}^\dagger, \hat{a})^2 - \tfrac{1}{4} \, . \tag{3.3}$$

This condition is equivalent to requiring $\lambda_+^2 \lambda_-^2 \geqslant 1/4$, which has a direct physical interpretation in the ball-and-stick diagram analyzed before: for any physical state, the uncertainty area (in quadrature units) must be greater than or equal to 1/4. For coherent states the noise is equally distributed in both quadratures, $\lambda_-^2 = \lambda_+^2 = 1/2$, so they are depicted by a minimal circle. In squeezed states, the fluctuations in one quadrature are reduced below the value 1/2, at the expense of the corresponding increased fluctuations in the other quadrature, such that they preserve the minimum area. Consequently, (3.3) is saturated by squeezed states.

In this regard, the condition of squeezing is just $\lambda_-^2 \leqslant 1/2$, which translates into

$$|V(\hat{a})| \geqslant C(\hat{a}^\dagger, \hat{a}) - 1/2 \, , \tag{3.4}$$

which completes (3.3).

Inequalities (3.3) and (3.4) can be represented in a very appealing way if we plot $C(\hat{a}^\dagger, \hat{a})$ as a function of the real and imaginary parts of $V(\hat{a})$, as done in Fig. 3. In these variables, the equality in (3.3) defines a hyperboloid with the vertex in the point (0,0,1/2), and all the moments about that hyperboloid are then possible. On the other hand, (3.4) defines a cone with the vertex in the same point (0,0,1/2): all points below the cone are squeezed.

Finally, we use once more Eq. (3.1), with $\hat{A} = \hat{a}^2$ and $\hat{B} = \hat{\mathbb{1}}$, to get $\langle \hat{a}^{\dagger 2} \hat{a}^2 \rangle \geqslant \langle \hat{a}^{\dagger 2} \rangle \langle \hat{a}^2 \rangle$. Assuming further the condition of zero complex amplitude, $\langle \hat{a} \rangle = 0$, we have

$$C(\hat{a}^{\dagger 2}, \hat{a}^2) \geqslant 2 \langle \hat{a}^\dagger \hat{a} \rangle + 1 \, , \tag{3.5}$$

which is saturated by the states spanned on the Hilbert subspace of the superposition of coherent states $|\pm\alpha\rangle$, a general solution of the eigenvalue problem $a^2|\psi\rangle = \alpha^2 |\psi\rangle$.

B. Intelligent states

We have been using the term extremal to loosely refer to those states for which the inequalities analyzed so far hold as equalities.

If the lower bound in an uncertainty relation is state dependent, states satisfying the equality in the uncertainty relation need not give a minimum in the uncertainty product. This is the case with our fundamental relation (2.4), so it requires a distinction between intelligent states [33] and minimum uncertainty product states [34].

The intelligent states are solutions of the non-Hermitian eigenvalue problem [35]

$$(\hat{n} - ir\,\hat{x}_\theta)|\Psi_r\rangle = \Omega|\Psi_r\rangle\,, \quad r \in \mathbb{R}\,, \tag{3.6}$$

where Ω is the eigenvalue. Although the solution to this equation has already been discussed in Ref. [15], we provide here a simplified alternative derivation. By introducing the complex parameter $\alpha = -ir/\sqrt{2}\,\exp(-i\theta)$, where θ is the phase of the quadrature $\hat{x}_\theta$, (3.6) reads

$$(\hat{a}^\dagger + \alpha^*)(\hat{a} - \alpha)|\Psi_\alpha\rangle = (\Omega - |\alpha|^2)|\Psi_\alpha\rangle\,. \tag{3.7}$$

Since $[\hat{a} - \alpha,(\hat{a}^\dagger + \alpha^*)^M] = M(\hat{a}^\dagger + \alpha^*)^{M-1}$ for every integer M, one quickly guesses that the intelligent states we are looking for are

$$|\Psi_\alpha\rangle = \mathscr{N}(a^\dagger + \alpha^*)^M|\alpha\rangle\,, \tag{3.8}$$

where $\mathscr{N}$ is a normalization constant and $|\alpha\rangle$ is a coherent state. We can also expand this expression in the Fock basis: using the generating function of the generalized Laguerre polynomials $L_m^a(x)$ [36],

$$(1+t)^M e^{-xt} = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(1+M)} L_n^{M-n}(x)\,, \tag{3.9}$$

we get

$$|\Psi_\alpha\rangle = \mathscr{N}\exp(-|\alpha|^2/2)\sum_{n=0}^{\infty}\frac{\sqrt{n!}}{M!}(\alpha)^{*M-n}L_n^{M-n}(-|\alpha|^2)\,|n\rangle\,. \tag{3.10}$$

These states were found in a different context by Yuen [37], who called them near-photon-number eigenstates. They are also called crescent states [38] because the contours of their Wigner function are sheared due typically to a Kerr nonlinearity [39,40]. It is worth stressing the close similarity of these states, written as in Eq. (3.10), with the expansion for Fock-displaced states [41], although their arguments differ in the sign, which introduces remarkable differences in the photon-number distribution.

For weak fields $|\alpha| \ll 1$, the crescent states reduce to the so-called M-photon-added coherent states [42]

$$|\Psi_\alpha\rangle \simeq \mathscr{N}\hat{a}^{M\dagger}|\alpha\rangle\,, \tag{3.11}$$

while in the strong-field limit $|\alpha| \gg 1$ they can be well approximated by the superposition of coherent and single-photon-added coherent states

$$|\Psi_\alpha\rangle \simeq \mathscr{N}(|\alpha\rangle + \gamma\hat{a}^\dagger|\alpha\rangle)\,, \tag{3.12}$$

with $\gamma = M/\alpha^*$. For this particular case, we get

$$\begin{aligned}\langle\hat{a}\rangle &= \alpha + \mathscr{N}(\gamma + |\gamma|^2\alpha)\,,\\ \langle\hat{n}\rangle &= 1 + |\alpha|^2 + \mathscr{N}(|\gamma|^2|\alpha|^2 - 1)\,,\end{aligned} \tag{3.13}$$

and the normalization constant is $\mathscr{N}^{-1} = 1 + \gamma^*\alpha + \gamma\alpha^* + |\gamma|^2(1+|\alpha|^2)$. The second-order moments can be analytically computed, although the expression is a bit involved and of no interest for our purposes here. In Fig. 4 we have plotted $V(\hat{n})$ versus $C(\hat{a}^\dagger,\hat{a})$, as we did in Fig. 2, for these states with varying values of γ. For comparison, we have included also the bound imposed by the weak uncertainty relation (2.13). As we can appreciate, the intelligent states are always very close to that bound.

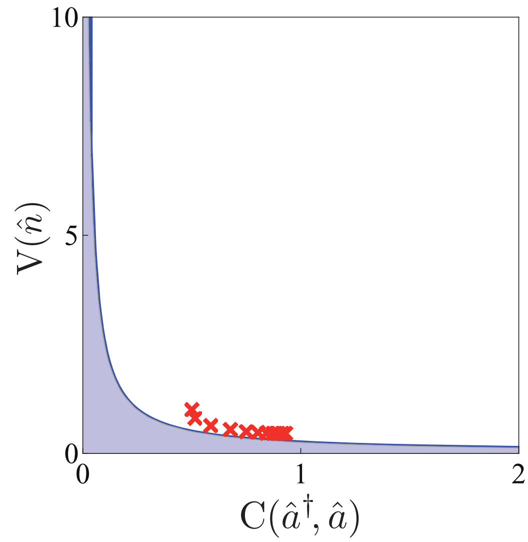

FIG. 4. (Color online) Same plot as in the right panel of Fig. 2. The solid line indicates the weak bound (2.13), while the crosses represent the approximate intelligent states (3.12) for several values of γ.

The final idea we wish to stress is that all these extremal states can be used as powerful tools to pinpoint important classes of states. Let us look at the crescent states (or the approximation of M-photon-added coherent states treated before). Since they are intelligent states for (2.4), the coefficient

$$G_1 := \frac{V(\hat{n})}{|\langle\hat{a}\rangle|^2}\left[\frac{\lambda_+\lambda_-}{\lambda(\varphi)}\right]^2 \geqslant 1 \tag{3.14}$$

quantifies how far a given state is from being intelligent.

Inequality (2.8) becomes trivial in the case of zero amplitude. In that case, however, inequality (3.5) provides a saturable bound. As discussed before, the extremal states are given by the linear superposition of coherent states $|\pm\alpha\rangle$, including Schrödinger-cat-like states $|\alpha\rangle + |-\alpha\rangle$. The performance measure suitable to check these states is

$$G_2 := \frac{C(\hat{a}^{\dagger 2},\hat{a}^2)}{2\langle\hat{n}\rangle + 1} = \frac{V(\hat{n}) + 4(\lambda_+^2 - 1/2)(\lambda_-^2 + 1/2)}{\langle\hat{n}\rangle} \geqslant 1\,, \tag{3.15}$$

which again provides a robust and simple alternative to more sophisticated methods. It is obvious that equivalent measures can be employed for the other extremal states explored here.

Although the inequalities reported above are fairly simple, they have interesting and not-yet-recognized consequences for quantum information processing. Actually, photon-added and catlike states are archetypes of non-Gaussian nonclassical feasible states [43,44]. Homodyne detection has been the only tool employed thus far to certify these states. However, the dimension of the reconstruction subspace predetermines the accuracy of the result.

The inequalities for G_1 and G_2 offer an intriguing alternative free from any assumptions of this kind: measuring the lower-order moments suffices to quantify how far the experimental data are from the ideal prediction. We emphasize that this really makes sense from an experimental viewpoint and concurs with previous achievements in quantum optics: photon-number states were characterized by sub-Poissonian noise, and squeezed states were characterized by quadrature-noise reduction.

The proposed benchmarking for photon-added and catlike states is more demanding, for it requires the control of both photon number and quadrature variances. It is probably not by chance that all interesting and feasible quantum states are in some sense extremal.

Finally, we mention that a complete test along these lines would require addressing all the moments [45], which is much more demanding than our simple procedure.

IV. CONCLUDING REMARKS

In short, we have formulated a tight uncertainty relation for photon number and rotated quadratures, which can be considered a sensible and timely approach to the canonical pair number phase. We have also constructed intelligent states for this uncertainty relation, retrieving the well-known crescent states. This saturable inequality, along with some other obtained from a systematic application of the Cauchy-Schwarz inequalities to all the second-order moments of the variables involved, can serve as a handy toolbox for nonclassical state diagnosis, an alternative to the more onerous and laborious quantum tomography.

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