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Perpetual motion and driven dynamics of a mobile impurity in a quantum fluid

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We study the dynamics of a mobile impurity in a quantum fluid at zero temperature. Two related settings are considered. In the first setting, the impurity is injected in the fluid with some initial velocity $\mathbf{v}_0$, and we are interested in its velocity at infinite time, $\mathbf{v}_\infty$. We derive a rigorous upper bound on $|\mathbf{v}_0 - \mathbf{v}_\infty|$ for initial velocities smaller than the generalized critical velocity. In the limit of vanishing impurity-fluid coupling, this bound amounts to $\mathbf{v}_\infty = \mathbf{v}_0$, which can be regarded as a rigorous proof of the Landau criterion of superfluidity. In the case of a finite coupling, the velocity of the impurity can drop, but not to zero; the bound quantifies the maximal possible drop. In the second setting, a small constant force is exerted upon the impurity. We argue that two distinct dynamical regimes exist—backscattering oscillations of the impurity velocity and saturation of the velocity without oscillations. For fluids with $v_{c\mathrm{L}} = v_s$ (where $v_{c\mathrm{L}}$ and v_s are the Landau critical velocity and sound velocity, respectively), the latter regime is realized. For fluids with $v_{c\mathrm{L}} < v_s$, both regimes are possible. Which regime is realized in this case depends on the mass of the impurity, a nonequilibrium quantum phase transition occurring at some critical mass. Our results are equally valid in one, two, and three dimensions.

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Introduction. What happens to an impurity particle injected in a quantum fluid at zero temperature? According to the Landau criterion of superfluidity [1] generalized to account for motion of a particle of a finite mass [2], if the initial velocity of the impurity v_0 is less than the (mass-dependent) generalized critical velocity v_c, the impurity keeps moving forever without dissipation.[1] However, the kinematical argument beyond the generalized Landau criterion [1,2] is nonrigorous: It is based on the assumption that the *kinetic* energy is conserved, which is only approximately valid. Generally speaking, this argument does not exclude the possibility that corrections to the above approximation build up with time in such a way that the velocity of the impurity does relax to a zero or nonzero value in the long run [3–6]. Indeed, numerical and seminumerical calculations for specific systems have shown that the velocity does drop below v_0 even when $v_0 < v_c$ [7,8]. As a rule, numerical calculations are limited to finite times and therefore cannot unambiguously provide an infinite-time asymptotic value of the velocity, v_∞. In particular, an important qualitative question—whether the impurity eventually stops—often remains unanswered.

This issue has been recently addressed in the context of a specific model: An upper bound on $|v_0 - v_\infty|$ has been rigorously derived for the impurity injected in the one-dimensional (1D) gas of free fermions [9]. The first goal of this Rapid Communication is to provide an analogous bound valid for an arbitrary quantum fluid in any dimensionality. We rigorously prove that $|\mathbf{v}_0 - \mathbf{v}_\infty|$ is bounded from above for $|\mathbf{v}_0| < v_c$, with the bound depending on the dispersion of the fluid, strength of the coupling between the impurity and the fluid, mass of the impurity, and its initial velocity. In the limit of vanishing impurity-fluid coupling, the bound reduces to $\mathbf{v}_\infty = \mathbf{v}_0$, in accordance with the generalized Landau criterion of superfluidity [1,2]. In the case of finite interaction, the bound quantifies the maximal possible drop of the velocity.

The second question we address is as follows: What happens to an impurity immersed in a quantum fluid at zero temperature and pulled by a small constant force? This question was previously studied for impurities in superfluid helium [10–12] and, recently, in 1D fluids [13–17]. It was found that impurities in helium exhibit sawtooth velocity oscillations emerging from backscattering on rotons [10–12]. Similar *backscattering oscillations* (BO) have been found in 1D Tonks-Girardeau gas, but only for sufficiently heavy impurities [16]. For lighter impurities, another dynamical regime has been observed—*saturation of the velocity without oscillations* (SWO). Basing on the same kinematical constraint which underlies the generalized Landau argument [1,2], we investigate how general quantum fluids can be classified with respect to the regimes of driven dynamics. We find that BO and SWO are the only two generic regimes. A criterion determining which one is realized for a particular fluid and impurity is derived.

It is worth emphasizing that all of the methods and results presented in this Rapid Communication are universally valid both for one-dimensional fluids and higher-dimensional fluids, despite the well-known dramatic difference between the former and the latter with respect to the structure of elementary excitations [18]. This constitutes the major advancement over recent works [7,9,13–17,19,20] focused on 1D fluids which explicitly invoked special features of physics in one dimension.

Setup and notations. We consider a single impurity particle immersed in a quantum fluid. The Hamiltonian of the combined impurity-fluid system reads $\hat{H} = \hat{H}_\mathrm{f} + \hat{H}_\mathrm{i} + \hat{U}$, where $\hat{H}_\mathrm{f}$, $\hat{H}_\mathrm{i}$, and $\hat{U}$ describe the fluid, the impurity, and the impurity-fluid interaction, respectively. $\hat{H}_\mathrm{f}$, $\hat{H}_\mathrm{i}$, and $\hat{U}$ are translationally invariant and isotropic (the latter requirement can be dropped at the price of the results and derivation being more bulky). An eigenstate of $\hat{H}_\mathrm{f}$ with an energy E_f is denoted

[1]A necessary condition for $v_c > 0$ is that the dispersion of the fluid is not identically zero, which we assume throughout this Rapid Communication. This means that we consider two- and three-dimensional superfluids and generic one-dimensional fluids, but not, e.g., Fermi liquids. In practical terms, our consideration can be relevant for superfluid helium and metastable quantum fluids realized in ultracold-atom experiments.

by $|E_{\rm f}\rangle$. Each $|E_{\rm f}\rangle$ is also an eigenstate of the momentum. The dispersion of the fluid, $\varepsilon(q)$, is defined as a minimal eigenenergy which corresponds to a given momentum $\mathbf{q}$ with $|\mathbf{q}| = q$.

We use a special notation, $|{\rm GS}\rangle$, for the ground state of the fluid. We set the ground-state energy of the fluid to zero and assume that the momentum in the ground state is zero. This implies $\varepsilon(0) = 0$ and $\varepsilon(q) \geqslant 0$. The speed of sound is defined as $v_s \equiv \varepsilon'(0)$. Note that we do not impose any restrictions on the strength of interactions between the elementary excitations of the fluid.

The Hamiltonian of the impurity reads $\hat{H}_{\rm i} = \hat{\mathbf{P}}_{\rm i}^2/(2m)$, where $\hat{\mathbf{P}}_{\rm i}$ is the momentum of the impurity. Interaction $\hat{U}$ is pairwise with an interaction potential $U(r)$. We call the interaction *everywhere repulsive* whenever

$$U(r) \geqslant 0 \;\forall\, r. \tag{1}$$

We denote product eigenstates of $\hat{H}_{\rm f} + \hat{H}_{\rm i}$ by $|E_{\rm f},\mathbf{v}\rangle \equiv |E_{\rm f}\rangle \otimes |\mathbf{v}\rangle$, where $|\mathbf{v}\rangle$ is the plane wave of the impurity with the momentum $m\mathbf{v}$. Initially, the impurity-fluid system is in the product state $|{\rm GS},\mathbf{v}_0\rangle = |{\rm GS}\rangle \otimes |\mathbf{v}\rangle$, i.e., the impurity moves in the fluid at zero temperature with velocity $\mathbf{v}_0$.[2] Since the total momentum is an integral of motion, in what follows we restrict all operators to the subspace with the total momentum $m\mathbf{v}_0$.

The quantity we are interested in is the velocity of the impurity at infinite time. It is defined as

$$\mathbf{v}_\infty \equiv \frac{1}{m} \lim_{t\to\infty} \frac{1}{t} \int_0^t dt' \langle {\rm GS},\mathbf{v}_0| e^{i\hat{H}t'} \hat{\mathbf{P}}_{\rm i} e^{-i\hat{H}t'} |{\rm GS},\mathbf{v}_0\rangle. \tag{2}$$

Expanding the initial state in eigenstates $|E\rangle$ of the total Hamiltonian $\hat{H}$, and integrating out oscillating exponents, one obtains

$$\mathbf{v}_\infty = \frac{1}{m} \sum_{|E\rangle} |\langle {\rm GS},\mathbf{v}_0|E\rangle|^2 \langle E|\hat{\mathbf{P}}_{\rm i}|E\rangle. \tag{3}$$

Note that if $\hat{H}$ has degenerate eigenvalues, one should adjust the eigenbasis to diagonalize the matrix $\langle E'|{\rm GS},\mathbf{v}_0\rangle\langle {\rm GS},\mathbf{v}_0|E\rangle$ in every degenerate subspace.

Perpetual motion. We start from reviewing kinematical arguments which lead to the notion of critical velocity [1,2]. Consider an impurity with a velocity $\mathbf{v}_0$ which scatters off the fluid which is initially in its ground state. Assume that the impurity cannot form a bound state with particles of the fluid. Assume further that the final state of the impurity-fluid system is a product eigenstate of noninteracting Hamiltonian $\hat{H}_{\rm f} + \hat{H}_{\rm i}$, with $\mathbf{q}$ and $E_{\rm f} \geqslant \varepsilon(q)$ being, respectively, the final momentum and energy of the fluid. If one disregards the contribution of the impurity-fluid coupling to its energy, then conservation laws lead to

$$v_0 q \geqslant \mathbf{v}_0\mathbf{q} = E_{\rm f} + \frac{q^2}{2m} \geqslant \varepsilon(q) + \frac{q^2}{2m}, \tag{4}$$

[2]This initial state can be realized in ultracold-atom experiments by accelerating a noninteracting impurity inside the atomic cloud and switching the impurity-atom interaction by means of the Feshbach resonance afterwards.

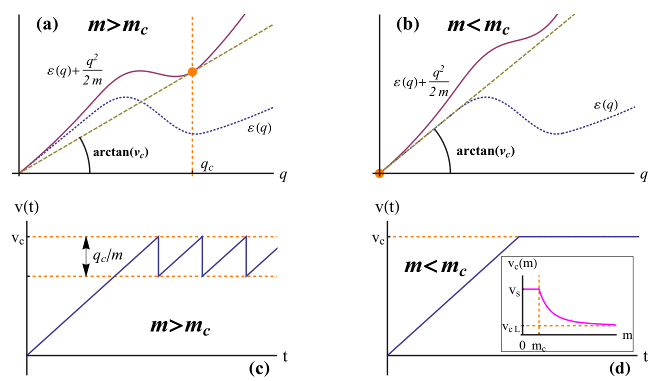

FIG. 1. (Color online) (a),(b) Geometrical illustration of definitions of generalized critical velocity v_c, given by Eq. (5), and critical momentum transfer q_c, given by Eq. (12). (a) v_c is smaller than the sound velocity v_s for $m > m_c$, while (b) $v_c = v_s$ for $m < m_c$. The thick orange dot marks the position of the critical momentum transfer q_c, which is finite for $m > m_c$ but vanishes for $m < m_c$. (c),(d) Velocity of the impurity pulled by a small constant force vs time. (c) Backscattering oscillations occur for $m > m_c$. (d) Velocity of the impurity saturates at v_c without oscillations for $m < m_c$. Inset: Generalized critical velocity as a function of the impurity mass. In the limit of $m \to \infty$, the generalized critical velocity approaches the Landau critical velocity $v_{c\rm L}$.

where $v_0 \equiv |\mathbf{v}_0|$. If v_0 is sufficiently small, $v_0 < v_c$, then for all $\mathbf{q} \neq 0$, the inequality (4) cannot be fulfilled. The generalized critical velocity v_c is defined as [2]

$$v_c \equiv \inf_q \frac{\varepsilon(q) + \frac{q^2}{2m}}{q}. \tag{5}$$

Physically, v_c is the minimal velocity which allows the impurity to create real excitations of the fluid, in the approximation of the noninteracting final impurity-fluid state. The geometrical sense of the generalized critical velocity can be seen from Fig. 1: The line $v_c q$ is a tangent to the curve $\varepsilon(q) + \frac{q^2}{2m}$.

Originally, Landau defined the critical velocity in the limit $m \to \infty$ [1]:

$$v_{c\rm L} \equiv \inf_q \left(\varepsilon(q)/q\right). \tag{6}$$

Note that $v_{c\rm L}$ is an attribute of the fluid alone, while v_c is an attribute of the impurity-fluid system.

It is worth emphasizing that the definition of the generalized critical velocity (5), although motivated by the Landau argument, stands alone and will be used beyond the scope of this argument in what follows.

Clearly, the argument by Landau reviewed above is not rigorous: The impurity-fluid interaction is largely disregarded, with its role being merely to justify why the transition from the initial to a final state occurs at all. Our aim is to derive a rigorous relation between $\mathbf{v}_0$ and $\mathbf{v}_\infty$. To this end, we prove the following.

Theorem. Consider an impurity particle immersed in a quantum fluid. Initially, the system is prepared in the product state $|{\rm GS},\mathbf{v}_0\rangle$ with the initial velocity of the impurity $v_0 \equiv |\mathbf{v}_0| < v_c$. The difference between the initial and infinite-time velocities of the impurity is bounded from above according to

$$|\mathbf{v}_0 - \mathbf{v}_\infty| \leqslant \frac{1}{m(v_c - v_0)}\left(\langle \mathrm{GS},\mathbf{v}_0|\hat{U}|\mathrm{GS},\mathbf{v}_0\rangle - \sum_{|E\rangle} |\langle \mathrm{GS},\mathbf{v}_0|E\rangle|^2 \langle E|\hat{U}|E\rangle\right). \tag{7}$$

If the interaction between the impurity and the fluid is everywhere repulsive, i.e., the condition (1) is fulfilled, then a more transparent bound holds:

$$|\mathbf{v}_0 - \mathbf{v}_\infty| \leqslant \frac{\overline{U}}{m(v_c - v_0)}, \tag{8}$$

where $\overline{U} \equiv \int d\mathbf{r}\rho U(|\mathbf{r}|)$ and ρ is the number density of the particles of the fluid.

This theorem generalizes an analogous result obtained in [9] for a specific one-dimensional fluid.

Proof. According to (3),

$$|\mathbf{v}_0 - \mathbf{v}_\infty| = \left|\sum_{|E\rangle}\sum_{|E_\mathrm{f},\mathbf{v}\rangle} (\mathbf{v}_0 - \mathbf{v})|\langle E|E_\mathrm{f},\mathbf{v}\rangle|^2 |\langle \mathrm{GS},\mathbf{v}_0|E\rangle|^2\right| \leqslant \sum_{|E\rangle}\left(\sum_{|E_\mathrm{f},\mathbf{v}\rangle} |\mathbf{v}_0 - \mathbf{v}||\langle E|E_\mathrm{f},\mathbf{v}\rangle|^2\right)|\langle \mathrm{GS},\mathbf{v}_0|E\rangle|^2. \tag{9}$$

The sums are performed over the eigenstates $|E\rangle$ of $\hat{H}$ and over the eigenstates $|E_\mathrm{f},\mathbf{v}\rangle$ of $\hat{H}_\mathrm{f} + \hat{H}_\mathrm{i}$ with the total momentum $m\mathbf{v}_0$.

The key step is to notice that according to (5),

$$|\mathbf{v}_0 - \mathbf{v}| \leqslant \frac{1}{m(v_c - v_0)}\left(E_\mathrm{f} + \frac{m\mathbf{v}^2}{2} - \frac{m\mathbf{v}_0^2}{2}\right) \tag{10}$$

for any $|E_\mathrm{f},\mathbf{v}\rangle$ with the total momentum $m\mathbf{v}_0$. This inequality is of pure kinematical origin. It leads to

$$\sum_{|E_\mathrm{f},\mathbf{v}\rangle} |\mathbf{v}_0 - \mathbf{v}||\langle E|E_\mathrm{f},\mathbf{v}\rangle|^2 \leqslant \frac{1}{m(v_c - v_0)}\sum_{|E_\mathrm{f},\mathbf{v}\rangle}\langle E|\hat{H}_\mathrm{f} + \hat{H}_\mathrm{i} - \frac{\mathbf{v}_0^2}{2m}|E_\mathrm{f},\mathbf{v}\rangle\langle E_\mathrm{f},\mathbf{v}|E\rangle = \frac{1}{m(v_c - v_0)}\left(E - \frac{\mathbf{v}_0^2}{2m} - \langle E|\hat{U}|E\rangle\right). \tag{11}$$

By substituting Eq. (11) into Eq. (9), one obtains the desired bound (7).

If the impurity-fluid coupling is everywhere repulsive, one obtains the bound (8) from the bound (7) by omitting the second term in the brackets in the right-hand side (rhs) of (7) and rewriting the first term according to $\langle \mathrm{GS},\mathbf{v}_0|\hat{U}|\mathrm{GS},\mathbf{v}_0\rangle = \overline{U}$. ■

In the remainder of the present section, we discuss the above theorem. First, we stress that the bounds (7) and (8) hold for an arbitrary interacting quantum fluid in arbitrary dimensions, in contrast to an earlier result [9] valid for a one-dimensional gas of free fermions. Remarkably, interactions between elementary excitations of the fluid renormalize $\varepsilon(p)$ but do not enter the bounds explicitly. Moreover, $\varepsilon(p)$ itself enters the bounds only through v_c.

Consider implications of the theorem in the weak impurity-fluid coupling limit. To define the latter, we introduce a family of interaction potentials $U_\gamma(r) = \gamma U_1(r)$ parameterized by the dimensionless coupling γ. The weak-coupling limit amounts to considering small γ (i.e., expanding all quantities of interest around $\gamma = 0$) after the thermodynamic limit ($N \to \infty$, $V = N/\rho$, ρ fixed) is taken. The physical meaning of this limit is that the interaction energy is small compared to the total energy per particle, but large compared to the level spacing.

The Landau criterion of superfluidity [1] (generalized for impurities of finite mass [2]) can be rigorously proved in the weak-coupling limit by virtue of the bound (7). To this end, if the interaction is not everywhere repulsive, we invoke an additional, rather natural assumption that $\langle E|\hat{U}_1|E\rangle \geqslant -C$ for any $|E\rangle$, where $C \geqslant 0$ is some constant independent on N and $|E\rangle$. For example, if bound states of the impurity particle and particles of the fluid exist, C is expected to be of the order of the largest binding energy among all such "molecules." The case of everywhere repulsive interaction amounts to $C = 0$. The bound (7) complemented by the aforementioned assumption immediately leads to the Landau's statement $\mathbf{v}_\infty = \mathbf{v}_0 + O(\gamma)$ for $|\mathbf{v}_0| < v_c$ in the weak-coupling limit.

It is worth emphasizing that a straightforward perturbation theory in γ does not lead to a correct many-body overlap $|\langle \mathrm{GS},\mathbf{v}_0|E\rangle|^2$ (see a thorough discussion of this point in [21]) and, as a consequence, does not permit a universal calculation of v_∞ directly from Eq. (3). This problem does not emerge when treating the rhs of the bound (7) because the interaction term $\hat{U}$ enters the latter explicitly.

Since the bound (7) invokes exact many-body eigenstates, its immediate application beyond the perturbative regime is possible only for integrable systems. These include (i) an impurity in a 1D gas of free fermions or infinitely repulsive bosons [22] and (ii) an impurity in a 1D gas of bosons, with masses of the impurity and host boson being equal, as well as boson-boson and boson-impurity couplings being equal (bosonic Yang-Gaudin model [23,24]). In the former model, it is possible to calculate v_∞ directly by means of Eq. (3) [25] (see also [20]). In the latter, more sophisticated model, an analogous analytical calculation would likely be much more intricate (if ever possible) since calculating overlaps $|\langle \mathrm{GS},\mathbf{v}_0|E\rangle|^2$ within Bethe ansatz is a hard task. On the other hand, application of the bound (7) should be feasible in this model since it requires a much simpler calculation of a matrix element of a local operator. In the nonintegrable cases, the bound (7) should be supplemented by some approximate method for calculating $\langle E|\hat{U}|E\rangle$ (e.g., perturbation theory, as is exemplified by the proof of the Landau criterion presented above).

Now we turn to the bound (8). Though valid for a narrower class of interactions, it has the advantage of simplicity compared to the bound (7) and can be easily applied without resorting to any approximations and limits. An additional benefit of the bound (8) is that it obviates two important points. First, the bound holds equally well for a finite fluid and in the thermodynamic limit. Second, inequality (8) represents a nontrivial bound even for long-range interactions, provided the interaction potential decreases with distance faster than $1/r^D$, with D being the dimensionality of the system. The latter requirement ensures that $\overline{U}$ does not diverge at large distances. We expect that both observations generically hold for the bound (7) as well.

Possible divergence of $\overline{U}$ deserves further discussion. It can also emerge at small r. In particular, it prevents us from considering hard sphere impurity-fluid interaction. Divergence in $\overline{U}$ implies that the initial state $|\text{GS},\mathbf{v}_0\rangle$ has divergent energy and thus the problem is ill formulated from the outset. How to correctly formulate the problem in this situation is an interesting open question.

We exemplify the usage of the bound (8) in one and three dimensions. In the case of one dimension, we consider the point-like repulsive impurity-fluid potential $U(x) = (U_0/\rho)\delta(x)$, with $U_0 > 0$ to obtain $|\mathbf{v}_0 - \mathbf{v}_\infty| \leqslant U_0[m(v_c - v_0)]^{-1}$. In the context of ultracold-atom experiments, this potential is an excellent low-energy approximation to any real impurity-fluid coupling with positive scattering length a, with U_0 being a function of a and transverse confinement energy [26]. This result has been obtained earlier for a special case of an impurity in a 1D gas of free fermions [9]; here it is proven for an arbitrary interacting 1D fluid.

In the case of three dimensions, we consider a "square" potential $U(r) = U_0\theta(r_0 - r)$. In this case, the bound reads $|\mathbf{v}_0 - \mathbf{v}_\infty| \leqslant (4\pi/3)r_0^3\rho U_0[m(v_c - v_0)]^{-1}$. In the limit when the interaction range r_0 is much larger than the scattering length $a \simeq 2\mu U_0 r_0^3/(3\hbar^2)$ (with μ being reduced mass), the bound can be expressed through the scattering length: $|\mathbf{v}_0 - \mathbf{v}_\infty| \lesssim 2\pi\hbar^2 a\rho[m\mu(v_c - v_0)]^{-1}$.

It is instructive to compare the above theorem with a rigorous result obtained in [19]: The expectation value of the impurity velocity in the momentum-dependent ground state is equal to the slope of the *total* dispersion of the impurity-fluid system, which is generically nonzero. Thus, Ref. [19] proves the very possibility of the perpetual motion of an impurity in a quantum fluid. However, it does not relate the initial velocity of the injected impurity, v_0, to its asymptotic velocity, v_∞, in contrast to the theorem presented above.

Dynamics of driven impurity. In the present section, we consider an impurity weakly coupled to a fluid and driven by a small constant force. The kinematical reasoning summarized in the beginning of the previous section can be extended to the case with driving. This was done for mobile impurities in superfluid helium in Refs. [10–12]. We study a problem in a wider context of an arbitrary quantum fluid.

Consider the impurity to be initially at rest. The force accelerates it freely until its velocity reaches v_c. At this instant, the impurity acquires a chance to scatter off the fluid. It is clear from Eq. (4) that the scattering channel which opens first is the backscattering. In this process, the impurity loses some momentum q_c which is transferred to the fluid. The critical momentum transfer q_c delivers the minimum in Eq. (5):

$$v_c q_c = \varepsilon(q_c) + \frac{q_c^2}{2m}. \quad (12)$$

The geometrical meaning of q_c is illustrated in Figs. 1(a) and 1(b): the line $v_c q$ touches the curve $\varepsilon(q) + \frac{q^2}{2m}$ in the point $(q_c, v_c q_c)$. Note that q_c is unrelated to $m v_c$.

Up to this point, our presentation has closely followed Refs. [10–12]. The central observation which allows us to go further is that the behavior of the impurity depends crucially on whether or not q_c is zero. Consider first the case $q_c > 0$ [see Fig. 1(a)], which is relevant, in particular, for impurities in helium [10–12]. After the first scattering, the velocity of the impurity drops by $\Delta v = q_c/m$, and the impurity starts to freely accelerate until its velocity again reaches v_c, after which the whole cycle is repeated. This is how backscattering oscillations emerge [10–12].

Consider now the case when $q_c = 0$; see Fig. 1(b). This case was not considered in [10–12] since it cannot be realized with realistic impurities in superfluid helium (see below). In this case, as soon as the velocity of the impurity reaches v_c, the impurity starts to dissipate the pumped energy by producing infrared excitations of the fluid. In the limit of small force, this leads to the saturation of its velocity at v_c without oscillations (SWO).

One can see that whether or not q_c is zero governs which of the two generic regimes, SWO or BO, is realized for a particular fluid and impurity. Note that $q_c = 0$ ($q_c > 0$) whenever $v_c = v_s$ ($v_c < v_s$); see Fig. 1. The relations between v_c and v_s, in turn, are determined by the Landau critical velocity of the fluid, $v_{c\mathrm{L}}$, and the mass of the impurity, m. As a result, in the fluid with $v_{c\mathrm{L}} = v_s$ (e.g., in the Bogoliubov gas of weakly coupled bosons), only SWO is possible, regardless of the value of m. In contrast, in the fluid with $v_{c\mathrm{L}} < v_s$, both SWO and BO are possible, depending on the mass of the impurity: BO emerge in the case of a heavy impurity, $m > m_c$, while SWO takes place for a light impurity, $m < m_c$. The critical mass m_c is determined from the equation $v_c(m_c) = v_s$, in which we explicitly indicate the dependence of the generalized critical velocity on the mass of the impurity; see Eq. (5) and the inset in Fig. 1. The amplitude of BO generically experiences a jump from a finite value to zero at $m = m_c$. Thus, if one regards m as a tunable parameter, the transition over m_c is a nonequilibrium quantum phase transition. Conditions discriminating between the two dynamical regimes are summarized in Table I. Note that SWO was not observed in superfluid helium since sufficiently light impurities were lacking.

TABLE I. Conditions determining which of the two dynamical regimes—backscattering oscillations (BO) or saturation without oscillations (SWO)—is realized in a specific fluid for a specific mass of the impurity.

		$v_{c\mathrm{L}} < v_s$	
	$v_{c\mathrm{L}} = v_s$	$m < m_c$	$m > m_c$
Regime	SWO	SWO	BO

The existence of two dynamical regimes separated by a nonequilibrium quantum phase transition is consistent with the results of the detailed study of a specific 1D fluid [16].

BO get damped at finite forces since the direction (for $D > 1$) and the value (for any dimensionality) of the momentum transfer vary from one scattering event to another. In Ref. [16], a kinetic theory for an impurity in the Tonks-Girardeau gas has been developed and the damping rate has been calculated. This theory can be generalized to arbitrary fluids, which is, however, beyond the scope of this Rapid Communication.

The physical picture we put forward differs significantly from the picture developed in Refs. [13–15] for 1D systems. The method of [13–15] is based on adiabatically following the total dispersion of the impurity-fluid system $\mathcal{E}(p)$. Since $\mathcal{E}(p)$ is periodic in one dimension, the authors of Refs. [13–15] conclude that Bloch-like oscillations of the velocity of the impurity develop, provided $\mathcal{E}(p)$ is a smooth function. This approach leaves no room for the SWO regime, in conflict with the results reported here and in Ref. [16]. We note, however, that a key ingredient of the argument of Refs. [13–15], i.e., adiabaticity, cannot, as a rule, be maintained for many-body gapless systems in the thermodynamic limit [27–30]. Although this issue has triggered an active discussion [31,32], it is not resolved so far and requires further study [33]. Note that the sawtooth oscillations in a 1D system have also been discussed in Ref. [15], but in the limit of strong force and only provided $\mathcal{E}(p)$ has a cusp (see also a precursory work [34]). These oscillations differ from those discussed here in amplitude and maximal velocity. We emphasize that smoothness of $\mathcal{E}(p)$ plays no role in our arguments, in contrast to Refs. [15,34].

Summary and concluding remarks. To summarize, we have studied two related settings. In the first setting, a mobile impurity is injected with some initial velocity v_0 in a quantum fluid at zero temperature. We have rigorously derived the upper bounds (7) and (8) on the difference between the initial and the asymptotic velocities of the impurity, $|\mathbf{v}_0 - \mathbf{v}_\infty|$, valid for $|\mathbf{v}_0|$ less than the mass-dependent generalized critical velocity v_c.

These bounds imply that while the velocity of the impurity can drop, it does not, generally speaking, drop to zero. This is consistent with the result of Ref. [3]: The impurity injected in the Bose-Einstein condensate creates a finite number of quasiparticles before relaxing to a steady state. On the other hand, our result disproves a suggestion of Refs. [5,6] (see also [4]) that perpetual motion of an impurity in a superfluid is nonexistent in the thermodynamic limit due to the Casimir-like friction force.

We note that at any finite temperature T, the infinite-time velocity is most likely to vanish. Results (7) and (8) remain relevant at low but nonzero temperatures if understood as bounds on the velocity at an intermediate time scale which is much less than the thermal relaxation time scale, $\sim\hbar^7 m(v_s/k_B T)^{2+2D} a^{-2D}$, where a is the scattering length [35–38]. For $D=3$, one gets relaxation time scale $\sim 1\,\mathrm{s}(\frac{m}{m_{\mathrm{Rb}}})(\frac{v_s}{1\,\mathrm{mm/s}})^8(\frac{T}{100\,\mathrm{nK}})^{-8}(\frac{a}{10\,\mathrm{nm}})^{-6}$ with $m_{\mathrm{Rb}} = 85.47$ amu and other reference values relevant for ultracold-atom experiments [39].

In the second setting, an impurity is pulled by a small constant force. We have demonstrated that, in general, two dynamical regimes can occur—backscattering oscillations of the impurity velocity (BO) or velocity saturation without oscillations (SWO). For fluids with $v_{c\mathrm{L}} = v_s$, SWO is the only possible regime. For fluids with $v_{c\mathrm{L}} < v_s$, SWO occurs for light impurities while BO occur for heavy impurities, with the two regimes being separated by a nonequilibrium quantum phase transition at some critical mass; see Table I and the inset in Fig. 1.

Our treatment of the first problem is valid for any strength of impurity-fluid interaction; however, the weaker is the interaction, the tighter are the bounds. Our treatment of the second problem is valid in the leading order of the weak-coupling limit only. However, it is not necessarily the bare coupling which should be weak: If one is able to find a renormalizing unitary transformation which takes into account the dressing of the impurity in a particular fluid and leads to a small effective coupling, this suffices to validate our treatment.

Note added. In a recent paper [40], the concept of mass-dependent generalized critical velocity of a mobile impurity in a Fermi superfluid is studied in great detail. In particular, the nonanalyticity of v_c as a function of mass is discussed.

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