Majorization approach to entropic uncertainty relations for coarse-grained observables

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We improve the entropic uncertainty relations for position and momentum coarse-grained measurements. We derive the continuous, coarse-grained counterparts of the discrete uncertainty relations based on the concept of majorization. The entropic inequalities obtained involve two Rényi entropies of the same order, and thus go beyond the standard scenario with conjugated parameters. In a special case describing the sum of two Shannon entropies, the majorization-based bounds significantly outperform the currently known results in the regime of larger coarse graining, and might thus be useful for entanglement detection in continuous variables.

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I. INTRODUCTION

The optimal entropic uncertainty relation for two conjugate continuous variables (position and momentum) has been known for almost 40 years [1]. One decade later, entropic formulation of the uncertainty principle has as well been developed in discrete settings [2,3]. Even though the topic of entropic uncertainty relations (EURs) has a long history (for a detailed review, see [4,5]), one can observe a recent increase of interest within the quantum information community, leading to several improvements [6–17], and even a deep asymptotic analysis of different bounds [18]. This is quite understandable, because the entropic uncertainty relations have various applications, for example in entanglement detection [19–23], security of quantum protocols [24,25], quantum memory [26,27], or as an ingredient of Einstein-Podolsky-Rosen steering criteria [28,29]. Moreover, the recent discussion [30] about the original Heisenberg idea of uncertainty led to the entropic counterparts of the noise-disturbance uncertainty relation [31,32] (also obtained with quantum memory [33]).

My favorite example of an entropic description of uncertainty [34–36] is situated between the continuous and the discrete scenarios. Continuous position and momentum variables, when studied with the help of coarse-grained measurements, lead to discrete probability distributions. This particular formulation of the uncertainty principle was recognized long ago [37–39] to faithfully capture the spirit of position-momentum duality. It also carries a deep physical insight, since the coarse-grained version of the Heisenberg uncertainty relation is nontrivial for any coarse graining (given in terms of two widths Δ and δ in the positions and momenta, respectively) provided that both widths are finite [40]. On the practical level, coarse-grained entropic relations are experimentally useful for entanglement [23,41] and steering detection [28] in continuous-variable schemes. The aim of this paper is thus to strengthen the theoretical and experimental tools based on coarse-grained EURs by taking advantage of the recent improvements in discrete entropic inequalities, in particular, the one based on majorization [12].

Let me start with a brief description of the entropic uncertainty landscape, with a special emphasis on the majorization approach developed recently. The standard position-momentum scenario deals with the sum of the continuous Shannon (or in general Rényi) entropies $-\int dz\rho(z)\ln\rho(z)$ calculated for both densities $\rho(x)=|\psi(x)|^2$ and $\tilde{\rho}(p)=|\tilde{\psi}(p)|^2$ describing positions and momenta, respectively. The position and momentum wave functions are mutually related by a Fourier transformation. The discrete EURs rely on the notion of the Rényi entropy of order α,

$$H_\alpha[P]=\frac{1}{1-\alpha}\ln\sum_i P_i^\alpha, \tag{1}$$

and the sum inequalities of the general form

$$H_\alpha[P(A;\varrho)]+H_\beta[P(B;\varrho)]\geqslant B_{\alpha\beta}(A,B) \tag{2}$$

valid for any density matrix ϱ and two nondegenerate observables A and B. If by $|a_i\rangle$ and $|b_j\rangle$ we denote the eigenstates of the two observables in question, the associated probability distributions entering (2) are

$$P_i(A;\varrho)=\langle a_i|\varrho|a_i\rangle \quad\text{and}\quad P_j(B;\varrho)=\langle b_j|\varrho|b_j\rangle. \tag{3}$$

The lower bound $B_{\alpha\beta}$ does not depend on ϱ, but only on the unitary matrix $U_{ij}=\langle a_i|b_j\rangle$. For instance, the most widely recognized result by Maassen and Uffink [3] gives the bound $-2\ln\max_{i,j}|U_{ij}|$, valid whenever

$$\frac{1}{\alpha}+\frac{1}{\beta}=2. \tag{4}$$

The pair (α,β) constrained as in Eq. (4) is often referred to as the conjugate parameters.

A. Majorization entropic uncertainty relations

In the majorization approach one looks for the probability vectors $Q(A,B)$ and $W(A,B)$ which majorize the tensor product [9,10] and the *direct sum* [12] of the involved distributions (3):

$$P(A;\varrho)\otimes P(B;\varrho)\prec Q(A,B), \tag{5}$$

$$P(A;\varrho)\oplus P(B;\varrho)\prec\{1\}\oplus W(A,B). \tag{6}$$

The majorization relation $x\prec y$ between any two D-dimensional probability vectors implies that for all $n\leqslant D$ we have $\sum_{k=1}^n x_k^\downarrow\leqslant\sum_{k=1}^n y_k^\downarrow$, with a necessary equality when

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$n = D$. In agreement with the usual notation, the symbol $\downarrow$ denotes decreasing order, which means that $(x^{\downarrow})_k \geqslant (x^{\downarrow})_l$, for all $k \leqslant l$. In the case when the vectors compared in (5) and (6) are of different size, the shorter vector should be completed by a proper number of coordinates equal to 0. The tensor product $x \otimes y$ (also called the Kronecker product) is a D^2-dimensional probability vector with the coefficients equal to

$$x_1 y_1, x_1 y_2, \ldots, x_1 y_D, \ldots, x_D y_1, x_D y_2, \ldots, x_D y_D, \tag{7}$$

while the direct sum $x \oplus y$ is a $2D$-dimensional probability vector given by

$$x_1, x_2, \ldots, x_D, y_1, y_2, \ldots, y_D. \tag{8}$$

One of the most important properties of the Rényi entropy of any order α is its additivity,

$$H_\alpha[x] + H_\alpha[y] = H_\alpha[x \otimes y]. \tag{9}$$

Moreover, in the special case of the Shannon entropies ($H_1[\cdot] \equiv H[\cdot]$) one easily finds that

$$H[x] + H[y] = H[x \oplus y]. \tag{10}$$

Since the presumed majorization relations (5) and (6) are valid for every ϱ, the Schur concavity of the Rényi (Shannon) entropy together with (9) and (10) immediately leads to the corresponding bounds $B_{\alpha\alpha} = H_\alpha[Q]$ [9,10] and $B_{11} = H[W]$ [12]. Due to the subadditivity property of the function $\ln(1+z)$ the validity of the latter bound can be extended [12] to the range $\alpha \leqslant 1$ (in that case the function $\sum_i z_i^\alpha$ is also Schur concave), i.e., $B_{\alpha\alpha}^{\alpha \leqslant 1} = H_\alpha[W]$. On the other hand, when $\alpha > 1$, this bound can be appropriately modified to the weaker form [12]

$$B_{\alpha\alpha}^{\alpha>1}[W] = \frac{2}{1-\alpha}\left[\ln\left(1 + \sum_i W_i\right) - \ln 2\right]. \tag{11}$$

The complete families of the vectors $Q(A,B)$ and $W(A,B)$ fulfilling (5) and (6) have been explicitly constructed in [9,10] and [12], respectively. The aim of the present paper is to obtain the counterpart of the majorizing vector $W(A,B)$ applicable to the position-momentum coarse-grained scenario described in detail in the forthcoming Sec. I B. In Sec. II we derive this vector using the sole idea of majorization, so that we shall omit here the detailed prescription established in [12]. We restrict further discussion to the direct-sum approach, since for $\alpha \leqslant 1$ (this case covers the sum of two Shannon entropies), the *direct-sum entropic uncertainty relation* is always stronger than the corresponding tensor-product EUR [12].

B. Entropic uncertainty relations for coarse-grained observables

The last set of ingredients we shall introduce contains the coarse-grained probabilities together with their EURs. Due to coarse graining, the continuous densities $\rho(x)$ and $\tilde{\rho}(p)$ become the discrete probabilities

$$q_k^\Delta = \int_{k^-\Delta}^{k^+\Delta} dx\, \rho(x), \quad p_l^\delta = \int_{l^-\delta}^{l^+\delta} dp\, \tilde{\rho}(p), \tag{12}$$

with $k^\pm = k \pm 1/2$, $l^\pm = l \pm 1/2$, and $k,l \in \mathbb{Z}$. The sum of the Rényi entropies $H_\alpha[q^\Delta]$ and $H_\beta[p^\delta]$ calculated for the probabilities (12) is lower bounded by [40]

$$B_{\alpha\beta}(\Delta,\delta) = \max\left[\mathcal{B}_\alpha(\Delta\delta/\hbar); \mathcal{R}(\Delta\delta/\hbar)\right], \tag{13}$$

where [36]

$$\mathcal{B}_\alpha(\gamma) = -\frac{1}{2}\left(\frac{\ln\alpha}{1-\alpha} + \frac{\ln\beta}{1-\beta}\right) - \ln(\gamma/\pi) \tag{14}$$

and [40]

$$\mathcal{R}(\gamma) = -\ln(\gamma/2\pi) - 2\ln R_{00}(\gamma/4,1) \geqslant 0. \tag{15}$$

Once more the above results are valid only for conjugate parameters (4), so that we label the bound (14) only by the index α. The function $R_{00}(\xi,\eta)$ is the "00" radial prolate spheroidal wave function of the first kind [42]. When $\gamma \ll 1$, the spheroidal term in (15) becomes negligible and we have

$$\mathcal{R}(\gamma) \approx \mathcal{B}_1(\gamma) + \ln 2 - 1, \tag{16}$$

so that the bound (14) dominates in this regime. In the opposite case, when $\gamma > e\pi \approx 8.54$, the bound (14) is negative, so starting from some smaller (α-dependent) value of γ the second bound $\mathcal{R}(\gamma)$ becomes significant.

II. DIRECT-SUM MAJORIZATION FOR COARSE-GRAINED OBSERVABLES

After the short but comprehensive introduction, we are in position to formulate the main result of this paper. Assume that a sum of any M position probabilities q^Δ and any N momentum probabilities p^δ is bounded by $1 + G_{MN}(\gamma)$, that is ($\gamma = \Delta\delta/\hbar$),

$$q_{k_1}^\Delta + \cdots + q_{k_M}^\Delta + p_{l_1}^\delta + \cdots + p_{l_N}^\delta \leqslant 1 + G_{MN}(\gamma), \tag{17}$$

for some indices $k_1 \neq k_2 \neq \cdots \neq k_M$ and $l_1 \neq l_2 \neq \cdots \neq l_N$. We implicitly assume here that $G_{MN}(\gamma)$ does not depend on the specific choice of the probabilities in the sum (it bounds any choice), and that $G_{MN}(\gamma) \leqslant 1$ since the left-hand side of (17) cannot exceed 2. Denote further

$$F_J(\gamma) = \max_{0 \leqslant M \leqslant J} G_{M,J-M}(\gamma). \tag{18}$$

Assume now that $F_J(\gamma)$, $J = 1,2,\ldots,\infty$ is an increasing sequence

$$F_{J+1}(\gamma) \geqslant F_J(\gamma). \tag{19}$$

If that happens, the construction of the vector $W(\gamma)$ applicable to the direct-sum majorization relation, i.e., such that $q^\Delta \oplus p^\delta \prec \{1\} \oplus W(\gamma)$, can be patterned after [12]:

$$W_i(\gamma) = F_{i+1}(\gamma) - F_i(\gamma), \tag{20}$$

for $i = 1,2,\ldots,\infty$. Due to (19) the coefficients $W_i(\gamma)$ are all non-negative, so that they form a probability vector. Note that $F_1(\gamma) \equiv 0$, since one picks up only a single probability ($M = 1$, $N = 0$ or $M = 0$, $N = 1$), and that $F_\infty(\gamma) \equiv 1$, because whenever the quantum state is localized (in position or momentum) in a single bin, the left-hand side of (17) is equal to 2. This is in accordance with an expectation that $W(\gamma)$ is the probability vector.

One can check by a direct inspection that

$$1 + \sum_{i=1}^{J-1} W_i^{\downarrow}(\gamma) \geqslant 1 + \sum_{i=1}^{J-1} W_i(\gamma) = 1 + F_J(\gamma), \tag{21}$$

which together with (17) and (18) is the essence of majorization. As in the case of discrete majorization [9,12], there is a whole family (labeled by $n = 2,\ldots,\infty$) of majorizing vectors $W^{(n)}(\gamma)$ given by the prescription $W_i^{(n)} \equiv W_i$ for $i < n$, $W_n^{(n)} = 1 - F_n$, and $W_i^{(n)} \equiv 0$ when $i > n$. In that notation, the basic vector (20) is equivalent to $W^{(\infty)}(\gamma)$, and the following majorization chain does hold:

$$W^{(2)} \succ W^{(3)} \succ \cdots \succ W^{(n)} \succ W^{(n+1)} \succ \cdots \succ W^{(\infty)} \equiv W. \tag{22}$$

The remaining task is to find the candidates for the coefficients $F_J(\gamma)$. To this end we shall define two sets:

$$X(\Delta) = \bigcup_{a=1}^{M} [k_a^- \Delta, k_a^+ \Delta], \qquad Y(\delta) = \bigcup_{b=1}^{N} [l_b^- \Delta, l_b^+ \Delta], \tag{23}$$

which are simply the unions of intervals associated with the probabilities present in (17). The measures of these sets are equal to $M\Delta$ and $N\delta$, respectively. Equation (17) rewritten in terms of the above sets simplifies to the form

$$\int_{X(\Delta)} dx\, \rho(x) + \int_{Y(\delta)} dp\, \tilde{\rho}(p) \leqslant 1 + G_{MN}(\gamma). \tag{24}$$

Following Lenard [37], we shall further introduce two projectors $\hat{\mathcal{Q}}$ and $\hat{\mathcal{P}}$, such that for any function $f(x)$, the function $(\hat{\mathcal{Q}} f)(x)$ has its support equal to $X(\Delta)$ and the Fourier transform of the function $(\hat{\mathcal{P}} f)(x)$ is supported in $Y(\delta)$. If both $X(\Delta)$ and $Y(\delta)$ are intervals, then according to Theorem 4 from [39] [this theorem in fact formalizes the content of Eq. (17) from [38]] the formal candidate for $G_{MN}(\gamma)$ is the square root of the largest eigenvalue λ_0 of the compact, positive operator $\hat{\mathcal{Q}}\hat{\mathcal{P}}\hat{\mathcal{Q}}$. Due to Proposition 11 (including the discussion around it) from [37], the above statement remains valid for any sets $X(\Delta)$ and $Y(\delta)$. As concluded by Lenard, this is a generalization of the seminal results by Landau and Pollak [43], who for the first time quantified uncertainty using spheroidal functions. It happens [44], however, that λ_0 has the largest value exactly in the interval case, so that it can always be upper bounded by the eigenvalue found by Landau and Pollak:

$$\lambda_0 \leqslant \frac{\xi}{2\pi\hbar}\left[R_{00}(\xi/4\hbar,1)\right]^2, \tag{25}$$

with ξ being the product of the measures of the two sets in question, that is, $\xi = (M\Delta)(N\delta)$. Since the right-hand side of (25) is an increasing function of ξ, we can easily find the maximum in (18). The maximal value of MN with fixed $M + N$ is given by possibly equal contributions of both numbers. Since M and N are integers we finally get

$$F_J(\gamma) = \sqrt{\frac{\gamma \lceil J/2\rceil \lfloor J/2\rfloor}{2\pi}}\, R_{00}\left(\frac{\gamma \lceil J/2\rceil \lfloor J/2\rfloor}{4},1\right), \tag{26}$$

where $\lceil\cdot\rceil$ and $\lfloor\cdot\rfloor$ denote the integer-valued ceiling and floor functions, respectively.[1] If J is odd then $\lceil J/2\rceil \lfloor J/2\rfloor = (J^2-1)/4$, and $\lceil J/2\rceil \lfloor J/2\rfloor = J^2/4$ in the simpler case when J is an even number. Note that the functions (26) form an increasing sequence as desired.

The final result of the above considerations is thus the family of majorization entropic uncertainty relations ($n = 2,\ldots,\infty$):

$$H_\alpha[q^\Delta] + H_\alpha[p^\delta] \geqslant \mathcal{R}_\alpha^{(n)}(\Delta\delta/\hbar) \equiv H_\alpha[W^{(n)}(\Delta\delta/\hbar)], \tag{27}$$

valid for $\alpha \leqslant 1$. As mentioned in Sec. I A the case of the Shannon entropy directly follows from (10), while the range $\alpha < 1$ is obtained due to the subadditivity of $\ln(1+z)$. In the case $\alpha > 1$ we need to replace the majorization bound according to (11), and obtain $\mathcal{R}_\alpha^{(n)}(\gamma) \equiv B_{\alpha\alpha}^{\alpha>1}[W^{(n)}(\gamma)]$.

III. DISCUSSION

A comparison of the previous bounds (14) and (15) with the majorization results is presented in Fig. 1, for the case of the Shannon entropy ($\alpha = 1 = \beta$). I depicted the first three majorization-based bounds (black, solid lines) since they are sufficient to capture the whole content of our uncertainty relations. First of all, only the bound for $n = 2$ is slightly weaker than the remaining majorization bounds in the regime of larger γ, while there is no difference between $n = 3$, $n = 4$, and other (not presented) values of n. For $\gamma \to \infty$, all the black curves exhibit the same behavior, so that one can take advantage of the asymptotic expansion [45]

$$\frac{\gamma}{2\pi}\left[R_{00}(\gamma/4,1)\right]^2 \sim 1 - 2\sqrt{\pi\gamma}\, e^{-\gamma/2}, \tag{28}$$

in order to show that

$$\mathcal{R}_1^{(n)}(\gamma) \sim \frac{\sqrt{\pi}}{2}\gamma^{3/2} e^{-\gamma/2} \tag{29}$$

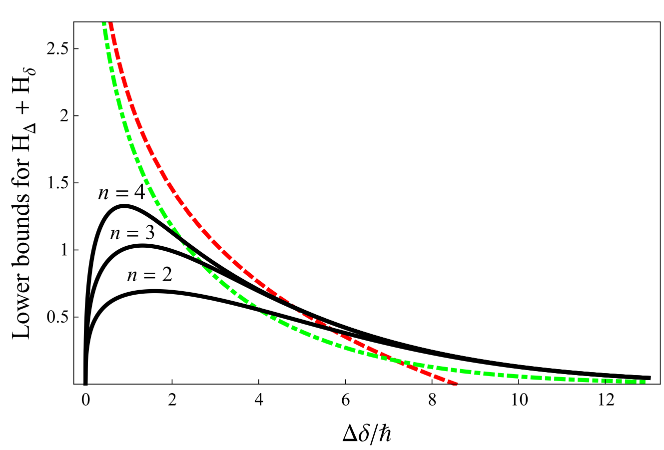

FIG. 1. (Color online) As a comparison, I plot the previously known lower bounds $\mathcal{B}_1$ (red dashed line) and $\mathcal{R}$ (green dash-dotted line), together with our majorization bounds (black solid lines), labeled by $n = 2,3,4$. By H_Δ and H_δ I denote $H_1[q^\Delta]$ and $H_1[p^\delta]$, respectively. From the value $\Delta\delta/\hbar \approx 4.8231$ at which the black line ($n = 4$) intersects the red dashed line, our bounds improve the previously known results.

[1] These functions may be defined as follows: $\lceil z\rceil = \min(i \in \mathbb{Z} : z \leqslant i)$ and $\lfloor z\rfloor = \max(j \in \mathbb{Z} : z \geqslant j)$.

for all $n = 2,\ldots,\infty$. The same expansion studied for the previous bound (15) leads to

$$\mathcal{R}(\gamma) \sim 2\sqrt{\pi\gamma}e^{-\gamma/2}. \tag{30}$$

The asymptotic value of (29) is larger than (30) by a divergent factor $\gamma/4$. Since the bound (15) for the sum of two Shannon entropies is always weaker than the pair $\mathcal{B}_1(\gamma)$ and $\mathcal{R}_1^{(3)}(\gamma)$, it is in this case sufficient to use only these two bounds. Obviously, the bound $\mathcal{R}(\gamma)$ remains useful (as being always non-negative) for the conjugated parameters (α,β) with $\alpha \neq \beta$, when the majorization bounds do not apply. Let me recall that in the limiting case $\alpha = 1/2$, $\beta = \infty$, the bound $\mathcal{R}(\gamma)$ is optimal and can be saturated for any value of γ.

While by increasing the number n we do not change the tail of the bound, we still substantially improve the area of small γ. Taking the limit $\gamma \to 0$, one can recognize that the optimal majorization bound $\mathcal{R}_1^{(\infty)}(\gamma)$ behaves as $-\frac{1}{2}\ln\gamma$, so it is still far below the bound $\mathcal{B}_1(\gamma)$. To show that property one needs to associate $i\sqrt{\gamma}$ in (20) with a continuous variable z, so that

$$W_i(\gamma) \to \sqrt{\gamma}\frac{d}{dz}\left[\frac{z}{2\sqrt{2\pi}}R_{00}(z^2/16,1)\right], \tag{31}$$

and use the definition of the Riemann integral. This kind of behavior is somewhat typical in the majorization approach to entropic uncertainty relations. In the discrete case, the tensor-product EUR (weaker than the direct-sum EUR used in this paper) can outperform the Maassen-Uffink result in more than 98% of cases [9], even for a small dimension of the Hilbert space equal to 5. But the Maassen-Uffink lower bound [3] always dominates when U_{ij} is sufficiently close to the Fourier matrix, so that both eigenbases of the observables A and B become mutually unbiased. The continuous limit $\gamma \to 0$ is of exactly the same sort, since the resulting continuous densities originate from the wave functions in position and momentum spaces, which are related by the Fourier transformation. Note that the behavior in the limit $\gamma \to 0$ thus does not permit us to derive counterparts of the continuous EURs [1,36,46], valid for $\beta = \alpha$.

IV. CONCLUSIONS

We have presented in Eq. (27) a direct-sum majorization entropic bound for coarse-grained observables in the case $\beta = \alpha$. In the Shannon case, the bound (13) holds as well and the comparison of all bounds is depicted in Fig. 1. The current bounds (black, solid lines) significantly improve the previously known results in the regime of $\gamma \geqslant 4.8231$ (this threshold value is an intersection point between the red dashed line and the black line labeled by $n = 4$). This regime of relevance ($\gamma \geqslant 4.8231$) is of practical importance. In [23], entanglement of a two-mode Gaussian state was experimentally confirmed with the coarse-graining widths $\Delta = 17\Delta_1$ and $\delta = 15\delta_1$, where $\Delta_1 = 0.0250$ mm and $\delta_1/\hbar = 1.546\ \text{mm}^{-1}$. To construct the entanglement criteria, one needs to put $\gamma = \Delta\delta/2\hbar$ inside the underlying uncertainty relation (the factor of 1/2 comes from different normalizations of the global quadratures), so that the above numbers reduce to the value $\gamma = 4.9279$. Even though we observe a tiny overlap between the regime in which our EUR outperforms the previous results and the parameters from [23], for slightly larger coarse graining, say $\gamma = 7$, the value of the bound increases by 60% because $\mathcal{R}_1^{(3)}(7)/\mathcal{B}_1(7) = 1.609$. This suggests, however, that with the current bound at hand, one could improve the performance of the entanglement criteria and possibly detect entanglement beyond the cases reported in [23]. The better detection ability might become important when dealing with multipartite entanglement [47], since due to the increasing number of degrees of freedom, the coarse-grained measurements might appear to be the one feasible experimental method [48].

In the discrete scenario with almost mutually unbiased bases the Maassen-Uffink bound always outperforms the majorization approach. However it can still be improved with the help of the monotonicity property of the relative entropy [11], or by combining the former approach with majorization techniques [12]. In the continuous case this type of analysis is far more difficult, since we actually do not have at our disposal a unitary matrix U such that $a_k = \sum_l U_{kl}b_l$ and a_k,b_l are the probability amplitudes reproducing (12),

$$q_k^{\Delta} = |a_k|^2, \quad p_l^{\delta} = |b_l|^2. \tag{32}$$

From the beginning we deal with the *per se* probabilities q_k^{Δ} and p_l^{δ}. This fundamental difference can be overcome if one introduces an additional degree of freedom [40,49,50] corresponding to the orthonormal bases on the intervals $[k_a^-\Delta,k_a^+\Delta]$ and $[l_b^-\Delta,l_b^+\Delta]$. Even though this approach produces the valid unitary matrix U, the remaining optimization required by [11] becomes a challenging task.

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