Fidelity and trace-norm distances for quantifying coherence

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We investigate the coherence measures induced by fidelity and trace norm, based on the coherence quantification recently proposed by Baumgratz *et al.* [T. Baumgratz, M. Cramer, and M. B. Plenio, Phys. Rev. Lett. **113**, 140401 (2014)]. We show that the fidelity of coherence does not in general satisfy the monotonicity requirement as a measure of coherence under the subselection of the measurement condition. We find that the trace norm of coherence can act as a measure of coherence for qubits and some special class of qutrits with some restrictions on the incoherent operators, while the general case needs to be explored further.

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I. INTRODUCTION

Coherence arising from quantum superposition plays a central role in quantum mechanics. Quantum coherence is an important subject in quantum theory and quantum information science and is a common necessary condition for both entanglement and other types of quantum correlations. It has been shown that a good definition of coherence depends not only on the state of the system ρ, but also on a fixed basis for the quantum system [1]. Up to now, several themes of coherence have been considered, such as witnessing coherence [2], catalytic coherence [3], the thermodynamics of quantum coherence [4], and the role of coherence in biological system [5]. There has been no well-accepted efficient method for quantifying coherence until recently. Girolami proposed a measure of quantum coherence based on the Wigner-Yanase-Dyson skew information [6]. It is implementable in theoretical as well as experimental schemes with the current technology. Baumgratz *et al.* introduced a rigorous framework of quantification of coherence and proposed several measures of coherence, which are based on the well-behaved metrics including the l_p norm, relative entropy, trace norm, and fidelity [1]. The quantification of coherence promoted in a unified and rigorous framework thus stimulated many further considerations about quantum coherence [7–11].

From the view point of the definition, one can straightforwardly quantify the coherence in a given basis by measuring the distance between the quantum state ρ and its nearest incoherent state. This property is similar to that of the well-studied measures of the quantum correlations, e.g., entanglement and quantum discord [12–14]. We remark that the coherence measures are to be applied to all quantum systems, in comparison with quantum correlation measures, which naturally involve multiple parties [7]. We know that several basic criteria are proposed that should be satisfied by any measure of the entanglement [12,14]. In comparison, the coherence measures presented by Baumgratz *et al.* also need to satisfy four necessary criteria [1]. Consider a finite-dimensional Hilbert space $\mathcal{H}$ with $d = \dim(\mathcal{H})$. We note that $\mathcal{I}$ is a set of quantum states, called incoherent states, that are diagonal in a fixed basis $\{|i\rangle\}_{i=1}^{d}$. Then any proper measure of the coherence C must satisfy the following conditions.

(i) $C(\rho) \geqslant 0$ for all quantum states ρ and $C(\rho) = 0$ if and only if $\rho \in \mathcal{I}$.

(ii a) Monotonicity under all the incoherent completely positive and trace-preserving (ICPTP) maps Φ: $C(\rho) \geqslant C(\Phi(\rho))$, where $\Phi(\rho) = \sum_n K_n \rho K_n^\dagger$ and $\{K_n\}$ is a set of Kraus operators, which satisfies $\sum_n K_n^\dagger K_n = \mathbb{I}$ with $K_n \mathcal{I} K_n^\dagger \subset \mathcal{I}$.

(ii b) Monotonicity for average coherence under subselection based on measurement outcomes: $C(\rho) \geqslant \sum_n p_n C(\rho_n)$, where $\rho_n = \frac{K_n \rho K_n^\dagger}{p_n}$ and $p_n = \mathrm{Tr}(K_n \rho K_n^\dagger)$ for all $\{K_n\}$ with $\sum_n K_n^\dagger K_n = \mathbb{I}$ and $K_n \mathcal{I} K_n^\dagger \subset \mathcal{I}$.

(iii) Nonincreasing under mixing of quantum states: $\sum_n p_n C(\rho_n) \geqslant C(\sum_n p_n \rho_n)$ for any ensemble $\{p_n, \rho_n\}$.

As shown in [1], the condition (ii b) is important as it allows for subselection based on measurement outcomes, a process available in well-controlled quantum experiments. It has been shown that the quantum relative entropy and l_1 norm satisfy all conditions (i)–(iii). The squared Hilbert-Schmidt norm satisfies conditions (i), (ii a), (iii), but not (ii b). However, it is still an open question whether some other coherence measures satisfy (ii b), for example, the fidelity and trace distance. In this paper we show that the measure of coherence induced by the fidelity does not satisfy condition (ii b) and an explicit example is presented. It is still unknown whether the trace norm of coherence obeys (ii b) in general; however, we consider some special circumstances. If we restrict the incoherent operators K_n to the 2×2 and 3×2 matrices, we show that the trace norm of coherence satisfies condition (ii b). For some special qutrits, if we restrict the incoherent operators K_n to the 3×3 matrices, the trace norm of coherence satisfies this condition.

This paper is organized as follows. In Sec. II we illustrate that the fidelity of coherence is not a good measure for quantum coherence by presenting an example in which condition (ii b) is not satisfied. In Sec. III we show that condition (ii b) can be satisfied in qubit and some special qutrits for the trace norm of coherence with some restrictions on the incoherent operators K_n. We summarize our results in Sec. IV.

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II. FIDELITY OF COHERENCE

In quantum information theory, the fidelity is a measure of the distance between quantum states ρ and σ; it is defined by [15]

$$F(\rho,\sigma) = [\mathrm{Tr}\sqrt{\rho^{1/2}\sigma\rho^{1/2}}]^2. \tag{1}$$

It is known that the fidelity is non-negative and it is nondecreasing under completely positive and trace-preserving (CPTP) maps $\mathcal{E}$, i.e., $F(\mathcal{E}(\rho),\mathcal{E}(\sigma)) \geqslant F(\rho,\sigma)$. Then we find that the fidelity-induced distance $D(\rho,\sigma) = 1 - \sqrt{F(\rho,\sigma)}$ is monotonic under CPTP maps and $F(\rho,\sigma) = 1$ if and only if $\rho = \sigma$. Hence, the fidelity of coherence can be defined as

$$C_f(\rho) := \min_{\delta\in\mathcal{I}} D(\rho,\delta) = 1 - \sqrt{\max_{\delta\in\mathcal{I}} F(\rho,\delta)}. \tag{2}$$

It is easy to verify that the fidelity of coherence fulfills (i), (ii a), and (iii) [1]. However, it is not clear whether condition (ii b) is satisfied. To test this condition, without loss of generality, we consider the simple one-qubit system. It is known that the fidelity has a more explicit formula for the one-qubit system. By using a Bloch sphere representation, quantum states ρ and δ can be expressed as [16]

$$\rho = \frac{\mathbb{I} + \mathbf{r}\cdot\sigma}{2}, \quad \delta = \frac{\mathbb{I} + \mathbf{s}\cdot\sigma}{2}, \tag{3}$$

where $\mathbb{I}$ is the identity operator, $\mathbf{r} = (r_x,r_y,r_z)$ and $\mathbf{s} = (s_x,s_y,s_z)$ are the Bloch vectors, and $\sigma = (\sigma_x,\sigma_y,\sigma_z)$ is a vector of Pauli matrices. Then the fidelity between qubits ρ and σ has an elegant form

$$F(\rho,\delta) = \frac{1}{2}[1 + \mathbf{r}\cdot\mathbf{s} + \sqrt{(1-|\mathbf{r}|^2)(1-|\mathbf{s}|^2)}], \tag{4}$$

where $\mathbf{r}\cdot\mathbf{s}$ is the inner product of $\mathbf{r}$ and $\mathbf{s}$, and $|\mathbf{r}|$ and $|\mathbf{s}|$ are the magnitudes of $\mathbf{r}$ and $\mathbf{s}$, respectively. Since δ is an incoherent state, then the Bloch vector $\mathbf{s}$ can be expressed as $\mathbf{s} = (0,0,s_z)$. This helps us simplify Eq. (4),

$$F(\rho,\delta) = \frac{1}{2}\left[1 + r_z s_z + \sqrt{\left(1 - r_x^2 - r_y^2 - r_z^2\right)\left(1 - s_z^2\right)}\right]. \tag{5}$$

To optimize $F(\rho,\delta)$ over all incoherent states, we take the derivative with respect to the parameter s_z,

$$\frac{dF(\rho,\delta)}{ds_z} = \frac{1}{2}\left[r_z - \sqrt{\left(1 - r_x^2 - r_y^2 - r_z^2\right)}\frac{s_z}{\sqrt{1-s_z^2}}\right]. \tag{6}$$

After some simple algebraic operation, we obtain

$$\max_{\delta\in\mathcal{I}} F(\rho,\delta) = \frac{1}{2}\left[1 + \sqrt{\left(1 - r_x^2 - r_y^2\right)}\right]. \tag{7}$$

Therefore, we give an analytical expression of the fidelity of coherence for the one-qubit system, namely,

$$\begin{aligned} C_f(\rho) &= 1 - \sqrt{\max_{\delta\in\mathcal{I}} F(\rho,\delta)} \\ &= 1 - \frac{\sqrt{2}}{2}\sqrt{1 + \sqrt{\left(1 - r_x^2 - r_y^2\right)}}. \end{aligned} \tag{8}$$

This implies that the state ρ_{diag} that is generated by removing all the off-diagonal elements and leaving the diagonal elements in the density operator ρ is not necessarily optimal for the fidelity of coherence in the one-qubit system. Thus, in general, we claim

$$\min_{\delta\in\mathcal{I}}[1 - \sqrt{F(\rho,\delta)}] \neq 1 - \sqrt{F(\rho,\rho_{\mathrm{diag}})}. \tag{9}$$

This makes the subselection process hard to verify. To simplify the calculation, we choose some peculiar incoherent operations.

Now we give an example to show that condition (ii b) is violated. We know that the depolarizing, the phase damping, and the amplitude damping channels are the qubit incoherent operations. We consider the amplitude dampinglike operation, which is an ICPTP map; its operation elements are

$$K_1 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}, \tag{10}$$

where $|a|^2 = 1$ and $|b|^2 + |c|^2 = 1$. After applying this channel on the one-qubit state ρ, we obtain the output state

$$\rho_1 = \begin{pmatrix} \frac{|a|^2(1+r_z)}{|a|^2(1+r_z)+|b|^2(1-r_z)} & \frac{ab^*(r_x-ir_y)}{|a|^2(1+r_z)+|b|^2(1-r_z)} \\ \frac{a^*b(r_x+ir_y)}{|a|^2(1+r_z)+|b|^2(1-r_z)} & \frac{|b|^2(1-r_z)}{|a|^2(1+r_z)+|b|^2(1-r_z)} \end{pmatrix} \tag{11}$$

with the probability

$$p_1 = \mathrm{Tr}(K_1\rho K_1^\dagger) = \tfrac{1}{2}[|a|^2(1+r_z) + |b|^2(1-r_z)]. \tag{12}$$

To obtain the quantity $C_f(\rho_1)$, we transform ρ_1 to the Bloch representation and define $\hat{\mathbf{r}} = \{\hat{r}_x,\hat{r}_y,\hat{r}_z\}$,

$$\begin{aligned} \hat{r}_x &= \frac{ab^*(r_x - ir_y) + a^*b(r_x + ir_y)}{|a|^2(1+r_z) + |b|^2(1-r_z)}, \\ \hat{r}_y &= \frac{-i[a^*b(r_x + ir_y) - ab^*(r_x - ir_y)]}{|a|^2(1+r_z) + |b|^2(1-r_z)}, \\ \hat{r}_z &= \frac{|a|^2(1+r_z) - |b|^2(1-r_z)}{|a|^2(1+r_z) + |b|^2(1-r_z)}. \end{aligned} \tag{13}$$

By using Eq. (8) we obtain the fidelity of coherence for ρ_1 as

$$\begin{aligned} C_f(\rho_1) &= 1 - \frac{\sqrt{2}}{2}\sqrt{1 + \sqrt{\left(1 - \hat{r}_x^2 - \hat{r}_y^2\right)}} \\ &= 1 - \frac{\sqrt{2}}{2}\sqrt{1 + \sqrt{1 - \frac{4|a|^2|b|^2\left(r_x^2 + r_y^2\right)}{[|a|^2(1+r_z) + |b|^2(1-r_z)]^2}}}. \end{aligned} \tag{14}$$

Setting $|b|^2 = \frac{1}{4}$, $|c|^2 = \frac{3}{4}$, $r_x^2 + r_y^2 = \frac{1}{2}$, and $r_z = -\frac{\sqrt{2}}{2}$ and substituting them into Eqs. (8), (12), and (14), we then have

$$\begin{aligned} p_1 C_f(\rho_1) &= \frac{10 - 3\sqrt{2}}{16}\left[1 - \frac{\sqrt{2}}{2}\sqrt{1 + \sqrt{1 - \frac{32}{(10-3\sqrt{2})^2}}}\right] \\ &\approx 0.082\,73 \end{aligned} \tag{15}$$

and

$$C_f(\rho) = 1 - \sqrt{\frac{1}{2}\left[1 + \frac{\sqrt{2}}{2}\right]} \approx 0.076\,12. \tag{16}$$

Note that the operation K_2 makes $C_f(\rho_2) = 0$. Thus, we obtain

$$\sum_{i=1}^{2} p_i C_f(\rho_i) = p_1 C_f(\rho_1) > C_f(\rho). \tag{17}$$

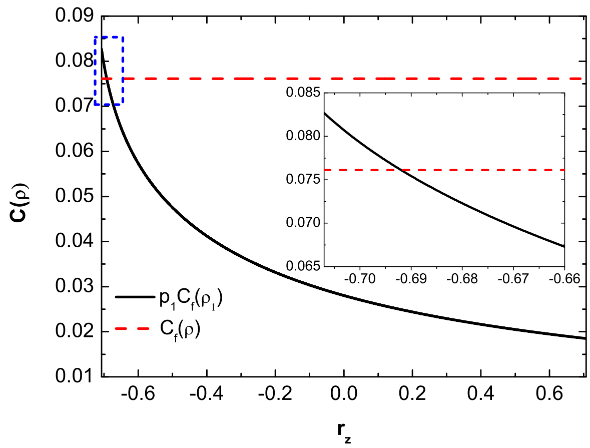

FIG. 1. (Color online) Comparison between $C_f(\rho)$ and $p_1C_f(\rho_1)$. The black solid line is obtained by taking $|a|^2=1$, $|b|^2=\frac{1}{4}$, $|c|^2=\frac{3}{4}$, $r_x^2+r_y^2=\frac{1}{2}$, and $-\frac{\sqrt{2}}{2}\leqslant r_z\leqslant\frac{\sqrt{2}}{2}$ in Eqs. (12) and (14). The red dashed line is obtained by taking the same values in Eq. (8).

From the above example, we then conclude that condition (ii b), i.e., $C_f(\rho)\geqslant\sum_n p_nC_f(\rho_n)$, is not generally true for the measure of coherence induced by fidelity. If the Bloch vector $\mathbf{r}=(r_x,r_y,r_z)$ satisfies $r_x^2+r_y^2+r_z^2\leqslant 1$ and we suppose that $-\frac{\sqrt{2}}{2}\leqslant r_z\leqslant\frac{\sqrt{2}}{2}$, then we can find many examples to illustrate that the fidelity of coherence does not satisfy condition (ii b), as shown in Fig. 1.

III. TRACE NORM OF COHERENCE

Next we investigate whether the trace norm may play the role of the measure to quantify coherence. In this section we would present progress in this direction.

The trace norm of coherence is defined as

$$C_{\rm tr}(\rho):=\min_{\delta\in\mathcal{I}} D(\rho,\delta), \tag{18}$$

where $D(\rho,\delta)=\mathrm{Tr}|\rho-\delta|$ is the trace norm between quantum states ρ and δ. For the trace norm of coherence, we list some basic formulas, which can help us judge whether the trace norm of coherence satisfies condition (ii b).

First, we consider the one-qubit states. For the incoherent states, we know that $s_x=s_y=0$ and the trace norm of coherence can be simplified as

$$\begin{aligned}C_{\rm tr}(\rho)&=\min_{\delta\in\mathcal{I}}|\mathbf{r}-\mathbf{s}|=\min_{\delta\in\mathcal{I}}\sqrt{r_x^2+r_y^2+(r_z-s_z)^2}\\&=\|\rho-\rho_{\rm diag}\|_{\rm tr}=\sqrt{r_x^2+r_y^2}.\end{aligned}\tag{19}$$

Note that $C_{\rm tr}(\rho)$ has the same form of expression as the l_1 norm of coherence $C_{l_1}(\rho)=\sum_{i,j,i\neq j}|\rho_{i,j}|$ for the one-qubit case. If we restrict the incoherent operators K_n to 2×2 matrices, then $C_{\rm tr}(\rho)$ satisfies condition (ii b). Here we simply conclude that the trace norm can act as a coherence measure for a qubit when the incoherent operators K_n are all of dimension 2.

For the one-qutrit quantum system, the eigenvalues of the qutrit density matrices have complex expressions. It seems difficult to estimate the optimal incoherent state. Fortunately, we find some special density matrices whose optimal incoherent states can be obtained.

Theorem. For the three classes of qutrit states

$$\rho_X=\begin{pmatrix}a_{11}&0&a_{13}\\0&a_{22}&0\\a_{13}^*&0&a_{33}\end{pmatrix}, \tag{20}$$

$$\rho_Y=\begin{pmatrix}a_{11}&a_{12}&0\\a_{12}^*&a_{22}&0\\0&0&a_{33}\end{pmatrix}, \tag{21}$$

and

$$\rho_Z=\begin{pmatrix}a_{11}&0&0\\0&a_{22}&a_{23}\\0&a_{23}^*&a_{33}\end{pmatrix}, \tag{22}$$

the optimal incoherent state of the trace norm of coherence is of the form $\rho_{\rm diag}$.

Proof. We only prove the case of state ρ_X; states ρ_Y and ρ_Z are completely analogous. Since all qutrit incoherent states have the form

$$\delta=\begin{pmatrix}x&0&0\\0&y&0\\0&0&z\end{pmatrix}, \tag{23}$$

then we can easily obtain the eigenvalues for $\rho_X-\delta$,

$$\begin{aligned}\lambda_1&=a_{22}-y,\\\lambda_2&=\frac{y-a_{22}}{2}-\frac{\sqrt{(2x+y-2a_{11}-a_{22})^2+4|a_{13}|^2}}{2},\\\lambda_3&=\frac{y-a_{22}}{2}+\frac{\sqrt{(2x+y-2a_{11}-a_{22})^2+4|a_{13}|^2}}{2}.\end{aligned}\tag{24}$$

We know that $\rho_X-\delta$ is a normal matrix; its singular values are the modulus of the eigenvalues for $\rho_X-\delta$. We then have

$$||\rho_X-\delta||_{\rm tr}=|\lambda_1|+|\lambda_2|+|\lambda_3|. \tag{25}$$

In order to minimize $||\rho_X-\delta||_{\rm tr}$ over all the incoherent states, we consider the following four cases.

Case 1. When $\frac{y-a_{22}}{2}\geqslant\frac{\sqrt{(2x+y-2a_{11}-a_{22})^2+4|a_{13}|^2}}{2}$ and $a_{22}\leqslant y$, we can simplify Eq. (25) as

$$\begin{aligned}||\rho_X-\delta||_{\rm tr}&=2y-2a_{22}\\&\geqslant 2\sqrt{(2x+y-2a_{11}-a_{22})^2+4|a_{13}|^2}\\&\geqslant 2\sqrt{|a_{13}|^2}\\&=||\rho_X-\rho_{\rm diag}||_{\rm tr}.\end{aligned}\tag{26}$$

Case 2. When $\frac{y-a_{22}}{2}\leqslant\frac{\sqrt{(2x+y-2a_{11}-a_{22})^2+4|a_{13}|^2}}{2}$ and $a_{22}\leqslant y$, similar to case 1, we have

$$\begin{aligned}||\rho_X-\delta||_{\rm tr}&=y-a_{22}+\sqrt{(2x+y-2a_{11}-a_{22})^2+4|a_{13}|^2}\\&\geqslant 2\sqrt{|a_{13}|^2}\\&=||\rho_X-\rho_{\rm diag}||_{\rm tr}.\end{aligned}\tag{27}$$

Case 3. When $\frac{y-a_{22}}{2} \leqslant \frac{\sqrt{(2x+y-2a_{11}-a_{22})^2+4|a_{13}|^2}}{2}$, $y \leqslant a_{22}$, and $\frac{y-a_{22}}{2} + \frac{\sqrt{(2x+y-2a_{11}-a_{22})^2+4|a_{13}|^2}}{2} \geqslant 0$, we have

$$\begin{aligned}||\rho_X - \delta||_{\text{tr}} &= a_{22} - y + \sqrt{(2x+y-2a_{11}-a_{22})^2 + 4|a_{13}|^2} \\ &\geqslant 2\sqrt{|a_{13}|^2} \\ &= ||\rho_X - \rho_{\text{diag}}||_{\text{tr}}. \end{aligned} \tag{28}$$

Case 4. When $\frac{y-a_{22}}{2} \leqslant \frac{\sqrt{(2x+y-2a_{11}-a_{22})^2+4|a_{13}|^2}}{2}$, $y \leqslant a_{22}$, and $\frac{y-a_{22}}{2} + \frac{\sqrt{(2x+y-2a_{11}-a_{22})^2+4|a_{13}|^2}}{2} \leqslant 0$, we have

$$\begin{aligned}||\rho_X - \delta||_{\text{tr}} &\geqslant 2\sqrt{(2x+y-2a_{11}-a_{22})^2 + 4|a_{13}|^2} \\ &\geqslant 2\sqrt{|a_{13}|^2} \\ &= ||\rho_X - \rho_{\text{diag}}||_{\text{tr}}. \end{aligned} \tag{29}$$

Through the above analysis, we obtain that the trace norm of coherence for ρ_X has the optimal incoherent state ρ_{diag}. ■

According to the above theorem, we can then obtain an analytical expression of the trace norm of coherence for ρ_X as

$$C_{\text{tr}}(\rho_X) = D_{\text{tr}}(\rho_X, \rho_{\text{diag}}) = 2|a_{13}|. \tag{30}$$

Note that $C_{\text{tr}}(\rho_X)$ also has the same form of expression as the l_1 norm of coherence $C_{l_1}(\rho) = \sum_{i,j,i\neq j} |\rho_{i,j}|$ for ρ_X.

Next we explain in detail that $C_{\text{tr}}(\rho_X)$ satisfies (ii b) when the Kraus operators are restricted to 3×3 matrices. Suppose that we have a 3×3 matrix K that has the form

$$K = \begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{pmatrix}.$$

If K can act as an incoherent operator, it means that $K\mathcal{I}K^\dagger \subset \mathcal{I}$. Here we set

$$\delta = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix} \in \mathcal{I},$$

where $x + y + z = 1$ and $x, y, z \in \mathbb{R}^+$. Then we have

$$K\mathcal{I}K^\dagger = \begin{pmatrix} |k_{11}|^2x + |k_{12}|^2y + |k_{13}|^2z & k_{11}k_{21}^*x + k_{12}k_{22}^*y + k_{13}k_{23}^*z & k_{11}k_{31}^*x + k_{12}k_{32}^*y + k_{13}k_{33}^*z \\ k_{11}^*k_{21}x + k_{12}^*k_{22}y + k_{13}^*k_{23}z & |k_{21}|^2x + |k_{22}|^2y + |k_{23}|^2z & k_{21}k_{31}^*x + k_{22}k_{32}^*y + k_{23}k_{33}^*z \\ k_{11}^*k_{31}x + k_{12}^*k_{32}y + k_{13}^*k_{33}z & k_{21}^*k_{31}x + k_{22}^*k_{32}y + k_{23}^*k_{33}z & |k_{31}|^2x + |k_{32}|^2y + |k_{33}|^2z \end{pmatrix}. \tag{31}$$

If $K\mathcal{I}K^\dagger \subset \mathcal{I}$, all of the off-diagonal elements must be equal to zero. Thus, we have

$$k_{11}k_{21}^*x + k_{12}k_{22}^*y + k_{13}k_{23}^*z = 0, \tag{32}$$

$$k_{11}k_{31}^*x + k_{12}k_{32}^*y + k_{13}k_{33}^*z = 0, \tag{33}$$

$$k_{21}k_{31}^*x + k_{22}k_{32}^*y + k_{23}k_{33}^*z = 0. \tag{34}$$

In order to obtain the form of K that satisfies $K\mathcal{I}K^\dagger \subset \mathcal{I}$, we choose some parameters x, y, and z for Eqs. (32)–(34). Those equations can be considered in the form

$$\begin{aligned} k_{11}k_{21}^* = 0, \quad k_{12}k_{22}^* = 0, \quad k_{13}k_{23}^* = 0, \\ k_{11}k_{31}^* = 0, \quad k_{12}k_{32}^* = 0, \quad k_{13}k_{33}^* = 0, \\ k_{21}k_{31}^* = 0, \quad k_{22}k_{32}^* = 0, \quad k_{23}k_{33}^* = 0. \end{aligned} \tag{35}$$

After some simple algebraic operation, we obtain 27 solutions of Eq. (35) (see the Appendix for the explicit form of those matrices).

We note that the elements k_{ij} of those 27 matrices can be equal to zero; then it is straightforward to verify that those 27 matrices can map ρ_X (as well as ρ_Y and ρ_Z) to the form of ρ_X, ρ_Y, or ρ_Z. Here we do not consider the completely positive property for those matrices, so all of the incoherent operators of dimension 3 are the subset of those 27 matrices. We conclude that trace norm can act as a coherence measure for those special qutrits when the incoherent operators K_n are all of dimension 3.

Interestingly, if we restrict the incoherent operators to 3×2 matrices of the form

$$K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \\ k_{31} & k_{32} \end{pmatrix}, \tag{36}$$

by similar calculations we find that there are nine matrices that satisfy $K\mathcal{I}K^\dagger \subset \mathcal{I}$, which are listed as follows:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ k_{31} & k_{32} \end{pmatrix}, \quad \begin{pmatrix} k_{11} & 0 \\ 0 & 0 \\ 0 & k_{32} \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ k_{21} & 0 \\ 0 & k_{32} \end{pmatrix},$$

$$\begin{pmatrix} 0 & k_{12} \\ 0 & 0 \\ k_{31} & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & k_{22} \\ k_{31} & 0 \end{pmatrix}, \quad \begin{pmatrix} k_{11} & k_{12} \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} k_{11} & 0 \\ 0 & k_{22} \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & k_{12} \\ k_{21} & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ k_{12} & k_{22} \\ 0 & 0 \end{pmatrix}.$$

We can also verify that those matrices can map the qubit to the form of ρ_X, ρ_Y, or ρ_Z. So we can also conclude that the trace norm can act as a coherence measure in this case.

Recently, Bromley *et al*. obtain that the trace norm of coherence C_{tr} coincides with the l_1 norm of coherence C_{l_1} for two-qubit Bell-diagonal states. It is shown that the equivalence between C_{tr} and C_{l_1} cannot be extended to general two-qubit states [17].

IV. CONCLUSION

In this paper we have shown that the fidelity of coherence does not satisfy condition (ii b) by presenting an example. We conclude that the measure of coherence induced by fidelity in general is not a good measure for quantifying coherence. For the trace norm of coherence, we have shown that the qubit states and some special qutrit states can satisfy condition (ii b) with some restrictions on the incoherent operators. Our results show that the trace norm of coherence is equivalent to the l_1 norm of coherence for qubits and special qutrits. Whether the coherence measure induced by the trace norm can be applied for general quantum states needs further exploration. Our findings complement the results of coherence quantification in Ref. [1].

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APPENDIX

Following Eq. (35), we find that the form of matrix K may take the following 27 forms:

$$\begin{pmatrix}0&0&0\\0&0&0\\k_{31}&k_{32}&k_{33}\end{pmatrix},\quad \begin{pmatrix}k_{11}&0&0\\0&0&0\\0&k_{32}&k_{33}\end{pmatrix},\quad \begin{pmatrix}0&0&0\\k_{21}&0&0\\0&k_{32}&k_{33}\end{pmatrix},$$

$$\begin{pmatrix}0&k_{12}&0\\0&0&0\\k_{31}&0&k_{33}\end{pmatrix},\quad \begin{pmatrix}0&0&0\\0&k_{22}&0\\k_{31}&0&k_{33}\end{pmatrix},\quad \begin{pmatrix}0&0&k_{13}\\0&0&0\\k_{31}&k_{32}&0\end{pmatrix},$$

$$\begin{pmatrix}0&0&0\\0&0&k_{23}\\k_{31}&k_{32}&0\end{pmatrix},\quad \begin{pmatrix}k_{11}&k_{12}&0\\0&0&0\\0&0&k_{33}\end{pmatrix},\quad \begin{pmatrix}k_{11}&0&0\\0&k_{22}&0\\0&0&k_{33}\end{pmatrix},$$

$$\begin{pmatrix}0&k_{12}&0\\k_{21}&0&0\\0&0&k_{33}\end{pmatrix},\quad \begin{pmatrix}0&0&0\\k_{21}&k_{22}&0\\0&0&k_{33}\end{pmatrix},\quad \begin{pmatrix}k_{11}&0&k_{13}\\0&0&0\\0&k_{32}&0\end{pmatrix},$$

$$\begin{pmatrix}k_{11}&0&0\\0&0&k_{23}\\0&k_{32}&0\end{pmatrix},\quad \begin{pmatrix}0&0&k_{13}\\k_{21}&0&0\\0&k_{32}&0\end{pmatrix},\quad \begin{pmatrix}0&0&0\\k_{21}&0&k_{23}\\0&k_{32}&0\end{pmatrix},$$

$$\begin{pmatrix}0&k_{12}&k_{13}\\0&0&0\\k_{31}&0&0\end{pmatrix},\quad \begin{pmatrix}0&k_{12}&0\\0&0&k_{23}\\k_{31}&0&0\end{pmatrix},\quad \begin{pmatrix}0&0&k_{13}\\0&k_{22}&0\\k_{31}&0&0\end{pmatrix},$$

$$\begin{pmatrix}0&0&0\\0&k_{22}&k_{23}\\k_{31}&0&0\end{pmatrix},\quad \begin{pmatrix}k_{11}&k_{12}&k_{13}\\0&0&0\\0&0&0\end{pmatrix},\quad \begin{pmatrix}k_{11}&k_{12}&0\\0&0&k_{23}\\0&0&0\end{pmatrix},$$

$$\begin{pmatrix}k_{11}&0&k_{13}\\0&k_{22}&0\\0&0&0\end{pmatrix},\quad \begin{pmatrix}k_{11}&0&0\\0&k_{22}&k_{23}\\0&0&0\end{pmatrix},\quad \begin{pmatrix}0&k_{12}&k_{13}\\k_{21}&0&0\\0&0&0\end{pmatrix},$$

$$\begin{pmatrix}0&k_{12}&0\\k_{21}&0&k_{23}\\0&0&0\end{pmatrix},\quad \begin{pmatrix}0&0&k_{13}\\k_{21}&k_{22}&0\\0&0&0\end{pmatrix},\quad \begin{pmatrix}0&0&0\\k_{21}&k_{22}&k_{23}\\0&0&0\end{pmatrix}.$$

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