Charge asymmetry in the differential cross section of high-energy bremsstrahlung in the field of a heavy atom

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The distinction between the charged particle and antiparticle differential cross sections of high-energy bremsstrahlung in the electric field of a heavy atom is investigated. The consideration is based on the quasiclassical approximation to the wave functions in the external field. The charge asymmetry (the ratio of the antisymmetric and symmetric parts of the differential cross section) arises due to the account for the first quasiclassical correction to the differential cross section. All evaluations are performed with the exact account of the atomic field. We consider in detail the charge asymmetry for electrons and muons. For electrons, the nuclear size effect is not important while for muons this effect should be taken into account. For the longitudinal polarization of the initial charged particle, the account for the first quasiclassical correction to the differential cross section leads to the asymmetry in the cross section with respect to the replacement $\varphi \to -\varphi$, where φ is the azimuth angle between the photon momentum and the momentum of the final charged particle.

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I. INTRODUCTION

The theoretical investigation of high-energy bremsstrahlung and high-energy particle-antiparticle photoproduction in the electric field of a heavy nucleus or atom has a long history because of the importance of these processes for various applications; for the latter process see reviews in Refs. [1,2]. These processes should be taken into account when considering electromagnetic showers in detectors, they also give the significant part of the radiative corrections in many cases. Therefore, it is necessary to know the cross sections of these processes with high accuracy. In the Born approximation, the cross sections of both processes have been obtained for arbitrary energies of particles and for arbitrary atomic form factors [3,4] (see also Ref. [5]). The Coulomb corrections to the cross section, which are the difference between the exact in the parameter $\eta = Z\alpha$ cross section and the Born cross section, are very important (here Z is the atomic charge number, $\alpha = e^2 \approx 1/137$ is the fine-structure constant, e is the electron charge, $\hbar = c = 1$). There are formal expressions for the Coulomb corrections to the cross sections exact in η and energies of particles [6]. However, the numerical computations based on these expressions become more and more difficult when energies are increasing, and, for instance, the numerical results for e^+e^- photoproduction have been obtained so far only for the photon energy $\omega < 12.5$ MeV [7].

At high energies of initial particles, the final particle momenta usually have small angles with respect to the incident direction. In this case typical angular momenta, which provide the main contribution to the cross section, are large ($l \sim E/\Delta \gg 1$, where E is energy and Δ is the momentum transfer). This is why the quasiclassical approximation, based on the account of large angular momenta contributions, becomes applicable. In this approximation, the wave functions and the Green's functions of the Dirac equation in the external field have very simple forms, which drastically simplify their use in specific calculations. The wave functions, obtained in the leading quasiclassical approximation for the Coulomb field, are the famous Furry-Sommerfeld-Maue wave functions [8,9] (see also Ref. [5]). The quasiclassical Green's function have been derived in Ref. [10] for the case of a pure Coulomb field, in Ref. [11] for an arbitrary spherically symmetric field, in Ref. [12] for a localized field which generally possesses no spherical symmetry, and in Ref. [13] for combined strong laser and atomic fields.

In the leading quasiclassical approximation, the cross sections for pair photoproduction and bremsstrahlung have been obtained in Refs. [14–18]. The first quasiclassical corrections to the spectra of both processes, as well as to the total cross section of pair photoproduction, have been obtained in Refs. [19–22]. Recently, the first quasiclassical correction to the fully differential cross section was obtained in Ref. [23] for e^+e^- pair photoproduction and in Ref. [24] for $\mu^+\mu^-$ pair photoproduction. As a result, the charge asymmetry in these processes (the asymmetry of the cross section with respect to permutation of particle and antiparticle momenta) was predicted. This asymmetry is absent in the cross section calculated in the Born approximation and also in the cross section exact in the parameter η but calculated in the leading quasiclassical approximation. Thus, the charge asymmetry appears solely due to the quasiclassical corrections to the Coulomb corrections. The difference between the atomic field and the Coulomb field of a nucleus results in the modification of the cross sections (effect os screening). The influence of screening on the Coulomb corrections to e^+e^- pair photoproduction is small for the differential cross section and for the total cross section [15]. However, screening is important for the Born term. The quantitative investigation of the effect of screening on the Coulomb corrections to the photoproduction cross section is performed in Ref. [19].

The influence of screening on the bremsstrahlung cross section in an atomic field is more complicated. It is shown in Refs. [16,20] that the Coulomb corrections to the differential cross section are very susceptible to screening. However, the

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Coulomb corrections to the cross section integrated over the momentum of final charged particle (electron or muon) are independent of screening in the leading approximation over a small parameter $1/m_e r_{scr}$, where $r_{scr} \sim Z^{-1/3}(m_e\alpha)^{-1}$ is a screening radius and m_e is the electron mass. The quantitative investigation of the effect of screening on the Coulomb corrections to the spectrum of bremsstrahlung is performed in Ref. [20]. The differential cross section of bremsstrahlung, calculated in the leading quasiclassical approximation, is the same for e^+ and e^- (for μ^+ and μ^-). Therefore, to predict the charge asymmetry (the difference between the bremsstrahlung differential cross section for particles and antiparticles in the atomic field), one should perform calculations in the next-to-leading quasiclassical approximation. This is the main goal of our paper. The result is obtained exactly in the parameter η. Besides, for the case of muons the nuclear size effect is taken into account.

The bremsstrahlung differential cross section from high-energy charged particle in an atomic field, $d\sigma(\boldsymbol{p},\boldsymbol{q},\boldsymbol{k},\eta)$, can be written as

$$\begin{aligned} d\sigma(\boldsymbol{p},\boldsymbol{q},\boldsymbol{k},\eta) &= d\sigma_s(\boldsymbol{p},\boldsymbol{q},\boldsymbol{k},\eta) + d\sigma_a(\boldsymbol{p},\boldsymbol{q},\boldsymbol{k},\eta),\\ d\sigma_s(\boldsymbol{p},\boldsymbol{q},\boldsymbol{k},\eta) &= \frac{d\sigma(\boldsymbol{p},\boldsymbol{q},\boldsymbol{k},\eta) + d\sigma(\boldsymbol{p},\boldsymbol{q},\boldsymbol{k},-\eta)}{2},\\ d\sigma_a(\boldsymbol{p},\boldsymbol{q},\boldsymbol{k},\eta) &= \frac{d\sigma(\boldsymbol{p},\boldsymbol{q},\boldsymbol{k},\eta) - d\sigma(\boldsymbol{p},\boldsymbol{q},\boldsymbol{k},-\eta)}{2}, \end{aligned} \tag{1}$$

where $\boldsymbol{k}$ is the photon momentum, $\boldsymbol{p}$ and $\boldsymbol{q}$ are the initial and final charged particle momenta, respectively. Evidently, the bremsstrahlung differential cross section from high-energy antiparticle can be obtained from $d\sigma(\boldsymbol{p},\boldsymbol{q},\boldsymbol{k},\eta)$ by the replacement $\eta \to -\eta$, so that it is equal to $d\sigma_s(\boldsymbol{p},\boldsymbol{q},\boldsymbol{k},\eta) - d\sigma_a(\boldsymbol{p},\boldsymbol{q},\boldsymbol{k},\eta)$. In the leading quasiclassical approximation, all dependence on η of the matrix element is contained in the factor N_0 [see Eq. (31)]. Since $N_0 \to -N_0^*$ at $\eta \to -\eta$, the cross section calculated in the leading quasiclassical approximation, which is proportional to $|N_0|^2$, does not reveal the charge asymmetry. Thus, the charge asymmetry appears solely due to the first quasiclassical correction to the matrix element. We show that the antisymmetric part of the differential cross section, $d\sigma_a(\boldsymbol{p},\boldsymbol{q},\boldsymbol{k},\eta)$, is independent of screening in the kinematical region, which provides the main contribution to the antisymmetric part of the spectrum.

The paper is organized as follows. In Sec. II we derive the general expression for the quasiclassical matrix element of the process. In Sec. III we find in the quasiclassical approximation all structures of the Green's function of the squared Dirac equation for a charged particle in arbitrary localized potential and the corresponding structures of the wave functions. We obtain the leading terms and the first quasiclassical corrections as well. Using these wave functions, we derive in Sec. IV the matrix element of the process and the corresponding differential cross section for arbitrary localized potential and in the particular case of the pure Coulomb field. In Sec. V we investigate in detail the charge asymmetry in high-energy bremsstrahlung from electrons. In this case the nuclear size effect is not important. In Sec. VI we investigate the charge asymmetry in high-energy bremsstrahlung from muons, which is sensitive to the deviation at small distances of the nuclear atomic field from the pure Coulomb field. Finally, in Sec. VII the main conclusions of the paper are presented.

II. GENERAL DISCUSSION

The differential cross section of bremsstrahlung in the electric field of a heavy atom reads [5]

$$d\sigma = \frac{\alpha\omega q\varepsilon_q}{(2\pi)^4}\, d\Omega_{\boldsymbol{k}}\, d\Omega_{\boldsymbol{q}}\, d\omega |M|^2, \tag{2}$$

where $d\Omega_{\boldsymbol{k}}$ and $d\Omega_{\boldsymbol{q}}$ are the solid angles corresponding to the photon momentum $\boldsymbol{k}$ and the final charged particle momentum $\boldsymbol{q}$, $\omega = \varepsilon_p - \varepsilon_q$ is the photon energy, $\varepsilon_p = \sqrt{\boldsymbol{p}^2 + m^2}$, $\varepsilon_q = \sqrt{\boldsymbol{q}^2 + m^2}$, and m is the particle mass. Below we assume that $\varepsilon_p \gg m$ and $\varepsilon_q \gg m$. The matrix element M reads

$$M = \int d\boldsymbol{r}\, \bar{u}_{\boldsymbol{q}}^{(-)}(\boldsymbol{r})\, \boldsymbol{\gamma}\cdot\boldsymbol{e}^*\, u_{\boldsymbol{p}}^{(+)}(\boldsymbol{r}) \exp(-i\boldsymbol{k}\cdot\boldsymbol{r}), \tag{3}$$

where γ^μ are the Dirac matrices, $u_{\boldsymbol{p}}^{(+)}(\boldsymbol{r})$ and $u_{\boldsymbol{q}}^{(-)}(\boldsymbol{r})$ are the solutions of the Dirac equation in the external field, $\boldsymbol{e}$ is the photon polarization vector. The superscripts $(-)$ and $(+)$ remind us that the asymptotic forms of $u_{\boldsymbol{q}}^{(-)}(\boldsymbol{r})$ and $u_{\boldsymbol{p}}^{(+)}(\boldsymbol{r})$ at large $\boldsymbol{r}$ contain, in addition to the plane wave, the spherical convergent and divergent waves, respectively. The wave functions $u_{\boldsymbol{p}}^{(+)}(\boldsymbol{r})$ and $u_{\boldsymbol{q}}^{(-)}(\boldsymbol{r})$ have the form [24]

$$\begin{aligned} \bar{u}_{\boldsymbol{q}}^{(-)}(\boldsymbol{r}) &= \bar{u}_{\boldsymbol{q}}[f_0(\boldsymbol{r},\boldsymbol{q}) - \boldsymbol{\alpha}\cdot\boldsymbol{f}_1(\boldsymbol{r},\boldsymbol{q}) - \boldsymbol{\Sigma}\cdot\boldsymbol{f}_2(\boldsymbol{r},\boldsymbol{q})],\\ u_{\boldsymbol{p}}^{(+)}(\boldsymbol{r}) &= [g_0(\boldsymbol{r},\boldsymbol{p}) - \boldsymbol{\alpha}\cdot\boldsymbol{g}_1(\boldsymbol{r},\boldsymbol{p}) - \boldsymbol{\Sigma}\cdot\boldsymbol{g}_2(\boldsymbol{r},\boldsymbol{p})]u_{\boldsymbol{p}},\\ u_{\boldsymbol{p}} &= \sqrt{\frac{\varepsilon_p + m}{2\varepsilon_p}}\begin{pmatrix}\phi\\ \dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{\varepsilon_p+m}\phi\end{pmatrix},\\ u_{\boldsymbol{q}} &= \sqrt{\frac{\varepsilon_q + m}{2\varepsilon_q}}\begin{pmatrix}\chi\\ \dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{q}}{\varepsilon_q+m}\chi\end{pmatrix}, \end{aligned} \tag{4}$$

where ϕ and χ are spinors, $\boldsymbol{\alpha} = \gamma^0\boldsymbol{\gamma}$, $\boldsymbol{\Sigma} = \gamma^0\gamma^5\boldsymbol{\gamma}$, and $\boldsymbol{\sigma}$ are the Pauli matrices. The following relations hold:

$$\begin{aligned} &g_0(\boldsymbol{r},\boldsymbol{q}) = f_0(\boldsymbol{r},-\boldsymbol{q}), \quad \boldsymbol{g}_1(\boldsymbol{r},\boldsymbol{q}) = \boldsymbol{f}_1(\boldsymbol{r},-\boldsymbol{q}),\\ &\boldsymbol{g}_2(\boldsymbol{r},\boldsymbol{q}) = -\boldsymbol{f}_2(\boldsymbol{r},-\boldsymbol{q}). \end{aligned} \tag{5}$$

The wave functions in the atomic field can be found from the Green's function $D(\boldsymbol{r}_2,\boldsymbol{r}_1|\varepsilon)$ of the squared Dirac equation in this field using the relations [24]

$$\begin{aligned} &\frac{\exp ipr_1}{4\pi r_1} u_{\boldsymbol{p}}^{(+)}(\boldsymbol{r}_2) = -\lim_{r_1\to\infty} D(\boldsymbol{r}_2,\boldsymbol{r}_1|\varepsilon_p)u_{\boldsymbol{p}}, \quad \boldsymbol{p} = -p\boldsymbol{n}_1,\\ &\frac{\exp iqr_2}{4\pi r_2} \bar{u}_{\boldsymbol{q}}^{(-)}(\boldsymbol{r}_1) = -\lim_{r_2\to\infty} \bar{u}_{\boldsymbol{q}} D(\boldsymbol{r}_2,\boldsymbol{r}_1|\varepsilon_q), \quad \boldsymbol{q} = q\boldsymbol{n}_2, \end{aligned} \tag{6}$$

where $\boldsymbol{n}_1 = \boldsymbol{r}_1/r_1$, $\boldsymbol{n}_2 = \boldsymbol{r}_2/r_2$, and

$$\begin{aligned} D(\boldsymbol{r}_2,\boldsymbol{r}_1|\varepsilon) &= \langle \boldsymbol{r}_2|\frac{1}{\hat{\mathcal{P}}^2 - m^2 + i0}|\boldsymbol{r}_1\rangle\\ &= \langle \boldsymbol{r}_2|\{[\varepsilon - V(r)]^2 + \nabla^2 - m^2\\ &\quad + i\boldsymbol{\alpha}\cdot\nabla V(r) + i0\}^{-1}|\boldsymbol{r}_1\rangle. \end{aligned} \tag{7}$$

Here $\hat{\mathcal{P}} = \gamma^\mu \mathcal{P}_\mu, \mathcal{P}_\mu = [\varepsilon - V(r), i\nabla]$, and $V(r)$ is the atomic potential. It follows from Eq. (7) that the Green's function $D(\boldsymbol{r}_2, \boldsymbol{r}_1|\varepsilon)$ can be written as

$$D(\boldsymbol{r}_2, \boldsymbol{r}_1|\varepsilon) = d_0(\boldsymbol{r}_2,\boldsymbol{r}_1) + \boldsymbol{\alpha}\cdot\boldsymbol{d}_1(\boldsymbol{r}_2,\boldsymbol{r}_1) + \boldsymbol{\Sigma}\cdot\boldsymbol{d}_2(\boldsymbol{r}_2,\boldsymbol{r}_1). \tag{8}$$

It is convenient to calculate the matrix element for definite helicities of the particles. Let μ_p, μ_q, and λ be the signs of the helicities of initial charged particle, final charged particle, and photon, respectively. We fix the coordinate system so that $\boldsymbol{\nu} = \boldsymbol{k}/\omega$ is directed along z-axis and $\boldsymbol{q}$ lies in the xz plane with $q_x > 0$. Denoting helicities by the subscripts, we have

$$\begin{aligned}
\phi_{\mu_p} &= \frac{1+\mu_p\boldsymbol{\sigma}\cdot\boldsymbol{n}_p}{4\cos(\theta_p/2)}\begin{pmatrix}1+\mu_p\\1-\mu_p\end{pmatrix}\\
&\approx \frac{1}{4}\left(1+\frac{\theta_p^2}{8}\right)(1+\mu_p\boldsymbol{\sigma}\cdot\boldsymbol{n}_p)\begin{pmatrix}1+\mu_p\\1-\mu_p\end{pmatrix},\\
\chi_{\mu_q} &= \frac{1+\mu_q\boldsymbol{\sigma}\cdot\boldsymbol{n}_q}{4\cos(\theta_q/2)}\begin{pmatrix}1+\mu_q\\1-\mu_q\end{pmatrix}\\
&\approx \frac{1}{4}\left(1+\frac{\theta_q^2}{8}\right)(1+\mu_q\boldsymbol{\sigma}\cdot\boldsymbol{n}_q)\begin{pmatrix}1+\mu_q\\1-\mu_q\end{pmatrix},\\
\boldsymbol{e}_\lambda &= \frac{1}{\sqrt{2}}(\boldsymbol{e}_x + i\lambda\boldsymbol{e}_y),
\end{aligned} \tag{9}$$

where θ_p and θ_q are the polar angles of the vectors $\boldsymbol{p}$ and $\boldsymbol{q}$, respectively. The unit vectors $\boldsymbol{e}_x$ and $\boldsymbol{e}_y$ are directed along $\boldsymbol{q}_\perp$ and $\boldsymbol{k}\times\boldsymbol{q}$, respectively, where the notation $\boldsymbol{X}_\perp = \boldsymbol{X} - (\boldsymbol{X}\cdot\boldsymbol{\nu})\boldsymbol{\nu}$ for any vector $\boldsymbol{X}$ is used. We also introduce the vectors $\boldsymbol{\theta}_p = \boldsymbol{p}_\perp/p$, $\boldsymbol{\theta}_q = \boldsymbol{q}_\perp/q$, and the matrix $\mathcal{F} = u_{\boldsymbol{p}\,\mu_p}\bar{u}_{\boldsymbol{q}\,\mu_q}$, which can be written as

$$\begin{aligned}
\mathcal{F} &= \frac{1}{8}\big(a_{\mu_p\mu_q} + \boldsymbol{\Sigma}\cdot\boldsymbol{b}_{\mu_p\mu_q}\big)[\gamma^0(1+PQ) + \gamma^0\gamma^5(P+Q)\\
&\quad + (1-PQ) - \gamma^5(P-Q)],\\
P &= \frac{\mu_p p}{\varepsilon_p + m}, \quad Q = \frac{\mu_q q}{\varepsilon_q + m},
\end{aligned} \tag{10}$$

where $a_{\mu_p\mu_q}$ and $\boldsymbol{b}_{\mu_p\mu_q}$ are defined from

$$\phi_{\mu_p}\chi^\dagger_{\mu_q} = \tfrac{1}{2}\big(a_{\mu_p\mu_q} + \boldsymbol{\sigma}\cdot\boldsymbol{b}_{\mu_p\mu_q}\big). \tag{11}$$

We obtain from Eq. (9)

$$\begin{aligned}
a_{\mu\mu} &= 1 - \frac{\theta_{pq}^2}{8} - \frac{i\mu}{4}\boldsymbol{\nu}\cdot[\boldsymbol{\theta}_p\times\boldsymbol{\theta}_q],\\
a_{\mu\bar{\mu}} &= \frac{\mu}{\sqrt{2}}\boldsymbol{e}_\mu\cdot\boldsymbol{\theta}_{pq},\\
\boldsymbol{b}_{\mu\mu} &= \left\{\mu\left[1 - \frac{1}{8}(\boldsymbol{\theta}_p+\boldsymbol{\theta}_q)^2\right] + \frac{i}{4}\boldsymbol{\nu}\cdot[\boldsymbol{\theta}_p\times\boldsymbol{\theta}_q]\right\}\boldsymbol{\nu}\\
&\quad + \frac{\mu}{2}(\boldsymbol{\theta}_p+\boldsymbol{\theta}_q) + \frac{i}{2}[\boldsymbol{\theta}_{pq}\times\boldsymbol{\nu}],\\
\boldsymbol{b}_{\mu\bar{\mu}} &= \sqrt{2}\boldsymbol{e}_\mu - \frac{1}{\sqrt{2}}(\boldsymbol{e}_\mu,\boldsymbol{\theta}_p+\boldsymbol{\theta}_q)\boldsymbol{\nu}, \quad \boldsymbol{e}_\mu = \frac{1}{\sqrt{2}}(\boldsymbol{e}_x + i\mu\boldsymbol{e}_y),
\end{aligned} \tag{12}$$

where $\boldsymbol{\theta}_{pq} = \boldsymbol{\theta}_p - \boldsymbol{\theta}_q$ and $\bar{\mu} = -\mu$. The matrix element M, Eq. (3), can be written as follows

$$\begin{aligned}
M = \int d\boldsymbol{r}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\,\mathrm{Tr}\{&(f_0 - \boldsymbol{\alpha}\cdot\boldsymbol{f}_1 - \boldsymbol{\Sigma}\cdot\boldsymbol{f}_2)\boldsymbol{\gamma}\\
&\cdot\boldsymbol{e}^*_\lambda(g_0 - \boldsymbol{\alpha}\cdot\boldsymbol{g}_1 - \boldsymbol{\Sigma}\cdot\boldsymbol{g}_2)\mathcal{F}\}.
\end{aligned} \tag{13}$$

Note that only the terms with $(P+Q)$ and $(1+PQ)$ in $\mathcal{F}$, Eq. (10), contribute to the matrix element (13) because it contains the odd number of the γ matrices.

In the quasiclassical approximation the relative magnitude of the functions f_0, $\boldsymbol{f}_{1,2}$, g_0, and $\boldsymbol{g}_{1,2}$ is different, so that

$$f_0 \sim l_c f_1 \sim l_c^2 f_2, \quad g_0 \sim l_c g_1 \sim l_c^2 g_2, \quad d_0 \sim l_c d_1 \sim l_c^2 d_2, \tag{14}$$

where $l_c \sim \varepsilon/\Delta \gg 1$ is the characteristic value of the angular momentum in the process, $\boldsymbol{\Delta} = \boldsymbol{q} + \boldsymbol{k} - \boldsymbol{p}$ is the momentum transfer. To find the distinction between the differential cross section of bremsstrahlung from particles and antiparticles, it is necessary to take into account the first quasiclassical corrections to the functions f_0, g_0, $\boldsymbol{f}_1$, and $\boldsymbol{g}_1$, while the functions $\boldsymbol{f}_2$ and $\boldsymbol{g}_2$ can be taken in the leading quasiclassical approximation. Let us introduce the quantities

$$\begin{aligned}
&(A_{00}, \boldsymbol{A}_{01}, \boldsymbol{A}_{10}, \boldsymbol{A}_{02}, \boldsymbol{A}_{20})\\
&\quad = \int d\boldsymbol{r}\,\exp(-i\boldsymbol{k}\cdot\boldsymbol{r})(f_0g_0, f_0\boldsymbol{g}_1, \boldsymbol{f}_1g_0, f_0\boldsymbol{g}_2, \boldsymbol{f}_2g_0).
\end{aligned} \tag{15}$$

In terms of these quantities, the matrix element M has the form

$$\begin{aligned}
M = \delta_{\mu_p\mu_q}\big[&\delta_{\lambda\mu_p}(\boldsymbol{e}^*_\lambda, -\boldsymbol{\theta}_q A_{00} - 2\boldsymbol{A}_{10} + 2\mu_p\boldsymbol{A}_{20})\\
&+ \delta_{\lambda\bar{\mu}_p}(\boldsymbol{e}^*_\lambda, -\boldsymbol{\theta}_p A_{00} + 2\boldsymbol{A}_{01} + 2\mu_p\boldsymbol{A}_{02})\big]\\
&- \frac{m\mu_p(p-q)}{\sqrt{2}\,pq}\delta_{\mu_q\bar{\mu}_p}\delta_{\lambda\mu_p}A_{00}.
\end{aligned} \tag{16}$$

Below we calculate all quantities in (16) for arbitrary atomic potential $V(r)$, which includes the effect of screening and the nuclear size effect as well.

III. GREEN'S FUNCTIONS AND WAVE FUNCTIONS

Let us consider the case of arbitrary central localized potential $V(r)$. We expand the Green's function $D(\boldsymbol{r}_2, \boldsymbol{r}_1|\varepsilon)$, Eq. (7), up to the second order with respect to the correction $\boldsymbol{\alpha}\cdot\nabla V(r)$:

$$\begin{aligned}
D(\boldsymbol{r}_2, \boldsymbol{r}_1|\varepsilon) &= \langle\boldsymbol{r}_2|\frac{1}{\mathcal{H}} - \frac{1}{\mathcal{H}}i\boldsymbol{\alpha}\cdot\nabla V(r)\frac{1}{\mathcal{H}}\\
&\quad + \frac{1}{\mathcal{H}}i\boldsymbol{\alpha}\cdot\nabla V(r)\frac{1}{\mathcal{H}}i\boldsymbol{\alpha}\cdot\nabla V(r)\frac{1}{\mathcal{H}}|\boldsymbol{r}_1\rangle,\\
\mathcal{H} &= \varepsilon^2 - m^2 - 2\varepsilon\varphi(r) + \nabla^2 + i0,\\
\varphi(r) &= V(r) - \frac{V^2(r)}{2\varepsilon}.
\end{aligned} \tag{17}$$

The function $D^{(0)}(\boldsymbol{r}_2, \boldsymbol{r}_1|\varepsilon) = \langle\boldsymbol{r}_2|\mathcal{H}^{-1}|\boldsymbol{r}_1\rangle$ is the Green's function of the Klein-Gordon equation. This function was found in the quasiclassical approximation with the first correction taken into account [12]:

$$D^{(0)}(\boldsymbol{r}_2,\boldsymbol{r}_1|\,\varepsilon) = \frac{ie^{i\kappa r}}{4\pi^2 r}\int d\boldsymbol{Q}\exp\left[iQ^2 - ir\int_0^1 dx\,V(\boldsymbol{R}_x)\right]\left\{1+\frac{ir^3}{2\kappa}\int_0^1 dx\int_0^x dy(x-y)\nabla_\perp V(\boldsymbol{R}_x)\cdot\nabla_\perp V(\boldsymbol{R}_y)\right\},$$
$$\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1,\quad \boldsymbol{R}_x = \boldsymbol{r}_1 + x\boldsymbol{r} + \boldsymbol{Q}\sqrt{\frac{2r_1r_2}{\kappa r}},\quad \kappa = \sqrt{\varepsilon^2 - m^2}, \tag{18}$$

where $\boldsymbol{Q}$ is a two-dimensional vector perpendicular to $\boldsymbol{r}$ and $\nabla_\perp$ is the component of the gradient perpendicular to $\boldsymbol{r}$. Within the same accuracy, $D^{(0)}(\boldsymbol{r}_2,\boldsymbol{r}_1|\varepsilon)$ coincides with the contribution $d(\boldsymbol{r}_2,\boldsymbol{r}_1)$ to the Green's function $D(\boldsymbol{r}_2,\boldsymbol{r}_1|\varepsilon)$, Eq. (8).

Using this formula and Eqs. (4), (6), and (8), we obtain the function $f_0(\boldsymbol{r},\boldsymbol{q})$,

$$f_0(\boldsymbol{r},\boldsymbol{q}) = -\frac{i}{\pi}e^{-i\boldsymbol{q}\cdot\boldsymbol{r}}\int d\boldsymbol{Q}\exp\left[iQ^2 - i\int_0^\infty dx\,V(\boldsymbol{r}_x)\right]\left\{1+\frac{i}{2\varepsilon_q}\int_0^\infty dx\int_0^x dy(x-y)\nabla_\perp V(\boldsymbol{r}_x)\cdot\nabla_\perp V(\boldsymbol{r}_y)\right\},$$
$$\boldsymbol{r}_x = \boldsymbol{r} + x\boldsymbol{n}_{\boldsymbol{q}} + \boldsymbol{Q}\sqrt{\frac{2r}{\varepsilon_q}},\quad \boldsymbol{Q}\cdot\boldsymbol{n}_{\boldsymbol{q}} = 0, \tag{19}$$

where $\nabla_\perp$ is the component of the gradient perpendicular to $\boldsymbol{n}_{\boldsymbol{q}} = \boldsymbol{q}/q$. Then we use the relation

$$i\nabla V(r) = \frac{1}{2\varepsilon}[\boldsymbol{p},\mathcal{H}] + \frac{i}{2\varepsilon}\nabla V^2(r), \tag{20}$$

and write the linear in $\nabla V(r)$ term in Eq. (17) as $\boldsymbol{\alpha}\cdot\boldsymbol{d}_1(\boldsymbol{r}_2,\boldsymbol{r}_1)$, where

$$\boldsymbol{d}_1(\boldsymbol{r}_2,\boldsymbol{r}_1) = -\frac{i}{2\varepsilon}(\nabla_1+\nabla_2)D^{(0)}(\boldsymbol{r}_2,\boldsymbol{r}_1|\,\varepsilon) + \delta\boldsymbol{d}_1(\boldsymbol{r}_2,\boldsymbol{r}_1),$$
$$\delta\boldsymbol{d}_1(\boldsymbol{r}_2,\boldsymbol{r}_1) = -\langle \boldsymbol{r}_2|\frac{1}{\mathcal{H}}\frac{i}{2\varepsilon}\nabla V^2(r)\frac{1}{\mathcal{H}}|\boldsymbol{r}_1\rangle. \tag{21}$$

If we replace $V(r)$ by $V(r)+\delta V(r)$ in the operator $\mathcal{H}$, where $\delta V(r) = -i\boldsymbol{\alpha}\cdot\nabla V^2(r)/(2\varepsilon)^2$, then we obtain from Eq. (18)

$$\delta\boldsymbol{d}_1(\boldsymbol{r}_2,\boldsymbol{r}_1) = -\frac{ie^{i\kappa r}}{16\pi^2\varepsilon^2}\int d\boldsymbol{Q}\exp\left[iQ^2 - ir\int_0^1 dx\,V(\boldsymbol{R}_x)\right]\times\int_0^1 dx\,\nabla V^2(\boldsymbol{R}_x), \tag{22}$$

where $\boldsymbol{R}_x$ is given in (18). Using Eqs. (4), (6), and (8), we find the function $\boldsymbol{f}_1(\boldsymbol{r},\boldsymbol{q})$,

$$\boldsymbol{f}_1(\boldsymbol{r},\boldsymbol{q}) = \frac{1}{2\varepsilon}(i\nabla - \boldsymbol{q})f_0(\boldsymbol{r},\boldsymbol{q}) + \delta\boldsymbol{f}_1(\boldsymbol{r},\boldsymbol{q}),$$
$$\delta\boldsymbol{f}_1(\boldsymbol{r},\boldsymbol{q}) = -\frac{i}{4\pi\varepsilon^2}e^{-i\boldsymbol{q}\cdot\boldsymbol{r}}\int d\boldsymbol{Q}\exp\left[iQ^2 - i\int_0^\infty dx\,V(\boldsymbol{r}_x)\right]\times\int_0^\infty dx\nabla V^2(\boldsymbol{r}_x), \tag{23}$$

where $\boldsymbol{r}_x$ is given in Eq. (19).

To transform the third term in (17), we replace $i\nabla V(r)$ by $\frac{1}{2\varepsilon}[\boldsymbol{p},\mathcal{H}]$. Then it follows from Eqs. (4), (6), and (8) that the function $\boldsymbol{d}_2(\boldsymbol{r}_2,\boldsymbol{r}_1)$ is

$$\boldsymbol{d}_2(\boldsymbol{r}_2,\boldsymbol{r}_1) = -\frac{i}{(2\varepsilon)^2}[\nabla_2\times\nabla_1]D^{(0)}(\boldsymbol{r}_2,\boldsymbol{r}_1|\,\varepsilon) + \delta\boldsymbol{d}_2(\boldsymbol{r}_2,\boldsymbol{r}_1),$$
$$\delta\boldsymbol{d}_2(\boldsymbol{r}_2,\boldsymbol{r}_1) = \boldsymbol{l}_{21}\,\langle \boldsymbol{r}_2|\frac{1}{\mathcal{H}}\frac{V'(r)}{2\varepsilon r}\frac{1}{\mathcal{H}}|\boldsymbol{r}_1\rangle, \tag{24}$$

where $\boldsymbol{l}_{21} = -(i/2)(\boldsymbol{r}_2\times\nabla_2 - \boldsymbol{r}_1\times\nabla_1)$ and $V'(r) = \partial V(r)/\partial r$. In (24) we use the relation $[\boldsymbol{l},\mathcal{H}] = 0$. If we replace $V(r)$ by $V(r)+\delta V(r)$ in the operator $\mathcal{H}$, where $\delta V(r) = r^{-1}V'(r)/(2\varepsilon)^2$, then we obtain from Eq. (18)

$$\delta\boldsymbol{d}_2(\boldsymbol{r}_2,\boldsymbol{r}_1) = \boldsymbol{l}_{21}\frac{e^{i\kappa r}}{16\pi^2\varepsilon^2}\int d\boldsymbol{Q}\times\exp\left[iQ^2 - ir\int_0^1 dx\,V(\boldsymbol{R}_x)\right]\times\int_0^1 dx\,\frac{V'(\boldsymbol{R}_x)}{R_x}. \tag{25}$$

Substituting this expression in (24), we finally find $\boldsymbol{d}_2(\boldsymbol{r}_2,\boldsymbol{r}_1)$,

$$\boldsymbol{d}_2(\boldsymbol{r}_2,\boldsymbol{r}_1) = -\frac{re^{i\kappa r}}{16\pi^2\varepsilon^2}\int d\boldsymbol{Q}\exp\left[iQ^2 - ir\int_0^1 dx\,V(\boldsymbol{R}_x)\right]\times\int_0^1 dx\int_0^x dy\,[\nabla V(\boldsymbol{R}_x)\times\nabla V(\boldsymbol{R}_y)]. \tag{26}$$

The corresponding function $\boldsymbol{f}_2(\boldsymbol{r},\boldsymbol{q})$ is

$$\boldsymbol{f}_2(\boldsymbol{r},\boldsymbol{q}) = -\frac{e^{-i\boldsymbol{q}\cdot\boldsymbol{r}}}{4\pi\varepsilon^2}\int d\boldsymbol{Q}\exp\left[iQ^2 - i\int_0^\infty dx\,V(\boldsymbol{r}_x)\right]\times\int_0^\infty dx\int_0^x dy\,[\nabla V(\boldsymbol{r}_x)\times\nabla V(\boldsymbol{r}_y)]. \tag{27}$$

For the Coulomb field $V_c(r) = -\eta/r$, we find from (19), (23), and (27)

$$f_0(\boldsymbol{r},\boldsymbol{q}) = F_A + (1+\boldsymbol{n}_{\boldsymbol{q}}\cdot\boldsymbol{n})F_C,$$
$$\boldsymbol{f}_1(\boldsymbol{r},\boldsymbol{q}) = (\boldsymbol{n}_{\boldsymbol{q}}+\boldsymbol{n})\eta F_B,$$
$$\boldsymbol{f}_2(\boldsymbol{r},\boldsymbol{q}) = -i\boldsymbol{\Sigma}\cdot[\boldsymbol{n}_{\boldsymbol{q}}\times\boldsymbol{n}]F_C, \tag{28}$$

where

$$F_A(\boldsymbol{r},\boldsymbol{q},\eta) = \exp\left(\frac{\pi\eta}{2} - i\boldsymbol{q}\cdot\boldsymbol{r}\right)\left[\Gamma(1-i\eta)F(i\eta,1,iz) + \frac{\pi\eta^2 e^{i\frac{\pi}{4}}}{2\sqrt{2qr}}\Gamma(1/2-i\eta)F(1/2+i\eta,1,iz)\right],$$
$$F_B(\boldsymbol{r},\boldsymbol{q},\eta) = -\frac{i}{2}\exp\left(\frac{\pi\eta}{2} - i\boldsymbol{q}\cdot\boldsymbol{r}\right)\left[\Gamma(1-i\eta)F(1+i\eta,2,iz) + \frac{\pi\eta^2 e^{i\frac{\pi}{4}}}{2\sqrt{2qr}}\Gamma(1/2-i\eta)F(3/2+i\eta,2,iz)\right],$$
$$F_C(\boldsymbol{r},\boldsymbol{q},\eta) = -\exp\left(\frac{\pi\eta}{2} - i\boldsymbol{q}\cdot\boldsymbol{r}\right)\frac{\pi\eta^2 e^{i\frac{\pi}{4}}}{8\sqrt{2qr}}\Gamma(1/2-i\eta)F(3/2+i\eta,2,iz),$$
$$z = (1+\boldsymbol{n}\cdot\boldsymbol{n}_q)qr, \quad \boldsymbol{n} = \frac{\boldsymbol{r}}{r}. \tag{29}$$

Here $\Gamma(x)$ is the Euler Γ function and $F(\alpha,\beta,x)$ is the confluent hypergeometric function. The results (28) and (29) are in agreement with that obtained in [23].

IV. CALCULATION OF THE MATRIX ELEMENT

The calculation of the quantities A_{00}, $\boldsymbol{A}_{01}$, $\boldsymbol{A}_{10}$, $\boldsymbol{A}_{02}$, and $\boldsymbol{A}_{20}$ (15) is performed in the same way as in Ref. [20]. We present details of this very tricky calculation in the Appendix. We obtain

$$A_{00} = \frac{1}{\omega m^4}\int d\boldsymbol{r}\exp[-i\boldsymbol{\Delta}\cdot\boldsymbol{r} - i\chi(\rho)]\left[i2\varepsilon_p\varepsilon_q\xi_p\xi_q(\boldsymbol{p}_\perp+\boldsymbol{q}_\perp) + m^2(\varepsilon_p\xi_p - \varepsilon_q\xi_q)\int_0^\infty dx\, x\,\nabla_\perp V(\boldsymbol{r}-x\boldsymbol{v})\right]\cdot\nabla_\perp V(\boldsymbol{r}),$$
$$\boldsymbol{A}_{01} = \frac{\varepsilon_q\xi_q}{\omega m^2}\int d\boldsymbol{r}\exp[-i\boldsymbol{\Delta}\cdot\boldsymbol{r} - i\chi(\rho)]\left[i\nabla_\perp V(\boldsymbol{r}) + \frac{\boldsymbol{\Delta}}{2\varepsilon_p}\int_0^\infty dx\, x\,\nabla_\perp V(\boldsymbol{r}-x\boldsymbol{v})\cdot\nabla_\perp V(\boldsymbol{r}) + \frac{i}{2\varepsilon_p}\nabla_\perp V^2(\boldsymbol{r})\right],$$
$$\boldsymbol{A}_{02} = -\frac{\varepsilon_q\xi_q}{2\omega\varepsilon_p m^2}\int d\boldsymbol{r}\exp[-i\boldsymbol{\Delta}\cdot\boldsymbol{r} - i\chi(\rho)]\int_0^\infty dx\,[\nabla V(\boldsymbol{r}-x\boldsymbol{v})\times\nabla V(\boldsymbol{r})],$$
$$\boldsymbol{A}_{10} = -\boldsymbol{A}_{01}(\varepsilon_q\leftrightarrow\varepsilon_p,\xi_q\leftrightarrow\xi_p), \quad \boldsymbol{A}_{20} = -\boldsymbol{A}_{02}(\varepsilon_q\leftrightarrow\varepsilon_p,\xi_q\leftrightarrow\xi_p),$$
$$\chi(\rho) = \int_{-\infty}^{\infty} V(z,\boldsymbol{\rho})dz, \quad \xi_p = \frac{m^2}{m^2+p_\perp^2}, \quad \xi_q = \frac{m^2}{m^2+q_\perp^2}. \tag{30}$$

Substituting Eq. (30) in Eq. (16), we find the matrix element M,

$$M = -\delta_{\mu_p\mu_q}(\varepsilon_p\delta_{\lambda\mu_p} + \varepsilon_q\delta_{\lambda\bar{\mu}_p})[N_0(\boldsymbol{e}_\lambda^*,\xi_p\boldsymbol{p}_\perp - \xi_q\boldsymbol{q}_\perp) + N_1(\boldsymbol{e}_\lambda^*,\varepsilon_p\xi_p\boldsymbol{p}_\perp - \varepsilon_q\xi_q\boldsymbol{q}_\perp)]$$
$$-\frac{1}{\sqrt{2}}m\mu_p\delta_{\mu_p\bar{\mu}_q}\delta_{\lambda\mu_p}(\varepsilon_p-\varepsilon_q)[N_0(\xi_p-\xi_q) + N_1(\varepsilon_p\xi_p - \varepsilon_q\xi_q)],$$
$$N_0 = \frac{2i}{\omega m^2\Delta_\perp^2}\int d\boldsymbol{r}\exp[-i\boldsymbol{\Delta}\cdot\boldsymbol{r} - i\chi(\rho)]\,\boldsymbol{\Delta}_\perp\cdot\nabla_\perp V(\boldsymbol{r}),$$
$$N_1 = \frac{1}{\omega m^2\varepsilon_p\varepsilon_q}\int d\boldsymbol{r}\exp[-i\boldsymbol{\Delta}\cdot\boldsymbol{r} - i\chi(\rho)]\int_0^\infty dx\, x\,\nabla_\perp V(\boldsymbol{r}-x\boldsymbol{v})\cdot\nabla_\perp V(\boldsymbol{r}). \tag{31}$$

Note that in Eq. (16) the contributions of $\boldsymbol{A}_{02}$ and $\boldsymbol{A}_{20}$ cancel out the contributions of the terms with $\nabla_\perp V^2(\boldsymbol{r})$ in $\boldsymbol{A}_{01}$ and $\boldsymbol{A}_{10}$ (30). The amplitude M is exact in the potential $V(r)$. It contains the leading quasiclassical contribution and the first quasiclassical correction as well. For high-energy bremsstrahlung from electrons in the field of a heavy atom, it is necessary to take into account the effect of screening. For high-energy bremsstrahlung from muons it is necessary to take also into account the finite nuclear radius R (nuclear size effect), because the muon Compton wavelength, $\lambda_\mu = 1/m_\mu = 1.87$ fm, is smaller than R, $R = 7.3$ fm for gold and $R = 7.2$ fm for lead, m_μ is the muon mass.

From (31) we have:

$$\sum_{\lambda\,\mu_q}|M|^2 = S_0 + S_1 + S_2, \quad S_0 = \frac{m^2|N_0|^2}{2}\left[\frac{\Delta^2}{m^2}(\varepsilon_p^2+\varepsilon_q^2)\xi_p\xi_q - 2\varepsilon_p\varepsilon_q(\xi_p-\xi_q)^2\right],$$
$$S_1 = \frac{m^2\mathrm{Re}N_0N_1^*}{2}\left\{\frac{\Delta^2}{m^2}(\varepsilon_p^2+\varepsilon_q^2)(\varepsilon_p+\varepsilon_q)\xi_p\xi_q + \left[(\varepsilon_p^2+\varepsilon_q^2)(\varepsilon_p-\varepsilon_q) - 4\varepsilon_p\varepsilon_q(\varepsilon_p\xi_p - \varepsilon_q\xi_q)\right](\xi_p-\xi_q)\right\},$$
$$S_2 = -\mu_p\mathrm{Im}N_0N_1^*\,\omega^2(\varepsilon_p+\varepsilon_q)\xi_p\xi_q\,[\boldsymbol{p}_\perp\times\boldsymbol{q}_\perp]\cdot\boldsymbol{v}. \tag{32}$$

The quantity S_0 is the even function of η, it contributes to the symmetric term $d\sigma_s(\boldsymbol{p},\boldsymbol{q},\boldsymbol{k},\eta)$ of the cross section (1). The quantity S_1 is the odd function of η, it contributes to the antisymmetric term $d\sigma_a(\boldsymbol{p},\boldsymbol{q},\boldsymbol{k},\eta)$ of the cross section (1). The quantity S_2 is the even function of η, it contributes to the symmetric term $d\sigma_s(\boldsymbol{p},\boldsymbol{q},\boldsymbol{k},\eta)$ of the cross section (1) which vanishes after averaging over the helicity μ_p of the initial electron. Note that the contribution of S_2 to the cross section is responsible for the effect of asymmetry with respect to the replacement $\varphi_i \to -\varphi_i$, where φ_i are the azimuth angles of the final particles in the frame where the z axis is directed along $\boldsymbol{p}$. Such asymmetry is absent in the cross section calculated in the leading quasiclassical approximation. We emphasize that the contributions S_1 and S_2 are nonzero due to accounting for the next-to-leading quasiclassical terms.

The coefficients N_0 and N_1 depend on the momenta $\boldsymbol{p}$, $\boldsymbol{q}$, and $\boldsymbol{k}$ via the momentum transfer $\boldsymbol{\Delta}$. Therefore, it is easy to find from (2) and (32) the cross section $d\sigma/d\omega d\boldsymbol{\Delta}_\perp$. A simple integration gives

$$\frac{d\sigma_s}{d\omega d\boldsymbol{\Delta}_\perp} = \frac{\alpha\omega\varepsilon_q m^4}{(2\pi)^3\varepsilon_p}|N_0|^2\Phi(\zeta), \quad \frac{d\sigma_a}{d\omega d\boldsymbol{\Delta}_\perp} = \frac{\alpha\omega\varepsilon_q(\varepsilon_p+\varepsilon_q)m^4}{(2\pi)^3\varepsilon_p}\mathrm{Re}N_0N_1^*\Phi,$$
$$\Phi = \frac{\ln(\zeta+\sqrt{1+\zeta^2})}{\zeta\sqrt{1+\zeta^2}}\left(\zeta^2\frac{\varepsilon_p^2+\varepsilon_q^2}{\varepsilon_p\varepsilon_q}+1\right)-1, \quad \zeta = \frac{\Delta_\perp}{2m}. \tag{33}$$

The function Φ has the following asymptotic forms:

$$\Phi = \left(\frac{\varepsilon_p^2+\varepsilon_q^2}{\varepsilon_p\varepsilon_q}-\frac{2}{3}\right)\zeta^2 \quad \text{at } \zeta \ll 1, \quad \Phi = \frac{\varepsilon_p^2+\varepsilon_q^2}{\varepsilon_p\varepsilon_q}\ln(2\zeta)-1 \quad \text{at } \zeta \gg 1. \tag{34}$$

V. CHARGE ASYMMETRY IN HIGH-ENERGY BREMSSTRAHLUNG FROM ELECTRONS

As is known, the main contribution to the Coulomb corrections to the symmetric part of the differential cross section of bremsstrahlung is given by the region $\Delta \sim \max(r_{scr}^{-1}, \Delta_{\min})$ [16,20], where $\Delta_{\min} = p - q - \omega \approx m^2\omega/2\varepsilon_q\varepsilon_p$. However, the main contribution to the charge asymmetry is given by the region $\Delta \gg \max(r_{scr}^{-1}, \Delta_{\min})$. In this region we can neglect the effect of screening, replace $V(r)$ by the Coulomb potential $V_c(r) = -\eta/r$, and neglect also $\Delta_\parallel$ in comparison with $\Delta_\perp$. A simple calculation gives for the coefficient N_0 and N_1 in (31):

$$N_0 = \frac{8\pi\eta(L\Delta)^{2i\eta}}{\omega m^2\Delta^2}\frac{\Gamma(1-i\eta)}{\Gamma(1+i\eta)},$$
$$N_1 = \frac{2\pi^2\eta^2(L\Delta)^{2i\eta}}{\omega m^2\varepsilon_p\varepsilon_q\Delta}\frac{\Gamma(1/2-i\eta)}{\Gamma(1/2+i\eta)}, \tag{35}$$

where $L \sim \min(\varepsilon_p/m^2, r_{scr})$. Note that the factor $(L\Delta)^{2i\eta}$ is irrelevant because it disappears in $|M|^2$. Then we obtain for the coefficients in S_0, S_1, and S_2 (32):

$$|N_0|^2 = \left(\frac{8\pi\eta}{\omega m^2\Delta^2}\right)^2, \quad \mathrm{Re}N_0N_1^* = \frac{\pi\mathrm{Re}g(\eta)\Delta}{4\varepsilon_p\varepsilon_q}|N_0|^2,$$
$$\mathrm{Im}N_0N_1^* = \frac{\pi\mathrm{Im}g(\eta)\Delta}{4\varepsilon_p\varepsilon_q}|N_0|^2,$$
$$g(\eta) = \eta\frac{\Gamma(1-i\eta)\Gamma(1/2+i\eta)}{\Gamma(1+i\eta)\Gamma(1/2-i\eta)}. \tag{36}$$

As it should, in the region $\Delta \sim m$ there are no Coulomb corrections to $d\sigma_s(\boldsymbol{p},\boldsymbol{q},\boldsymbol{k},\eta)$ calculated in the leading quasiclassical approximation. The Coulomb corrections to $d\sigma_a(\boldsymbol{p},\boldsymbol{q},\boldsymbol{k},\eta)$ (the term S_1) and $d\sigma_s(\boldsymbol{p},\boldsymbol{q},\boldsymbol{k},\eta)$ (the term S_2) are accumulated in the functions $\mathrm{Re}g(\eta)$ and $\mathrm{Im}g(\eta)$, respectively. These functions are shown in Fig. 1. At $\eta \ll 1$ we have $\mathrm{Re}g(\eta) \approx \eta$ and $\mathrm{Im}g(\eta) \approx -(4\ln 2)\eta^2$. It is seen from Fig. 1 that the functions $\mathrm{Re}g(\eta)$ and $\mathrm{Im}g(\eta)$ differ significantly from their small-η asymptotic forms already at very small η.

At $\omega \ll \varepsilon_p$, the ratio of the antisymmetric part of the cross section to the symmetric one,

$$\frac{S_1}{S_0} = \frac{\pi\Delta\mathrm{Re}g(\eta)}{2\varepsilon_p}, \tag{37}$$

increases with Δ/ε_p and can be more than ten percent. The ration S_2/S_0 is small at $\omega \ll \varepsilon_p$ because it is suppressed by the factor $(\omega/\varepsilon_p)^2$.

If $|\boldsymbol{p}_\perp| \gg m$ and $|\boldsymbol{q}_\perp| \gg m$, then

$$\frac{S_1}{S_0} = \frac{\pi\mathrm{Re}g(\eta)}{2\Delta}\boldsymbol{\Delta}\cdot\boldsymbol{\theta}_{qp},$$
$$\frac{S_2}{S_0} = \mu_p\frac{\pi\omega(\varepsilon_p+\varepsilon_q)\mathrm{Im}g(\eta)}{2(\varepsilon_p^2+\varepsilon_q^2)\Delta}[\boldsymbol{\Delta}\times\boldsymbol{\theta}_{qp}]\cdot\boldsymbol{\nu}, \tag{38}$$

where $\boldsymbol{\theta}_{qp} = \boldsymbol{p}_\perp/p - \boldsymbol{q}_\perp/q$. Thus, the azimuth asymmetry increases with ω and may be important.

Let us discuss the cross section $d\sigma/d\omega d\boldsymbol{\Delta}_\perp$ at $\Delta \gg \max(r_{scr}^{-1}, \Delta_{\min})$, Eq. (33):

$$\frac{d\sigma_s}{d\omega d\boldsymbol{\Delta}_\perp} = \frac{8\alpha\eta^2\varepsilon_q}{\pi\omega\varepsilon_p\Delta_\perp^4}\Phi,$$
$$\frac{d\sigma_a}{d\omega d\boldsymbol{\Delta}_\perp} = \frac{\pi\mathrm{Re}g(\eta)(\varepsilon_p+\varepsilon_q)\Delta}{4\varepsilon_p\varepsilon_q}\frac{d\sigma_s}{d\omega d\boldsymbol{\Delta}_\perp}. \tag{39}$$

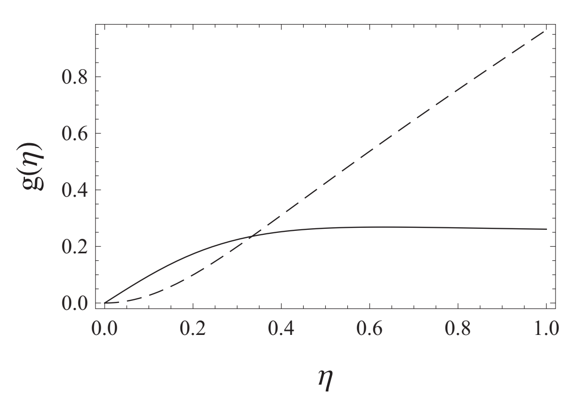

FIG. 1. The functions $\mathrm{Re}g(\eta)$ (solid curve) and $-\mathrm{Im}g(\eta)$ (dashed curve), Eq. (36).

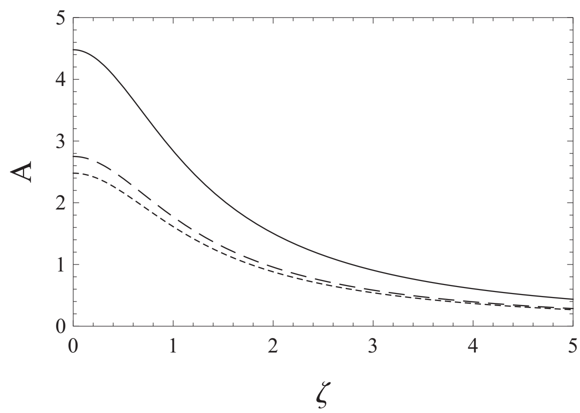

FIG. 2. Dependence of $A=\sigma_0^{-1}d\sigma_a/d\omega d\mathbf{\Delta}_\perp$ on $\zeta=\Delta_\perp/2m$, Eq. (39), for a few values of $t=\varepsilon_q/\varepsilon_p$; $\sigma_0=\alpha\eta^2\mathrm{Reg}(\eta)/(2m^2\omega\varepsilon_p\Delta_\perp)$: $t=0.25$ (solid curve), $t=0.5$ (dashed curve), and $t=0.75$ (dotted curve).

In Fig. 2 we show the dependence of $A=\sigma_0^{-1}d\sigma_a/d\omega d\mathbf{\Delta}_\perp$ on $\zeta=\Delta_\perp/2m$ for a few values of $t=\varepsilon_q/\varepsilon_p$; $\sigma_0=\alpha\eta^2\mathrm{Reg}(\eta)/(2m^2\omega\varepsilon_p\Delta_\perp)$. This figure confirms our statement that the main contribution to the antisymmetric part of the cross section is given by the region $\Delta\sim m$.

Performing integration over $\mathbf{\Delta}_\perp$ in (33), we obtain the antisymmetric correction to the spectrum

$$\frac{d\sigma_a}{d\omega}=\frac{\alpha\pi^3\eta^2\mathrm{Reg}(\eta)}{4m\omega\varepsilon_p^2}\left[2\frac{\varepsilon_p^2+\varepsilon_q^2}{\varepsilon_p\varepsilon_q}-1\right](\varepsilon_p+\varepsilon_q). \quad (40)$$

This result coincides with the corresponding result of Refs. [19,20].

In Fig. 3 we show the dependence of $A_1=\sigma_1^{-1}d\sigma_a/d\omega d\boldsymbol{k}_\perp$ on $\zeta_1=k_\perp/m$ for a few values of $t=\varepsilon_q/\varepsilon_p$; $\sigma_1=\alpha\eta^2\mathrm{Reg}(\eta)/(2m^3\omega\varepsilon_p)$. Here $\boldsymbol{k}_\perp$ is the component of $\boldsymbol{k}$ perpendicular to the vector $\boldsymbol{p}$, $\boldsymbol{k}_\perp=-\omega\boldsymbol{p}_\perp/p$. The result is obtained by numerical integration of the differential cross section $d\sigma_a(\boldsymbol{p},\boldsymbol{q},\boldsymbol{k},\eta)$ over $d\boldsymbol{q}_\perp$.

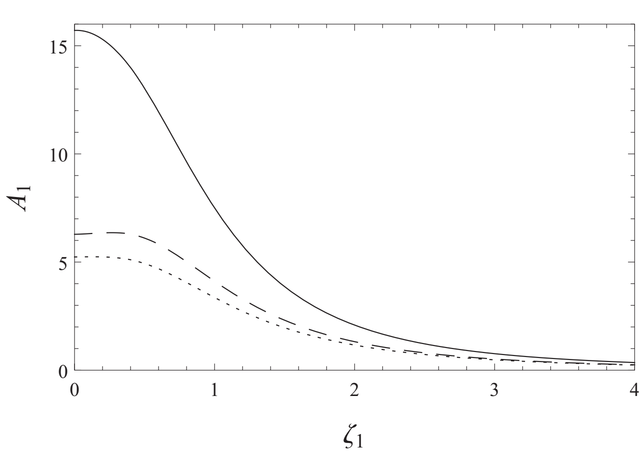

FIG. 3. Dependence of $A_1=\sigma_1^{-1}d\sigma_a/d\omega d\boldsymbol{k}_\perp$ on $\zeta_1=k_\perp/m$, for a few values of $t=\varepsilon_q/\varepsilon_p$; $\sigma_1=\alpha\eta^2\mathrm{Reg}(\eta)/(2m^3\omega\varepsilon_p)$: $t=0.25$ (solid curve), $t=0.5$ (dashed curve), and $t=0.75$ (dotted curve).

As is known (see, e.g., Ref. [5]), the cross section $d\sigma_{\mathrm{brem}}(\omega,\varepsilon_p,k_\perp)/d\omega d\boldsymbol{k}_\perp$ of bremsstrahlung (ε_p is the initial electron energy) can be obtained from the cross section $d\sigma_{\mathrm{photo}}(\omega,\epsilon_p,p_\perp)/d\varepsilon_p d\boldsymbol{p}_\perp$ of photoproduction (ε_p is the positron energy) by the relation

$$\frac{d\sigma_{\mathrm{brem}}(\omega,\epsilon_p,k_\perp)}{d\omega d\boldsymbol{k}_\perp}=\frac{\varepsilon_p^2 d\sigma_{\mathrm{photo}}(-\omega,-\epsilon_p,p_\perp)}{\omega^2 d\varepsilon_p d\boldsymbol{p}_\perp}, \quad (41)$$

where $p_\perp=pk_\perp/\omega$. The antisymmetric part of $d\sigma_{\mathrm{photo}}(\omega,\epsilon_p,p_\perp)/d\varepsilon_p d\boldsymbol{p}_\perp$ was obtained in Ref. [23]. Using Fig. 2 in that paper and Eq. (41), we find that our result shown in Fig. 3 is in agreement with the corresponding result in Ref. [23].

VI. CHARGE ASYMMETRY IN HIGH-ENERGY BREMSSTRAHLUNG FROM MUONS

The charge asymmetry in the differential cross section of high-energy $\mu^+\mu^-$ photoproduction in the electric field of a heavy atom was investigated in detail in Ref. [24]. The deviation of the nuclear electric field from the Coulomb field at small distances due to the finite nuclear radius R (nuclear size effect) is crucially important for the charge asymmetry. Though the Coulomb corrections to the total cross section are negligibly small, it was shown in Ref. [24] that the charge asymmetry is not negligible for selected final states of μ^+ and μ^-. In this section we study the charge asymmetry in the differential cross section of high-energy bremsstrahlung from muons in the field of a heavy atom.

We write the Fourier transformation $V_F(\Delta^2)$ of the potential $V(r)$ as

$$V_F(\Delta^2)=-\frac{4\pi\eta F(\Delta^2)}{\Delta^2}, \quad (42)$$

where $F(\Delta^2)$ is the form factor, which differs essentially from unity at $\Delta\gtrsim 1/R$ and $Q\lesssim 1/r_{scr}$. Let us first discuss the Coulomb corrections to the symmetric part of the cross section calculated in the leading quasiclassical approximation. In this case the cross section $d\sigma_s$ depends on the parameters of the field via the factor N_0 (31). In the Born approximation

$$N_{0B}=\frac{2i}{\omega m^2\Delta^2}\int d\boldsymbol{r}\exp(-i\mathbf{\Delta}\cdot\boldsymbol{r})\mathbf{\Delta}\cdot\nabla_\perp V(\boldsymbol{r})$$
$$=-\frac{2V_F(\Delta^2)}{\omega m^2}. \quad (43)$$

We define the Coulomb corrections $\mathcal{N}_0$ to the quantity $|N_0|^2$ as

$$\mathcal{N}_0=|N_0|^2-|N_{0B}|^2. \quad (44)$$

The quantity $\mathcal{N}_0$ vanishes at $r_{scr}^{-1}\ll\Delta\ll R^{-1}$ and has two peaks: at $\Delta\sim r_{scr}^{-1}$ and at $\Delta\sim R^{-1}$. The contributions of these peaks to the integral $\int\Delta^2\mathcal{N}_0 d\mathbf{\Delta}_\perp$ are [20]

$$\int\Delta^2\mathcal{N}_0 d\mathbf{\Delta}_\perp=\mp\frac{128\pi^3\eta^2 f(\eta)}{\omega^2 m^4},$$
$$f(\eta)=\mathrm{Re}\psi(1+i\eta)-\psi(1), \quad (45)$$

where $\psi(x)=d\ln\Gamma(x)/dx$, the negative contribution corresponds to the peak at $\Delta\sim r_{scr}^{-1}$, and the positive contribution corresponds to the peak at $\Delta\sim R^{-1}$. These two contributions

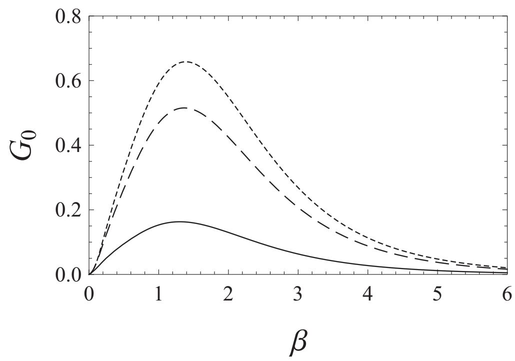

FIG. 4. Dependence of $G_0 = |N_0|^2/|N_{0B}|^2 - 1$ on $\beta = \Delta/\Lambda$ at $\beta \gg 1/(r_{scr}\Lambda)$ and a few values of η; $\eta = 0.34$ (Ag, solid curve), $\eta = 0.6$ (Pb, dashed curve), and $\eta = 0.67$ (U, dotted curve).

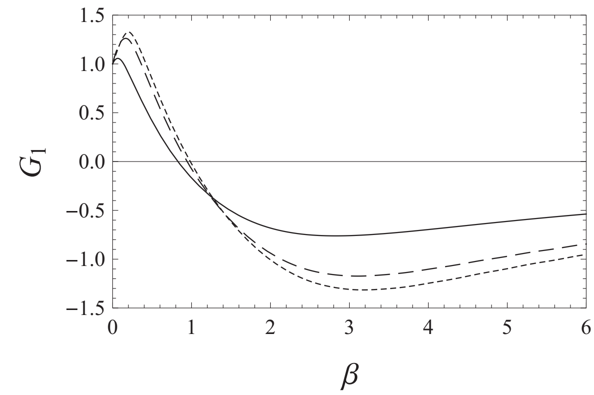

FIG. 5. Dependence of $G_1 = \Sigma_R^{-1}\mathrm{Re}N_0N_1^*/|N_0|^2$ on $\beta = \Delta/\Lambda$ (52), for a few values of η; $\eta = 0.34$ (Ag, solid curve), $\eta = 0.6$ (Pb, dashed curve), and $\eta = 0.67$ (U, dotted curve).

are the universal functions of η independent of the form of the potential in the regions $r \sim r_{scr}$ and $r \sim R$, while the function $\mathcal{N}_0$ is very sensitive to the form of the potential in these regions [16,20]. Since $m \ll R^{-1}$ for electrons, only the region $r \sim r_{scr}$ gives the Coulomb corrections to $d\sigma_s/d\omega$ [16],

$$\frac{d\sigma_C}{d\omega} = -\frac{4\alpha\eta^2 f(\eta)}{m^2\omega}\left(t^2 - \frac{2}{3}t + 1\right), \quad t = \frac{\varepsilon_q}{\varepsilon_p}. \tag{46}$$

For muons $m_\mu \gg R^{-1}$, so that the sum of the contributions from both peaks, $\Delta \gtrsim 1/R$ and $Q \lesssim 1/r_{scr}$, gives the total Coulomb corrections to $d\sigma_s/d\omega$. As a result, the total Coulomb corrections vanish, see Eq. (45). However, the Coulomb corrections to the differential cross section at $\Delta \sim R^{-1}$ are large. To illustrate this statement, we consider the form factor $F(\Delta^2)$ in the form

$$F(\Delta^2) = \frac{\Lambda^2}{\Delta^2 + \Lambda^2}, \tag{47}$$

where $\Lambda \sim 60\,\mathrm{MeV}$ for heavy nuclei. This form is valid for $\Delta \gg r_{scr}^{-1}$, where the factor N_0 is given by

$$N_0 = \frac{8\pi\eta}{\omega m^2\Delta^2}\int_0^\infty d\rho\, J_1(\rho)\left[1 - \frac{\rho}{\beta}K_1\left(\frac{\rho}{\beta}\right)\right] \times \exp\left\{-2i\eta\left[\ln\frac{\rho}{2} + K_0\left(\frac{\rho}{\beta}\right)\right]\right\},$$
$$N_{0B} = \frac{8\pi\eta}{\omega m^2\Delta^2(1+\beta^2)}, \quad \beta = \frac{\Delta}{\Lambda}. \tag{48}$$

Here $J_n(x)$ is the Bessel function and $K_n(x)$ is the modified Bessel function of the second kind. In Fig. 4 we show the dependence of $G_0 = |N_0|^2/|N_{0B}|^2 - 1$ on $\beta = \Delta/\Lambda$ for a few values of η.

Note that very narrow peak at $\Delta \sim r_{scr}^{-1}$ [$\delta\beta \sim 1/(r_{scr}\Lambda) \ll 1$] is not shown in this figure. The dependence of the peak on the shape of the atomic potential at $\Delta \sim r_{scr}^{-1}$ was investigated in detail in Ref. [20]. It is seen from Fig. 4 that the Coulomb corrections to $|N_0|^2$ are significant in the region $\Delta/\Lambda \sim 1$.

Let us consider the factor N_1 (31). In the lowest in η approximation it reads

$$N_{1B} = \frac{\mathcal{J}(\Delta)}{\omega m^2\varepsilon_p\varepsilon_q},$$
$$\mathcal{J}(\Delta) = \int\frac{d\boldsymbol{s}}{(2\pi)^3}[V_F(Q_+)V_F(Q_-) + (\Delta^2 - 4s_\parallel^2)V_F(Q_+)V_F'(Q_-)],$$
$$Q_\pm = (\boldsymbol{s} \pm \boldsymbol{\Delta}/2)^2, \quad s_\parallel = \boldsymbol{s}\cdot\boldsymbol{\Delta}/\Delta, \tag{49}$$

where $V_F'(Q) = \partial V_F(Q)/\partial Q$, see Ref. [24]. For the form factor (47), the explicit form of $\mathcal{J}(\Delta)$ is given in Ref. [24]:

$$\mathcal{J}(\Delta) = \frac{2\pi^2\eta^2}{\Delta}\mathcal{F}(\beta), \quad \beta = \frac{\Delta}{\Lambda},$$
$$\mathcal{F}(\beta) = 1 + \frac{2}{\pi}\arcsin\frac{\beta}{\sqrt{\beta^2+4}} - \frac{4}{\pi}\arcsin\frac{\beta}{\sqrt{\beta^2+1}} - \frac{12\beta}{\pi(\beta^2+4)(\beta^2+1)}. \tag{50}$$

The exact in η factor N_1 at $\Delta \gg r_{scr}^{-1}$ is given by

$$N_1 = \frac{2\pi^2\eta^2}{\omega m^2\varepsilon_p\varepsilon_q\Delta}\int_0^\infty\int_0^\infty dx d\rho\, \mathcal{F}(\beta x/\rho)\, J_0(\rho)J_0(x) \times \exp\left\{-2i\eta\left[\ln\frac{\rho}{2} + K_0\left(\frac{\rho}{\beta}\right)\right]\right\}. \tag{51}$$

To demonstrate the influence of the nuclear size effect on the ratios S_1/S_0 and S_2/S_0 (32), we plot in Figs. 5 and 6 the quantities G_1 and G_2,

$$G_1 = \frac{\mathrm{Re}N_0N_1^*}{|N_0|^2\Sigma_R}, \quad \Sigma_R = \frac{\pi\mathrm{Re}g(\eta)\Delta}{4\varepsilon_p\varepsilon_q},$$
$$G_2 = \frac{\mathrm{Im}N_0N_1^*}{|N_0|^2\Sigma_I}, \quad \Sigma_I = \frac{\pi\mathrm{Im}g(\eta)\Delta}{4\varepsilon_p\varepsilon_q}, \tag{52}$$

as a function of $\beta = \Delta/\Lambda$. For a pure Coulomb field $G_1 = G_2 = 1$ (36).

It is seen that the quantities G_1 and G_2 decrease rapidly with increasing β at $\beta \lesssim 1$.

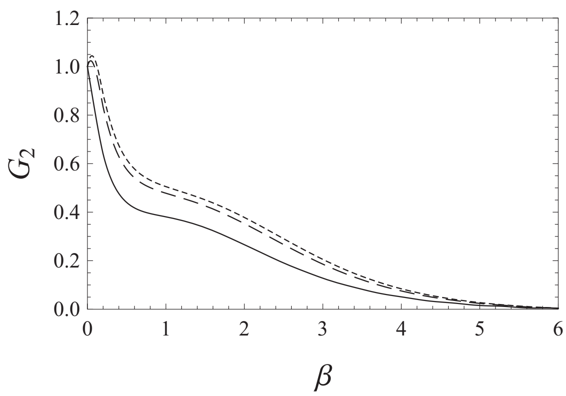

FIG. 6. Dependence of $G_2 = \Sigma_l^{-1}\mathrm{Im}N_0N_1^*/|N_0|^2$ on $\beta = \Delta/\Lambda$ (52), for a few values of η; $\eta = 0.34$ (Ag, solid curve), $\eta = 0.6$ (Pb, dashed curve), and $\eta = 0.67$ (U, dotted curve).

VII. CONCLUSION

We have investigated in detail the charge asymmetry in the process of high-energy bremsstrahlung in the field of a heavy atom. The charge asymmetry arises due to the account for the first quasiclassical correction to the differential cross section of the process. The results are exact in the parameters of the atomic field and are valid even for $\eta \sim 1$, they take into account the effect of screening and the nuclear size effect. The latter is important for high-energy bremsstrahlung from muons where the charge asymmetry is very sensitive to the shape of the nuclear form factor. It is shown that the Coulomb corrections essentially modify the charge asymmetry as compared with the leading in η result already for the relatively small η. In the experimental region of interest, where $\varepsilon_p \gg p_\perp \gg m$ and $\varepsilon_q \gg q_\perp \gg m$ but $\Delta \lesssim 1/R$, the asymmetry can be as large as a few tens of percent. For the longitudinal polarization of the initial charged particle, due to the account for the first quasiclassical correction, the differential cross section reveals the asymmetry with respect to the replacement $\varphi \to -\varphi$, where φ is the azimuth angle between the photon momentum $\boldsymbol{k}$ and the momentum $\boldsymbol{q}$ of the final charged particle in the frame where the z axis is directed along $\boldsymbol{p}$. Due to account for the first quasiclassical correction, our results for the differential cross section of high-energy bremsstrahlung have essentially higher precision than the famous results in Ref. [16] and should be taken into account in precision experiments and at data analysis in detectors.

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APPENDIX

In this Appendix, following the method of Ref. [20], we derive the expression (30) for the quantity A_{00},

$$A_{00} = \int d\boldsymbol{r}\, \exp(-i\boldsymbol{k}\cdot\boldsymbol{r}) f_0 g_0, \tag{A1}$$

where the function $f_0(\boldsymbol{r},\boldsymbol{q})$ is given in (19) and $g_0(\boldsymbol{r},\boldsymbol{p}) = f_0(\boldsymbol{r},-\boldsymbol{p})$. Other quantities in (30) are calculated in the same way. We split the integration region into two, $z > 0$ and $z < 0$, and denote the corresponding contributions to A_{00} as A_{00}^+ and A_{00}^-. For $z > 0$, the function f_0 has a simple eikonal form

$$f_0(\boldsymbol{r},\boldsymbol{q}) = e^{-i\boldsymbol{q}\cdot\boldsymbol{r}} \exp\left[-i\int_0^\infty dx\, V(\boldsymbol{r} + x\boldsymbol{n}_{\boldsymbol{q}})\right], \tag{A2}$$

so that

$$\begin{aligned} A_{00}^+ = &\int_{z>0} d\boldsymbol{r} \int \frac{d\boldsymbol{Q}}{i\pi} \exp\left\{iQ^2 - i\boldsymbol{\Delta}\cdot\boldsymbol{r} - i\int_0^\infty dx[V(\boldsymbol{r}_x) + V(\boldsymbol{r} + x\boldsymbol{n}_{\boldsymbol{q}})]\right\} \\ &\times \left[1 + \frac{i}{2\varepsilon_p}\int_0^\infty dx \int_0^x dy(x-y)\boldsymbol{\nabla}_\perp V(\boldsymbol{r}_x)\cdot\boldsymbol{\nabla}_\perp V(\boldsymbol{r}_y)\right], \end{aligned} \tag{A3}$$

where $\boldsymbol{r}_x = \boldsymbol{r} - x\boldsymbol{n}_{\boldsymbol{p}} + \boldsymbol{Q}\sqrt{2r/p}$. Within our accuracy we can replace the quantity $V(\boldsymbol{r} + x\boldsymbol{n}_{\boldsymbol{q}})$ in (A3) by $V(\boldsymbol{r} + x\boldsymbol{n}_{\boldsymbol{q}} + \boldsymbol{Q}\sqrt{2r/p})$, shift $\boldsymbol{\rho} \to \boldsymbol{\rho} - \boldsymbol{Q}\sqrt{2r/p}$, and take the integral over $\boldsymbol{Q}$. We obtain

$$\begin{aligned} A_{00}^+ = &\int_{z>0} d\boldsymbol{r}\, \exp\left\{-i\frac{z}{2p}\Delta_\perp^2 - i\boldsymbol{\Delta}\cdot\boldsymbol{r} - i\int_0^\infty dx[V(\boldsymbol{r} - x\boldsymbol{n}_{\boldsymbol{p}}) + V(\boldsymbol{r} + x\boldsymbol{n}_{\boldsymbol{q}})]\right\} \\ &\times \left[1 + \frac{i}{2\varepsilon_p}\int_0^\infty dx \int_0^x dy(x-y)\boldsymbol{\nabla}_\perp V(\boldsymbol{r} - x\boldsymbol{n}_{\boldsymbol{p}})\cdot\boldsymbol{\nabla}_\perp V(\boldsymbol{r} - y\boldsymbol{n}_{\boldsymbol{p}})\right]. \end{aligned} \tag{A4}$$

In the same way, we obtain

$$A_{00}^{-} = \int_{z<0} d\boldsymbol{r} \exp\left\{i\frac{z}{2q}\Delta_\perp^2 - i\boldsymbol{\Delta}\cdot\boldsymbol{r} - i\int_0^\infty dx[V(\boldsymbol{r}-x\boldsymbol{n_p}) + V(\boldsymbol{r}+x\boldsymbol{n_q})]\right\}$$
$$\times\left[1 + \frac{i}{2\varepsilon_q}\int_0^\infty dx\int_0^x dy(x-y)\nabla_\perp V(\boldsymbol{r}+x\boldsymbol{n_q})\cdot\nabla_\perp V(\boldsymbol{r}+y\boldsymbol{n_q})\right]. \quad \text{(A5)}$$

There are two overlapping regions of the momentum transfer Δ:

$$\text{I. } \Delta \ll \frac{m\omega}{\varepsilon_p} \quad \text{II. } \Delta \gg \Delta_{\min} = \frac{m^2\omega}{2\varepsilon_p\varepsilon_q}. \quad \text{(A6)}$$

In the first region, one can neglect the term proportional to $\Delta_\perp^2$ in the exponents in (A4) and (A5). Then the sum $A_{00} = A_{00}^{+} + A_{00}^{-}$ reads

$$A_{00} = \int d\boldsymbol{r} \exp\left\{-i\boldsymbol{\Delta}\cdot\boldsymbol{r} - i\int_0^\infty dx[V(\boldsymbol{r}-x\boldsymbol{n_p}) + V(\boldsymbol{r}+x\boldsymbol{n_q})\right\}$$
$$\times\left[1 + \frac{i}{2\varepsilon_p}\int_0^\infty dx\int_0^x dy(x-y)\nabla_\perp V(\boldsymbol{r}-x\boldsymbol{n_p})\cdot\nabla_\perp V(\boldsymbol{r}-y\boldsymbol{n_p})\right.$$
$$\left.+ \frac{i}{2\varepsilon_q}\int_0^\infty dx\int_0^x dy(x-y)\nabla_\perp V(\boldsymbol{r}+x\boldsymbol{n_q})\cdot\nabla_\perp V(\boldsymbol{r}+y\boldsymbol{n_q})\right]. \quad \text{(A7)}$$

In the prefactor we make the replacement $\boldsymbol{n_p},\boldsymbol{n_q} \to \boldsymbol{v}$, and in the exponent we take into account the linear term of expansion in $\boldsymbol{n_p} - \boldsymbol{v}$ and $\boldsymbol{n_q} - \boldsymbol{v}$ of the integral. Besides, in the arguments of the functions $V(\boldsymbol{r}+y\boldsymbol{v})$ and $V(\boldsymbol{r}-y\boldsymbol{v})$ we make the substitutions $z \to z - y$ and $z \to z + y$, respectively. After that we take the integral over y and obtain the contribution of the first region

$$A_{00} = \int d\boldsymbol{r} \exp[-i\boldsymbol{\Delta}\cdot\boldsymbol{r} - i\chi(\rho)]\nabla_\perp V(\boldsymbol{r})\cdot\left[i\frac{\boldsymbol{\theta}_{qp}}{\Delta_z^2} - \frac{\omega}{2\Delta_z\varepsilon_p\varepsilon_q}\int_0^\infty dx x\nabla_\perp V(\boldsymbol{r}-x\boldsymbol{v})\right],$$
$$\chi(\rho) = \int_{-\infty}^{\infty} V(z,\boldsymbol{\rho})dz, \quad \text{(A8)}$$

where $\Delta_z = \boldsymbol{v}\cdot\boldsymbol{\Delta}$ and $\boldsymbol{\theta}_{qp} = \boldsymbol{q}_\perp/q - \boldsymbol{p}_\perp/p$.

Now we pass to the calculation of A_{00} in the second region (A6). In Eq. (A4) for A_{00}^{+}, we make the replacement $\boldsymbol{n}_q \to \boldsymbol{n}_p$ and $z\Delta_\perp^2/2p \to (\boldsymbol{n}_q\cdot\boldsymbol{n})\Delta_\perp^2/2p$. The polar angle of $\boldsymbol{n}$ is small, and we can integrate in (A4) over the region $\boldsymbol{n}_q\cdot\boldsymbol{n} > 0$. After the integration over z, we obtain

$$A_{00}^{+} = \frac{1}{\boldsymbol{\Delta}\cdot\boldsymbol{n}_p + \Delta_\perp^2/2p}\int d\boldsymbol{\rho}\exp[-i\boldsymbol{\Delta}_\perp\cdot\boldsymbol{\rho} - i\chi(\rho)]\left[-i + \int_{-\infty}^\infty dz\int_0^\infty dx x\nabla_\perp V(\boldsymbol{r})\cdot\nabla_\perp V(\boldsymbol{r}-x\boldsymbol{n}_p)\right]. \quad \text{(A9)}$$

The calculation of A_{00}^{-} is performed quite similarly. As a result we have

$$A_{00}^{-} = \frac{1}{-\boldsymbol{\Delta}\cdot\boldsymbol{n}_q + \Delta_\perp^2/2q}\int d\boldsymbol{\rho}\exp[-i\boldsymbol{\Delta}_\perp\cdot\boldsymbol{\rho} - i\chi(\rho)]\left[-i + \int_{-\infty}^\infty dz\int_0^\infty dx x\nabla_\perp V(\boldsymbol{r})\cdot\nabla_\perp V(\boldsymbol{r}-x\boldsymbol{n}_q)\right]. \quad \text{(A10)}$$

Taking into account that

$$\boldsymbol{\Delta}\cdot\boldsymbol{n}_p + \Delta_\perp^2/2p = -\frac{m^2\omega}{2\varepsilon_p\varepsilon_q\xi_q}, \quad -\boldsymbol{\Delta}\cdot\boldsymbol{n}_q + \Delta_\perp^2/2q = \frac{m^2\omega}{2\varepsilon_p\varepsilon_q\xi_p}, \quad \xi_p = \frac{m^2}{m^2+\boldsymbol{p}_\perp^2}, \quad \xi_q = \frac{m^2}{m^2+\boldsymbol{q}_\perp^2},$$

we obtain for $A_{00} = A_{00}^{+} + A_{00}^{-}$ in the second region

$$A_{00} = \frac{1}{m^4\omega}\int d\boldsymbol{\rho}\exp[-i\boldsymbol{\Delta}_\perp\cdot\boldsymbol{\rho} - i\chi(\rho)]$$
$$\times\left[2i\varepsilon_p\varepsilon_q\xi_p\xi_q(\boldsymbol{p}_\perp + \boldsymbol{q}_\perp)\cdot\nabla_\perp\chi(\rho) + m^2(\varepsilon_p\xi_p - \varepsilon_q\xi_q)\int_{-\infty}^\infty dz\int_0^\infty dx x\nabla_\perp V(\boldsymbol{r}-x\boldsymbol{v})\cdot\nabla_\perp V(\boldsymbol{r})\right]. \quad \text{(A11)}$$

Now we can compare (A11) and (A8) and write the expression for A_{00}, which is valid in all region of Δ. We finally arrive at the following result:

$$A_{00} = \frac{1}{m^4\omega}\int d\boldsymbol{r}\exp[-i\boldsymbol{\Delta}\cdot\boldsymbol{r} - i\chi(\rho)]\left[2i\varepsilon_p\varepsilon_q\xi_p\xi_q(\boldsymbol{p}_\perp + \boldsymbol{q}_\perp) + m^2(\varepsilon_p\xi_p - \varepsilon_q\xi_q)\int_0^\infty dx x\nabla_\perp V(\boldsymbol{r}-x\boldsymbol{v})\right]\cdot\nabla_\perp V(\boldsymbol{r}). \quad \text{(A12)}$$

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