Quantum non-Markovianity based on the Fisher-information matrix

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With the development of quantum-information theory, there has been a flurry of investigations of quantum non-Markovian dynamics, and several significant measures for such dynamics have been proposed from various perspectives, such as the breakdown of dynamical divisibility, increase in the distinguishability between quantum states, increase in correlations between the system and an arbitrary ancillary, and so on. Motivated by the idea of exploiting the information content of parameters encoded in initial states, we propose a conceptually simple and physically intuitive characterization for non-Markovianity with the help of a quantum-Fisher-information matrix. The basic features are illustrated through several examples, and relations with other approaches are elucidated. A hierarchial aspect of quantum non-Markovianity is revealed.

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I. INTRODUCTION

Quantum non-Markovian evolutions are responsible for a wide variety of physically interesting effects, and have attracted more and more attention in both theory and practice in recent years [1–31]. Although classical Markovianity and non-Markovianity are well defined and widely studied in stochastic processes and random dynamics [32], their quantum extensions, i.e., quantum Markovianity and quantum non-Markovianity, remain elusive and subtle. A number of quantitative measures for non-Markovianity have been proposed based on different considerations, such as the derivation from divisibility [2,4,10], entanglement with the environment [4], states distinguishability [3,7,12], Fisher information [8], correlations with an ancilla [15], channel capacities [25], accessible information [26], local quantum uncertainty [28], and quantum interferometric power [30]. Each of the above characterizations captures a certain aspect of quantum non-Markovianity and exhibits some unique feature, and they do not coincide in general [11,19,29]. A universal characterization for quantum non-Markovianity is still lacking and might not exist.

In this work, we employ quantum Fisher information to propose an alternative characterization of quantum non-Markovianity for qubit systems, and indicate that the method can also be readily applied to treat higher dimensional situations. To gain an intuitive motivation, let us first recall the notion of quantum Fisher information, which is a central concept in quantum detection, estimation, and metrology [33–51]. The quantum Fisher information effectively characterizes the statistical distinguishability about parameters encoded in quantum states, and delimits the precision of parameter estimation in quantum scenarios through the celebrated Cramér-Rao inequality [35,36]. It also has intrinsic relation with the Bures distance in quantum state space [37,38]. Due to the subtle noncommutative nature of quantum theory, there are many different and useful versions of quantum Fisher information [33–42]. Here we will adopt the one based on the symmetric logarithmic derivative [35,36,38], which is the most important and significant version. This version of quantum Fisher information coincides with the maximum of the measurement-induced classical Fisher information [38]. Based on quantum Fisher information, we propose a simple and intuitive measure for quantum non-Markovianity. The main idea is to exploit the variation of the Fisher information of parameters encoded in quantum states in the course of evolution, and to formulate a witness of quantum non-Markovianity in terms of violation of the monotonicity of quantum Fisher information under conventional quantum Markovian dynamics. Thus we call a dynamics FI-Markovian if it always reduces the Fisher information of parameters encoded in quantum states, and FI-non-Markovian in the case of violation of this decreasing property. These will be made more precise in mathematical terms shortly.

It is worth mentioning that the approach to quantum non-Markovianity proposed here is quite different from the study in Ref. [8], although both involve the notion of quantum Fisher information. In Ref. [8], the quantum-Fisher-information flow is decomposed into the contributions from different dissipative channels for a class of non-Markovian master equations in time-local forms. Our main results consist in the introduction of an alternative notion of quantum Markovianity or non-Markovianity and making comparison with existing ones. We will show that while they coincide for many instances, they exhibit subtle differences in general cases, which are illustrated through several examples.

The remainder of the paper is organized as follows. In Sec. II, we introduce a formulation for quantum non-Markovianity from the perspective of quantum-Fisher-information matrix. In Sec. III, several examples are analyzed in detail in order to show the physical significance and relations of the proposed approach with other ones. We conclude with a summary in Sec. IV.

II. NON-MARKOVIANITY FROM FISHER-INFORMATION PERSPECTIVE

Consider a two-dimensional quantum system evolving under the dynamics $\Lambda = \{\Lambda_t : t \geqslant 0\}$ (a family of quantum

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operations, also called quantum channels, i.e., completely positive, linear, and trace-preserving maps on quantum state space). Without loss of generality, we assume that the initial system state has the typical form $\rho(0) = |\Psi\rangle\langle\Psi|$ with

$$|\Psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle, \tag{1}$$

where $\theta \in [0,\pi)$ and $\phi \in [0,2\pi)$ are parameters which may be regarded as encoding the amplitude and phase information, respectively, $|0\rangle$ and $|1\rangle$ are the eigenvectors of the Pauli spin matrix σ_3. The state of the quantum system evolves as $\rho(t) = \Lambda_t(\rho(0))$ for $t \geqslant 0$. We have suppressed the dependence on parameters θ and ϕ for notational simplicity.

To quantify the information content of all parameters contained in the state $\rho(t)$, we employ the quantum-Fisher-information matrix

$$F(t) = \begin{pmatrix} F_\theta(t) & F_{\theta\phi}(t) \\ F_{\phi\theta}(t) & F_\phi(t) \end{pmatrix}$$

with $F_\theta(t) = \mathrm{tr}\rho(t)L_\theta^2$, $F_\phi(t) = \mathrm{tr}\rho(t)L_\phi^2$, and

$$F_{\theta\phi}(t) = F_{\phi\theta}(t) = \tfrac{1}{2}\mathrm{tr}\rho(t)(L_\theta L_\phi + L_\phi L_\theta),$$

where L_θ and L_ϕ are the symmetric logarithmic derivatives for the parameters θ and ϕ defined by

$$\frac{\partial}{\partial\theta}\rho(t) = \frac{1}{2}[\rho(t)L_\theta + L_\theta\rho(t)],$$
$$\frac{\partial}{\partial\phi}\rho(t) = \frac{1}{2}[\rho(t)L_\phi + L_\phi\rho(t)],$$

respectively. Since $L_\theta^\dagger = L_\theta$ and $L_\phi^\dagger = L_\phi$, the quantum-Fisher-information matrix $F(t)$ is Hermitian. The essential feature of this matrix is that it sets a lower bound to the mean-square error of any unbiased estimator for the parameters through the Cramér-Rao inequality [35,36].

To focus on the dynamics (i.e., temporal evolution with time parameter t), we get off the dependence of the Fisher-information matrix on particular values of parameters θ and ϕ in the initial state by averaging, that is, we perform integration with respect to the uniform distribution $d\Omega = \frac{1}{4\pi}\sin\theta\, d\theta\, d\phi$ of the parameters $\theta \in [0,\pi)$, $\phi \in [0,2\pi)$ on the unit Bloch sphere, and thus come to the following averaged Fisher-information matrix

$$\bar{F}(t) = \begin{pmatrix} \bar{F}_\theta(t) & \bar{F}_{\theta\phi}(t) \\ \bar{F}_{\phi\theta}(t) & \bar{F}_\phi(t) \end{pmatrix}$$

with $\bar{F}_\theta(t) = \int F_\theta(t)\, d\Omega$, $\bar{F}_\phi(t) = \int F_\phi(t)\, d\Omega$, and $\bar{F}_{\theta\phi}(t) = \bar{F}_{\phi\theta}(t) = \int F_{\theta\phi}(t)\, d\Omega$. The integrations are over $\theta \in [0,\pi)$, $\phi \in [0,2\pi)$. The averaged Fisher-information matrix $\bar{F}(t)$ now depends explicitly only on the time t and the quantum dynamics $\Lambda = \{\Lambda_t : t \geqslant 0\}$, and can be served as an indicator to characterize the dynamics from the information perspective.

Motivated by the intuitive and plausible idea that Markovian dynamics usually leads to loss of information, and thus should never increase the information content of the parameters encoded in the states, it is desirable to define a quantum dynamics to be Markovian in the sense of decreasing quantum-Fisher-information (abbreviated as FI-Markovian) if the averaged quantum-Fisher-information matrix $\bar{F}(t)$ monotonously decreases with time $t \geqslant 0$. In other words, if the derivative matrix $\frac{d}{dt}\bar{F}(t)$ is always nonpositive definite, i.e., the eigenvalues of the Hermitian matrix $\frac{d}{dt}\bar{F}(t)$, denoted as $\lambda_1(t)$ and $\lambda_2(t)$, are nonpositive, then we say that $\Lambda = \{\Lambda_t : t \geqslant 0\}$ exhibits FI-Markovianity. Any violation of this monotonicity is an indication for FI-non-Markovianity.

To summarize, a dynamics $\Lambda = \{\Lambda_t : t \geqslant 0\}$ for qubit systems is defined to be FI-Markovian if $\frac{d}{dt}\bar{F}(t) \leqslant 0$ for all $t \geqslant 0$. Otherwise, it is defined to be FI-non-Markovian. In this context, a quantitative measure for FI-non-Markovianity may be defined in terms of $\lambda(t) = \max\{\lambda_1(t),\lambda_2(t)\}$ as

$$\mathcal{N}_{\mathrm{FI}}(\Lambda) = \int_{\lambda(t)>0} \lambda(t)\, dt.$$

Here we remark that one may, according to the context, define other measures in a similar spirit.

The question immediately arises as to what are the relationships between the present approach to quantum non-Markovianity and other ones, e.g., the Breuer-Laine-Piilo (BLP) characterization and the Luo-Fu-Song (LFS) characterization of quantum non-Markovianity [3,15]. We will illustrate their similarities and differences in the next section. We recall that the BLP characterization is proposed by Breuer, Laine and Piilo in terms of distinguishability (i.e., trace distance) between states which is monotonically decreasing under conventional Markovian dynamics [3], while the LFS characterization is from the point of view that the total correlations (i.e., quantum mutual information) between the system and an arbitrary ancillary always decrease under conventional Markovian dynamics [15]. A violation of the monotonicity is regarded as an indication for the corresponding characterization of quantum non-Markovianity. The corresponding measures for non-Markovianity, denoted as $\mathcal{N}_{\mathrm{BLP}}(\Lambda)$ and $\mathcal{N}_{\mathrm{LFS}}(\Lambda)$ for convenience, are defined, respectively, as

$$\mathcal{N}_{\mathrm{BLP}}(\Lambda) = \sup_{\rho,\tau} \int_{\frac{d}{dt}\mathrm{tr}|\Lambda_t(\rho-\tau)|>0} \frac{1}{2}\frac{d}{dt}\mathrm{tr}|\Lambda_t(\rho-\tau)|\, dt,$$

with optimization being over all pairs of initial states of the system, and

$$\mathcal{N}_{\mathrm{LFS}}(\Lambda) = \sup_{\rho^{sa}} \int_{\frac{d}{dt}I(\rho_t^{sa})>0} \frac{d}{dt}I\left(\rho_t^{sa}\right) dt,$$

where $\rho_t^{sa} = (\Lambda_t \otimes \mathbf{1})\rho^{sa}$, and the sup is over all bipartite states ρ^{sa} on $H \otimes H^a$, with H^a an arbitrary ancillary space and $\mathbf{1}$ the identity operator on it. To simplify the optimization involved in the last expression, a practical substitute of LFS characterization, denoted as LFS_0, is proposed as [15]

$$\mathcal{N}_{\mathrm{LFS}_0}(\Lambda) = \int_{\frac{d}{dt}I(\rho_t^{sa})>0} \frac{d}{dt}I\left(\rho_t^{sa}\right) dt,$$

where $\rho_t^{sa} = (\Lambda_t \otimes \mathbf{1})|\Phi\rangle\langle\Phi|$, and $|\Phi\rangle$ is an arbitrary maximally entangled state.

III. COMPARISON

Let us consider several typical examples and compare the Fisher-information characterization of quantum non-Markovianity with the above two well-established approaches, i.e., the BLP characterization [3] and the LFS_0 characterization [15].

Example 1. Consider the qubit quantum dynamics $\Lambda = \{\Lambda_t : t \geqslant 0\}$ with the evolving state $\rho(t) = \Lambda_t(\rho(0))$ described by the differential equation

$$\frac{d}{dt}\rho(t) = \gamma(t)[\sigma_3\rho(t)\sigma_3 - \rho(t)],$$

with $\int_0^t \gamma(s)\,ds \geqslant 0$ and σ_3 the third Pauli spin matrix.

Notice that the initial state of the system $\rho(0) = |\Psi\rangle\langle\Psi|$ is parametrized in Eq. (1), thus the evolving state can be expressed as

$$\rho(t) = \Lambda_t(\rho(0)) = \frac{1}{2}\begin{pmatrix} 1+\cos\theta & e^{-i\phi}f(t)\sin\theta \\ e^{i\phi}f(t)\sin\theta & 1-\cos\theta \end{pmatrix}$$

with $f(t) = e^{-2\int_0^t \gamma(s)\,ds}$. The quantum-Fisher-information matrix can be readily evaluated as

$$F(t) = \begin{pmatrix} 1 & 0 \\ 0 & f^2(t)\sin^2\theta \end{pmatrix},$$

and the averaged Fisher-information matrix is

$$\bar{F}(t) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{2}{3}f^2(t) \end{pmatrix}.$$

The derivative of the above matrix is

$$\frac{d}{dt}\bar{F}(t) = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{8}{3}\gamma(t)f^2(t) \end{pmatrix}.$$

According to the approach introduced in Sec. II, $\Lambda = \{\Lambda_t : t \geqslant 0\}$ is FI-Markovian if and only if $\gamma(t) \geqslant 0$ for all $t \geqslant 0$. Once $\gamma(t) < 0$ for some time t, the quantum dynamics exhibits FI-non-Markovianity. This result is in accordance with the BLP characterization and the LFS_0 characterization in view of the results in Ref. [15], that is, in this instance, the three approaches to quantum Markovianity coincide: This quantum dynamics is FI-Markovian if and only if it is BLP-Markovian, and which in turn is equivalent to LFS_0-Markovian.

The measure for FI-non-Markovianity in this case is

$$\mathcal{N}_{\rm FI}(\Lambda) = -\frac{8}{3}\int_{\gamma(t)<0} \gamma(t)f^2(t)\,dt.$$

Example 2. Consider the qubit quantum dynamics $\Lambda = \{\Lambda_t : t \geqslant 0\}$ with the evolving state given by the following master equation:

$$\frac{d}{dt}\rho(t) = -\frac{i}{2}s(t)[\sigma_+\sigma_-,\rho(t)] + \gamma(t)\left(\sigma_-\rho(t)\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-,\rho(t)\}\right),$$

where $s(t) = -2{\rm Im}\frac{\dot{G}(t)}{G(t)}$, $\gamma(t) = -2{\rm Re}\frac{\dot{G}(t)}{G(t)}$, and $G(t)$ is determined by

$$\dot{G}(t) = \frac{d}{dt}G(t) = -\int_0^t ds f(t-s)G(s), \quad G(0) = 1,$$

with $f(\tau) = \sum_k |g_k|^2 e^{i(\omega_0-\omega_k)\tau}$ a two-point correlation function.

The evolution of the density matrix $\rho(0) = |\Psi\rangle\langle\Psi|$ as in Eq. (1) reads

$$\rho(t) = \Lambda_t(\rho(0)) = \frac{1}{2}\begin{pmatrix} 2-|G(t)|^2(1-\cos\theta) & G^*(t)e^{-i\phi}\sin\theta \\ G(t)e^{i\phi}\sin\theta & |G(t)|^2(1-\cos\theta) \end{pmatrix}.$$

The quantum-Fisher-information matrix can be evaluated as

$$F(t) = \begin{pmatrix} |G(t)|^2 & 0 \\ 0 & |G(t)|^2\sin^2\theta \end{pmatrix},$$

and the corresponding averaged Fisher-information matrix is given by

$$\bar{F}(t) = \begin{pmatrix} |G(t)|^2 & 0 \\ 0 & \frac{2}{3}|G(t)|^2 \end{pmatrix}.$$

The derivative of the above matrix is

$$\frac{d}{dt}\bar{F}(t) = \frac{d}{dt}|G(t)|\begin{pmatrix} 2|G(t)| & 0 \\ 0 & \frac{4}{3}|G(t)| \end{pmatrix}.$$

It is clear that $\frac{d}{dt}\bar{F}(t) \leqslant 0$ is equivalent to $\frac{d}{dt}|G(t)| \leqslant 0$. Once there exists some time $t > 0$ such that $\frac{d}{dt}|G(t)| > 0$, the dynamics is FI-non-Markovian. Now combining the results in Ref. [15], we conclude that the three kinds of quantum non-Markovianity (i.e., FI-non-Markovianity, BLP-non-Markovianity, and LFS_0-non-Markovianity) all coincide.

The measure of FI-non-Markovianity is

$$\mathcal{N}_{\rm FI}(\Lambda) = 2\int_{\frac{d}{dt}|G(t)|>0} |G(t)|\frac{d}{dt}|G(t)|\,dt.$$

Example 3. Consider the dynamics $\Lambda = \{\Lambda_t : t \geqslant 0\}$ of a qubit system with

$$\rho(t) = \Lambda_t(\rho(0)) = \sum_{i=0}^{3} p_i(t)\sigma_i\rho(0)\sigma_i, \quad t \geqslant 0,$$

where $p_i(t) \geqslant 0$, $\sum_{i=0}^3 p_i(t) = 1$, $\sigma_0 = \mathbf{1}$ (identity matrix), and σ_i, $i = 1,2,3$, are the Pauli spin matrices. This dynamics is a random unitary dynamics (Pauli channel) and has been exploited in Ref. [19] to demonstrate the nonequivalence between LFS_0-Markovianity and BLP-Markovianity.

We are restricted to the case $p_i(t) = \alpha_i[1 - p_0(t)]$, $i = 1,2,3$, and $\alpha_1 = \alpha_2 = \alpha \in [0,1/2]$, then $\alpha_3 = 1 - 2\alpha \in [0,1]$. Put $H(\alpha) = -2\alpha\ln\alpha - (1-2\alpha)\ln(1-2\alpha)$ (natural logarithm) and $q(t) = 1 - p_0(t)$.

Our main comparison results may be summarized as follows: The quantum dynamics $\Lambda = \{\Lambda_t : t \geqslant 0\}$ is

(1) BLP-Markovian if and only if $\dot{q}(t) \geqslant 0$ and

$$q(t) \leqslant {\rm BLP}(\alpha) := \min\left\{\frac{1}{2(1-\alpha)}, \frac{1}{4\alpha}\right\}; \quad (2)$$

(2) FI-Markovian if and only if $\dot{q}(t) \geqslant 0$ and

$$q(t) \leqslant {\rm FI}(\alpha) := \min\left\{\frac{1}{2(1-\alpha)}, h(\alpha)\right\}, \quad (3)$$

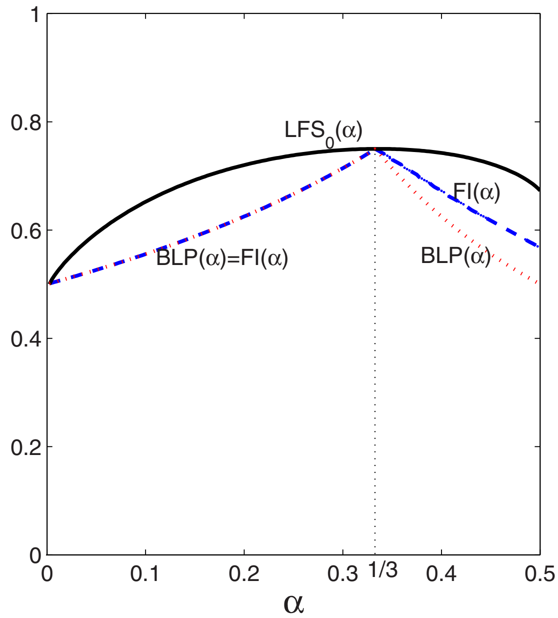

FIG. 1. (Color online) Graphs of BLP(α), FI(α), and LFS$_0$(α). Note that BLP(α) = FI(α) < LFS$_0$(α) for $\alpha \in (0,1/3)$, while BLP(α) < FI(α) < LFS$_0$(α) for $\alpha \in (1/3,1/2]$, and these three curves coincide when $\alpha = 0$ and $\alpha = 1/3$. All quantities are dimensionless.

where $h(\alpha)$ is implicitly given in the Appendix with the graph depicted in Fig. 2.

(3) LFS$_0$-Markovian if and only if $\dot{q}(t) \geqslant 0$ and

$$q(t) \leqslant \mathrm{LFS}_0(\alpha) := \frac{1}{1+e^{-H(\alpha)}}, \tag{4}$$

The graphs of the functions BLP(α), FI(α), and LFS$_0$(α) are depicted in Fig. 1 by dotted, dashed, and solid lines, respectively.

Before proceeding to the derivation of the above results, let us gain a more intuitive illustration and understanding of the comparison among the three kinds of quantum Markovianity and non-Markovianity. From Fig. 1, we have the following observations:

(1) When $\alpha \in [0,1/3]$, BLP(α) is equal to FI(α), and both are smaller than LFS$_0$(α). That is to say, in this interval, BLP-Markovianity coincides with FI-Markovianity, but differs from (and actually implies) LFS$_0$-Markovianity.

(2) When $\alpha \in (1/3,1/2]$, the three curves are different from each other, from which it follows the differences among the three characterizations of quantum Markovianity. Since BLP(α) < FI(α) < LFS$_0$(α), we have in this instance the following hierarchial relations: BLP-Markovianity implies FI-Markovianity, which in turn implies LFS$_0$-Markovianity, but the converse is not true. Put it alternatively, when $\alpha \in (1/3,1/2]$, LFS$_0$-non-Markovianity implies FI-non-Markovianity, which in turn implies BLP-non-Markovianity.

To summarize, in this example, BLP-Markovian is the most restrictive, and LFS$_0$-Markovian is the most general, while FI-Markovian lies in between. To illustrate more concretely the hierarchial structure of the three kinds of Markovianity, we consider the specific values of $\alpha = 1/4,\ 1/3,\ 2/5$, then

TABLE I. Comparisons between BLP-Markovianity, FI-Markovianity, and LFS$_0$-Markovianity [noting that all require $\dot{q}(t) \leqslant 0$].

Markovianity	$\alpha = 1/4$	$\alpha = 1/3$	$\alpha = 2/5$
BLP	$q(t) \leqslant 2/3$	$q(t) \leqslant 3/4$	$q(t) \leqslant 0.625$
FI	$q(t) \leqslant 2/3$	$q(t) \leqslant 3/4$	$q(t) \leqslant 0.672$
LFS$_0$	$q(t) \leqslant 0.739$	$q(t) \leqslant 3/4$	$q(t) \leqslant 0.742$

$$\mathrm{BLP}\left(\tfrac{1}{4}\right) = \mathrm{FI}\left(\tfrac{1}{4}\right) = \tfrac{2}{3} < \mathrm{LFS}_0\left(\tfrac{1}{4}\right) \approx 0.739,$$
$$\mathrm{BLP}\left(\tfrac{1}{3}\right) = \mathrm{FI}\left(\tfrac{1}{3}\right) = \mathrm{LFS}_0\left(\tfrac{1}{3}\right) = \tfrac{3}{4},$$
$$\mathrm{BLP}\left(\tfrac{2}{5}\right) = 0.625 < \mathrm{FI}\left(\tfrac{2}{5}\right) \approx 0.672 < \mathrm{LFS}_0\left(\tfrac{2}{5}\right) \approx 0.742.$$

The comparison is listed in Table I.

We have illustrated the difference and a hierarchical aspect involving our FI approach compared with two existing approaches to quantum non-Markovianity through specific examples. For other general dynamics, the detailed hierarchical structure may change, and the point is that these three kinds of non-Markovianity are different. It will be desirable to further investigate their relationships.

IV. CONCLUSION

We have employed quantum Fisher information to characterize quantum Markovianity and non-Markovianity from an information perspective. More precisely, we have introduced the notion of FI-Markovianity for quantum dynamics by exploiting the temporal decreasing property of quantum Fisher information of the parameters encoded in initial states. The physical significance, similarities, and differences of this approach with exciting ones, namely, the BLP and LFS$_0$ approaches, are illustrated and compared by virtue of several examples. Although these approaches coincide and thus yield the same characterization of quantum Markovianity in many special cases, they are different in general. For some random unitary dynamics, we have revealed the hierarchial relationships between them, and further illustrated that the FI-Markovianity stands in between the BLP-Markovianity and the LFS$_0$-Markovianity. The hierarchial structure may change for other general dynamics.

Quantum Fisher information is a fundamental quantity in quantum metrology with deep significance and wide applications. Quantum non-Markovianity may be an important resource for quantum tasks. Our Fisher-information approach to quantum non-Markovianity may shed alternative light on the engineering of quantum non-Markovianity for information processing.

We may consider more general parametrizations other than the canonical one in Eq. (1), but as long as the Jacobian between different parametrizations is nonsingular, the analysis and results will be similar. Finally we remark that although we have focused on the dynamics of qubit systems, it is plain that the method can be extended to higher dimensional cases, though the computation will be much more complicated.

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APPENDIX

Here we derive the criteria, inequalities (2), (3), and (4), for BLP-Markovianity, FI-Markovianity, and LFS_0-Markovianity, respectively. Inequalities (2) and (4) follow from Ref. [19] by straightforward manipulation of the expressions. We only need to establish inequality (3). We now investigate the condition for FI-Markovianity of the dynamics $\Lambda = \{\Lambda_t : t \geqslant 0\}$. The initial state $\rho(0) = |\Psi\rangle\langle\Psi|$ parametrized as in Eq. (1) will evolve to

$$\rho(t) = \Lambda_t(\rho(0)) = \frac{1}{2}\begin{pmatrix} 1 + r(t)\cos\theta & s(t)e^{-i\phi}\sin\theta \\ s(t)e^{i\phi}\sin\theta & 1 - r(t)\cos\theta \end{pmatrix},$$

with $r(t) = 1 - 4\alpha q(t)$, $s(t) = 1 - 2(1-\alpha)q(t)$. The quantum-Fisher-information matrix can be evaluated as

$$F(t) = \begin{pmatrix} F_\theta(t) & 0 \\ 0 & F_\phi(t) \end{pmatrix},$$

with

$$F_\theta(t) = 1 - \frac{u(t)}{1 + k(t)\cos^2\theta},$$
$$F_\phi(t) = s^2(t)\sin^2\theta,$$

where $u(t) = 8\alpha q(t)[1 - 2\alpha q(t)]$, and

$$k(t) = \frac{(3\alpha - 1)[1 - (1+\alpha)q(t)]}{(1-\alpha)[1 - (1-\alpha)q(t)]}.$$

The averaged quantum-Fisher-information matrix

$$\bar{F}(t) = \begin{pmatrix} \bar{F}_\theta(t) & 0 \\ 0 & \bar{F}_\phi(t) \end{pmatrix}$$

can be evaluated as

$$\bar{F}_\phi(t) = \frac{1}{2}\int_0^\pi s^2(t)\sin^3\theta\, d\theta = \frac{2}{3}s^2(t),$$

and

$$\bar{F}_\theta(t) = 1 - \frac{1}{2}\int_0^\pi \frac{u(t)}{1 + k(t)\cos^2\theta}\sin\theta\, d\theta, = \begin{cases} 1 - u(t), & k(t) = 0, \\ 1 + \frac{u(t)}{2}\int_0^\pi \frac{d\cos\theta}{1 + k(t)\cos^2\theta}, & k(t) \neq 0. \end{cases}$$

Furthermore, for the case $k(t) \neq 0$, using the following integral formula ($a > 0$)

$$\int \frac{dx}{ax^2 + b} = \begin{cases} \frac{1}{\sqrt{ab}}\arctan\sqrt{\frac{a}{b}}x + C, & b > 0, \\ \frac{1}{2\sqrt{-ab}}\ln\left|\frac{\sqrt{a}x - \sqrt{-b}}{\sqrt{a}x + \sqrt{-b}}\right| + C, & b < 0, \end{cases}$$

where C is an arbitrary constant, we get

$$\bar{F}_\theta(t) = \begin{cases} 1 - \frac{u(t)}{\sqrt{k(t)}}\arctan\sqrt{k(t)}, & k(t) > 0, \\ 1 - \frac{u(t)}{2\sqrt{-k(t)}}\ln\left|\frac{1 + \sqrt{-k(t)}}{-1 + \sqrt{-k(t)}}\right|, & k(t) < 0. \end{cases}$$

If we put $f(\alpha,q) = \frac{\partial \bar{F}_\theta}{\partial q}$, then from the derivative

$$\frac{d}{dt}\bar{F}_\theta(t) = f(\alpha,q)\dot{q}(t),$$

we conclude that $\frac{d}{dt}\bar{F}_\theta(t) \leqslant 0$ is equivalent to $\dot{q}(t) \geqslant 0$, $f(\alpha,q) \leqslant 0$, or $\dot{q}(t) \leqslant 0$, $f(\alpha,q) \geqslant 0$. The expression of $f(\alpha,q)$ can be derived as follows. If $k(t) > 0$,

$$f(\alpha,q) = -f_1(t)\frac{\arctan\sqrt{k(t)}}{\sqrt{k(t)}} + f_2(t)u(t)\left(\frac{\arctan\sqrt{k(t)}}{k(t)} - \frac{1}{\sqrt{k(t)}[1 + k(t)]}\right),$$

and if $k(t) < 0$,

$$f(\alpha,q) = \frac{-f_1(t)}{2\sqrt{-k(t)}}\ln|K(t)| - f_3(t)u(t)\left(\frac{\ln|K(t)|}{2k(t)} + \frac{1}{\sqrt{-k(t)}[1 + k(t)]}\right),$$

with

$$f_1(t) = 8\alpha - 32\alpha^2 q(t),$$
$$f_2(t) = \frac{\alpha(1 - 3\alpha)}{\sqrt{k(t)}(1-\alpha)[1 - (1-\alpha)q(t)]^2},$$
$$f_3(t) = \frac{\alpha(3\alpha - 1)}{\sqrt{-k(t)}(1-\alpha)[1 - (1-\alpha)q(t)]^2},$$
$$K(t) = \frac{1 + \sqrt{-k(t)}}{-1 + \sqrt{-k(t)}}.$$

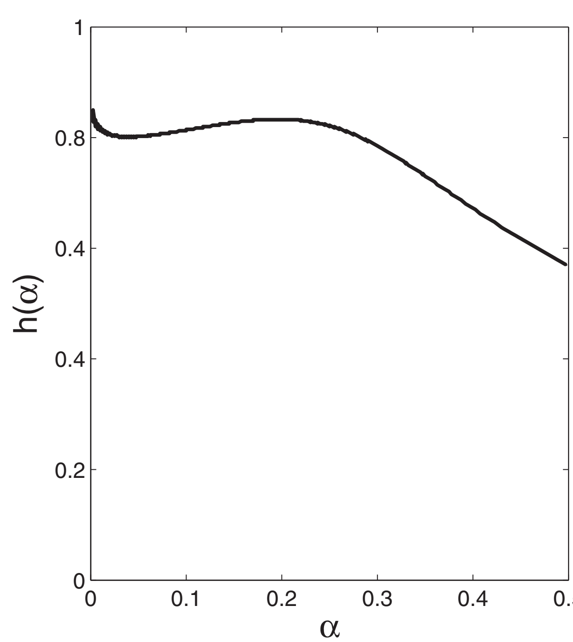

FIG. 2. Graph of $h(\alpha)$. The quantities are dimensionless.

It can be verified that $\frac{\partial f(\alpha,q)}{\partial q}$ is always non-negative, that is, $f(\alpha,q)$ is a monotonically increasing function of q with given $\alpha \in [0,1/2]$, and the condition $f(\alpha,q) \leqslant 0$ can be expressed as $q(t) \leqslant h(\alpha)$, which satisfies $f(\alpha,h(\alpha)) = 0$. The expression of $h(\alpha)$ is complicated, with its graph shown in Fig. 2 as the solid curve.

Moreover, from

$$\frac{d}{dt}\bar{F}_\phi(t) = -\frac{8}{3}(1-\alpha)(1-2(1-\alpha)q(t))\dot{q}(t),$$

it follows that $\frac{d}{dt}\bar{F}_\phi(t) \leqslant 0$ is equivalent to $\dot{q}(t) \geqslant 0$, $q(t) \leqslant \frac{1}{2(1-\alpha)}$ or $\dot{q}(t) \leqslant 0$, $q(t) \geqslant \frac{1}{2(1-\alpha)}$.

We conclude that the dynamics $\Lambda = \{\Lambda_t : t \geqslant 0\}$ is FI-Markovian if and only if $\dot{q}(t) \geqslant 0, q(t) \leqslant \min\{h(\alpha), \frac{1}{2(1-\alpha)}\}$ or $\dot{q}(t) \leqslant 0$, $q(t) \geqslant \max\{h(\alpha), \frac{1}{2(1-\alpha)}\}$. Since $\Lambda_0 = \mathbf{1}$, $q(0) = 0$, and $0 \leqslant q(t) \leqslant 1$, the above condition actually means that $\dot{q}(t) \geqslant 0$ and

$$q(t) \leqslant \min\left\{\frac{1}{2(1-\alpha)}, h(\alpha)\right\} = \mathrm{FI}(\alpha).$$

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