Asymmetric transmission through a flux-controlled non-Hermitian scattering center

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We study the possibility of asymmetric transmission induced by a non-Hermitian scattering center embedded in a one-dimensional waveguide, motivated by the aim of realizing quantum diodes in a non-Hermitian system. It is shown that a $\mathcal{PT}$-symmetric non-Hermitian scattering center always has symmetric transmission although the dynamics within the isolated center can be unidirectional, especially at its exceptional point. We propose a concrete scheme based on a flux-controlled non-Hermitian scattering center, which comprises a non-Hermitian triangular ring threaded by an Aharonov-Bohm flux. The analytical solution shows that such a complex scattering center acts as a diode at the resonant energy level of the spectral singularity, exhibiting perfect unidirectionality of the transmission. The connections between the phenomena of the asymmetric transmission and reflectionless absorption are also discussed.

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I. INTRODUCTION

Asymmetric transmission is of significant interest in the quantum analogues of electronic devices, such as quantum diode device, which is the key to quantum information processing in integrated circuits [1]. It is characterized by the nonreciprocal particle transport along the opposite directions. Recently, it has been reported that the unidirectional transport can be realized in practical systems [2–5]. A non-Hermitian Hamiltonian can possess peculiar features that have no Hermitian counterpart. A typical one is nonreciprocal dynamics, which has been observed in experiments [6]. However, it was not paid due attention by the physics community until the discovery of non-Hermitian Hamiltonians with parity-time symmetry, which have a real spectrum [7]. It has boosted the research on the complex extension of quantum mechanics on a fundamental level [8–18]. Recently, the concept of spectral singularity of a non-Hermitian system has gained a lot of attention [19–28], motivated by the possible physical relevance of this since the pioneering work of Mostafazadeh [29]. The majority of previous works focused on the non-Hermitian system in the absence of an external magnetic field [21,30–43].

The aim of this work is to study the possibility of asymmetric transmission induced by a non-Hermitian scattering center embedded in a one-dimensional waveguide, motivated by the recent investigation on the physical relevance of a spectral singularity. It is shown that a $\mathcal{PT}$-symmetric non-Hermitian scattering center always has symmetric transmission although the nonreciprocal dynamics within the isolated center is allowed, especially at its exceptional point [31,44]. We consider a $\mathcal{PF}$-symmetric non-Hermitian scattering center, which comprises a non-Hermitian triangular ring threaded by an Aharonov-Bohm flux, where $\mathcal{F}$ is the action of flipping the flux. We show that such a complex scattering center acts as a diode, which is characterized by the different performances of transmission coefficients along the opposite directions. Furthermore, it is found that the perfect unidirectionality of the transmission is a signature of the existence of a spectral singularity. And the criterion for spectral singularity by transfer matrix is not applicable to the present system due to the presence of the magnetic field.

This paper is organized as follows. In Sec. II, we present a general formalism for the scattering problem. In Sec. III, the Hamiltonian for asymmetric transmission is constructed and the analytical scattering solution is obtained. In Sec. IV, we study the connection between the perfect unidirectionality and the spectral singularity. Finally, we give a summary and discussion in Sec. V.

II. SYMMETRIC TRANSMISSION

In this section, we present a general formalism for one-dimensional scattering process of several types of scattering centers [45,46]. We will show that asymmetric transmission, which is the base of a quantum diode, cannot be realized via $\mathcal{P}$-, $\mathcal{T}$-, or $\mathcal{PT}$-symmetric non-Hermitian scattering centers. However, it may be possible via the non-Hermitian scattering center with an internal degree of freedom.

This term "one-dimensional" refers to the space domain of incident, reflected, and transmitted waves, rather than that of the scattering center. This constraint requires the asymptotic eigenfunctions to be one-dimensional plane waves. Consider a scattering problem for an arbitrary scattering center, which is schematically illustrated in Fig. 1(a). According to the above analysis, for the left and right incident waves, we have

$$\psi_{\mathrm{L}}^{k}(x)=\begin{cases}e^{ikx}+r_{\mathrm{L}}^{k}e^{-ikx}, & (x\ll 0)\\ t_{\mathrm{L}}^{k}e^{ikx}, & (x\gg 0)\end{cases}, \tag{1}$$

and

$$\psi_{\mathrm{R}}^{k}(x)=\begin{cases}t_{\mathrm{R}}^{k}e^{-ikx}, & (x\ll 0)\\ e^{-ikx}+r_{\mathrm{R}}^{k}e^{ikx}, & (x\gg 0)\end{cases}. \tag{2}$$

Combining $\psi_{\mathrm{R}}^{k}(x)$ and $\psi_{\mathrm{L}}^{-k}(x)$ into the form $t_{\mathrm{L}}^{-k}\psi_{\mathrm{R}}^{k}(x)-\psi_{\mathrm{L}}^{-k}(x)$ and comparing it to $-r_{\mathrm{L}}^{-k}\psi_{\mathrm{L}}^{k}(x)$, we have the following relations for the reflection and transmission amplitudes:

$$t_{\mathrm{L}}^{-k}t_{\mathrm{R}}^{k}+r_{\mathrm{L}}^{-k}r_{\mathrm{L}}^{k}=1, \tag{3}$$

$$t_{\mathrm{L}}^{-k}r_{\mathrm{R}}^{k}+r_{\mathrm{L}}^{-k}t_{\mathrm{L}}^{k}=0. \tag{4}$$

It still holds if we take L $\leftrightarrows$ R or $k\longrightarrow -k$.

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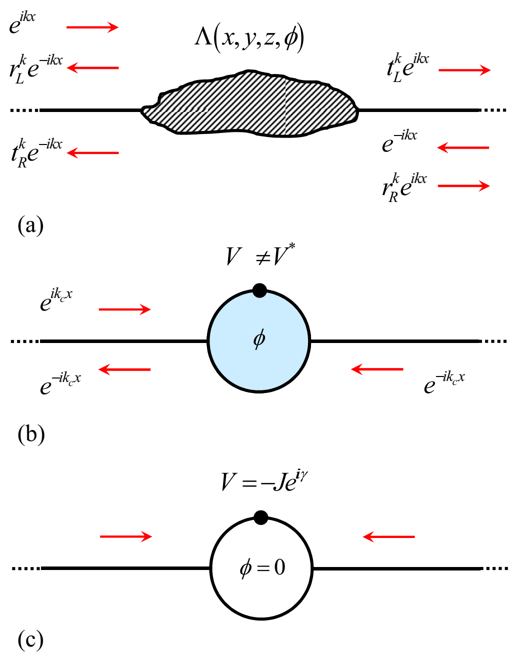

FIG. 1. (Color online) Sketch of one-dimensional scattering systems for incident waves from the left and right. (a) An arbitrary scattering center with three-dimensional structure and magnetic flux is represented by $\Lambda(x,y,z,\phi)$, which can be non-Hermitian or possesses certain symmetry. (b) A tight-binding scattering center which comprises a non-Hermitian triangular ring threaded by an Aharonov-Bohm flux ϕ. The non-Hermiticity arises from the on-site complex potential V. It turns out that the optimal ϕ and V can lead to perfect asymmetric transmission, the diode characteristic: $r_{\mathrm{R,L}}^k = t_{\mathrm{L}}^k = 0$, $|t_{\mathrm{R}}^k| = 1$.

In the following, we investigate the symmetry of the transmission for several types of scattering centers. For a scattering center with time reversal ($\mathcal{T}$) symmetry, where the time-reversal operator $\mathcal{T}$ has the function $\mathcal{T}i\mathcal{T}^{-1} = -i$, the $\mathcal{T}$ symmetry brings up additional constraints for the coefficients. One can obtain the following relations:

$$\left(t_{\mathrm{L}}^k\right)^* t_{\mathrm{R}}^k + \left(r_{\mathrm{L}}^k\right)^* r_{\mathrm{L}}^k = 1, \tag{5}$$

$$\left(t_{\mathrm{L}}^k\right)^* r_{\mathrm{R}}^k + \left(r_{\mathrm{L}}^k\right)^* t_{\mathrm{L}}^k = 0, \tag{6}$$

because the conjugations of Eqs. (1) and (2) are still the asymptotic eigenfunctions of the system. Together with the continuity of probability currents

$$\left|t_{\mathrm{L,R}}^k\right|^2 + \left|r_{\mathrm{L,R}}^k\right|^2 = 1, \tag{7}$$

we have the symmetry relations

$$t_{\mathrm{R}}^k = t_{\mathrm{L}}^k, \quad \left|r_{\mathrm{R}}^k\right| = \left|r_{\mathrm{L}}^k\right|. \tag{8}$$

This indicates that for a Hermitian scattering center, the asymmetry potential cannot lead to transmission asymmetry.

A natural question is whether a non-Hermitian scattering center can lead to the asymmetrical transmission. Before we answer this question we would like to point out that several types of non-Hermitian scattering centers cannot be candidates. This may provide guidance for the diode design. To begin with, a parity-symmetric non-Hermitian scattering center should exhibit symmetric reflection and transmission. Here the parity operator $\mathcal{P}$ has the function $\mathcal{P}x\mathcal{P}^{-1} = -x$. In addition to that, we will show that a $\mathcal{PT}$-symmetric non-Hermitian scattering center also possesses the transmission symmetry. For a $\mathcal{PT}$-symmetric scattering center, wave functions obtained by the $\mathcal{PT}$ action on Eqs. (1) and (2) are still the asymptotic eigenfunctions.

Comparing $\mathcal{PT}\psi_{\mathrm{R}}^k(x)(\mathcal{PT})^{-1}$ and $\psi_{\mathrm{L}}^{-k}(x)$, we have

$$\left(r_{\mathrm{R}}^k\right)^* = r_{\mathrm{L}}^{-k}, \quad \left(t_{\mathrm{R}}^k\right)^* = t_{\mathrm{L}}^{-k}, \tag{9}$$

which still holds if we take $\mathrm{L} \leftrightarrows \mathrm{R}$ or $k \longrightarrow -k$. Together with Eq. (3), we have

$$\left(t_{\mathrm{R}}^k\right)^* t_{\mathrm{R}}^k + \left(r_{\mathrm{R}}^k\right)^* r_{\mathrm{L}}^k = 1, \tag{10}$$

and

$$\left(t_{\mathrm{L}}^k\right)^* t_{\mathrm{L}}^k + \left(r_{\mathrm{R}}^k\right)^* r_{\mathrm{L}}^k = 1, \tag{11}$$

which lead to

$$\left|t_{\mathrm{R}}^k\right| = \left|t_{\mathrm{L}}^k\right|. \tag{12}$$

This result indicates that it is impossible to construct a diode, a scattering center allowing unidirectional flow, by a $\mathcal{PT}$-symmetric non-Hermitian scattering center in the framework of this paper, although it cannot tell us which type of structure meets the demand.

Now we consider the case where the scattering center has an internal degree of freedom ϕ as illustrated in Fig. 1(a). The Hamiltonian has $\mathcal{PF}$ symmetry, i.e.,

$$\mathcal{PF}H(\mathcal{PF})^{-1} = H, \tag{13}$$

where $\mathcal{F}$ is the ϕ-flip operator, defined as $\mathcal{F}H(\phi)\mathcal{F}^{-1} = H(-\phi)$. Applying the $\mathcal{PF}$ operator on the Eqs. (1) and (2), we obtain the new solutions of the Hamiltonian, which lead to the relations

$$t_{\mathrm{L}}^k(\phi) = t_{\mathrm{R}}^k(-\phi), \quad r_{\mathrm{L}}^k(\phi) = r_{\mathrm{R}}^k(-\phi). \tag{14}$$

In contrast to the systems with $\mathcal{P}$, $\mathcal{T}$, and $\mathcal{PT}$ symmetry, respectively, one cannot get the conclusion of the symmetric transmission. It opens the possibility of $t_L^k(\phi) \neq t_R^k(\phi)$, and a perfect diode for the specific k_c and ϕ_c, i.e.,

$$\left|t_{\mathrm{R}}^{k_c}(\phi_c)\right|^2 = 1, \quad t_{\mathrm{L}}^{k_c}(\phi_c) = 0, \quad r_{\mathrm{R,L}}^{k_c}(\phi_c) = 0, \tag{15}$$

does not contradict the general constraint relations in Eq. (14). In the following section, we propose a concrete example, a non-Hermitian $\mathcal{PF}$-symmetric scattering center embedded in a one-dimensional tight-binding network, which exhibits perfect unidirectionality.

III. PERFECT UNIDIRECTIONALITY

Inspired by the previous work [47] and the analysis in the above section, we start our design of a diode scattering center by the simplest geometry, a three-site cluster. The threading flux and single on-site complex potential can destroy the $\mathcal{P}$ symmetry as well as the $\mathcal{PT}$ symmetry. However, it possesses $\mathcal{PF}$ symmetry with the internal degree of freedom being the threading flux.

The Hamiltonian of the concerned scattering tight-binding network has the form

$$H = H_{\mathrm{L}} + H_{\mathrm{R}} + H_{\mathrm{Tri}}, \tag{16}$$

where

$$H_{\mathrm{L}} = -J \sum_{j=-\infty}^{-2} (a_{j+1}^{\dagger} a_j + \mathrm{H.c}), \tag{17}$$

$$H_{\mathrm{R}} = -J \sum_{j=1}^{\infty} (a_{j+1}^{\dagger} a_j + \mathrm{H.c}), \tag{18}$$

represent the left (HL) and right (HR) waveguides with real J, and

$$H_{\mathrm{Tri}} = -e^{i\phi/3} J (a_0^{\dagger} a_{-1} + a_1^{\dagger} a_0 + a_{-1}^{\dagger} a_1) + \mathrm{H.c} + V a_0^{\dagger} a_0, \tag{19}$$

describes a non-Hermitian scattering center, with the non-Hermiticity arising from the complex potential $V \neq V^*$. In these equations, the notation $a_j^{\dagger}$ and a_j are boson creation and annihilation operators, respectively, and H.c. represents the Hermitian conjugate of all hopping items. It is a triangular lattice threaded by a magnetic flux ϕ, which satisfies Eq. (13) and is schematically illustrated in Fig. 1(b). We note that zero ϕ leads to $\mathcal{P}$ symmetry, and real V leads to $\mathcal{PT}$ symmetry, both of which are the obstacles for the transmission asymmetry as shown in the above section. In this paper, we consider the case with complex potential $V = -Je^{i\gamma}$ [$\gamma \in (0,\pi)$].

We consider the left and right incident scattering processes. We focus our study on a single-particle subspace spanned by the basis $\{|j\rangle = a_j^{\dagger}|0\rangle\}$. The discrete versions of Eqs. (1) and (2) have the forms

$$\psi_{\mathrm{L}}^{k}(j) = \begin{cases} e^{ikj} + r_{\mathrm{L}}^{k} e^{-ikj}, & (j \ll 0) \\ t_{\mathrm{L}}^{k} e^{ikj}, & (j \gg 0) \end{cases}, \tag{20}$$

and

$$\psi_{\mathrm{R}}^{k}(j) = \begin{cases} t_{\mathrm{R}}^{k} e^{-ikj}, & (j \ll 0) \\ e^{-ikj} + r_{\mathrm{R}}^{k} e^{ikj}, & (j \gg 0) \end{cases}, \tag{21}$$

which correspond to the Bethe ansatz wave function. Employing the Bethe ansatz technique, we have

$$r_{\mathrm{L}}^{k} = r_{\mathrm{R}}^{k} = -\frac{\cos\phi + \cos k}{\Omega(k,\phi,\gamma)}, \tag{22}$$

$$t_{\mathrm{L}}^{k} = \frac{ie^{-i\phi/3} \sin k (e^{i\phi} + 2\cos k - e^{i\gamma})}{\Omega(k,\phi,\gamma)}, \tag{23}$$

$$t_{\mathrm{R}}^{k} = t_{\mathrm{L}}^{k}(\phi \to -\phi), \tag{24}$$

with

$$\Omega(k,\phi,\gamma) = e^{ik}[i \sin k (2\cos k - e^{i\gamma}) + e^{ik}\cos\phi + 1]. \tag{25}$$

For an incident wave with momentum $k_c = \gamma$ from left or right, from Eq. (22) we have

$$r_{\mathrm{L}}^{k_c} = r_{\mathrm{R}}^{k_c} = 0, \tag{26}$$

$$t_{\mathrm{R}}^{k_c} = e^{i\pi/3} e^{-i4\gamma/3}, \quad t_{\mathrm{L}}^{k_c} = 0, \tag{27}$$

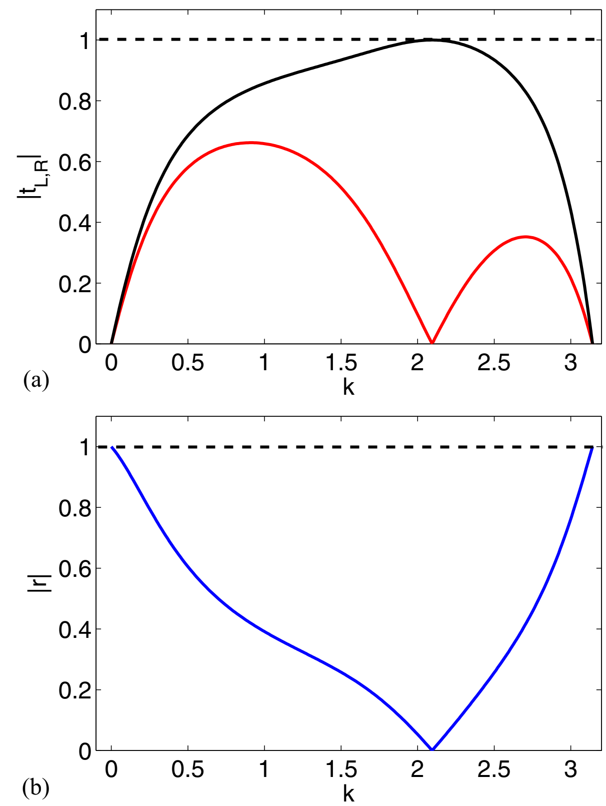

FIG. 2. (Color online) Absolute values of the (a) transmission and (b) reflection amplitude profiles as functions of the wave number k whose unit is radian (rad). It shows the perfect asymmetric behavior at $k_c = 2\pi/3$.

which exhibits perfect unidirectionality, when the flux ϕ takes the value $\phi_c = \pi - \gamma$. To illustrate the asymmetric transmission effect we consider the case with $\gamma = 2\pi/3$ and $\phi = \pi/3$. Figure 2 shows transmission and reflection profiles as functions of the wave number k. The perfect asymmetric behavior with $|r_{\mathrm{L,R}}^{k_c}| = |t_{\mathrm{L}}^{k_c}| = 0.0$ and $|t_{\mathrm{R}}^{k_c}| = 1.0$ at $k_c = 2\pi/3$, as expected, is observed. It shows that there is a relative wider region around k_c, within which the system still exhibits the diode characteristic approximately.

IV. SPECTRAL SINGULARITY

In this section, we will show that the occurrence of perfect unidirectionality is related to the presence of a spectral singularity. To this end, we consider the solution of the Hamiltonian

$$H^{\dagger} = H_{\mathrm{L}} + H_{\mathrm{R}} + H_{\mathrm{Tri}}^{\dagger}, \tag{28}$$

where

$$H_{\mathrm{L}}^{\dagger} = H_{\mathrm{L}} = -J \sum_{j=-\infty}^{-2} (a_{j+1}^{\dagger} a_j + \mathrm{H.c}), \tag{29}$$

$$H_{\mathrm{R}}^{\dagger} = H_{\mathrm{R}} = -J \sum_{j=1}^{\infty} (a_{j+1}^{\dagger} a_j + \mathrm{H.c}), \tag{30}$$

$$H_{\rm Tri}^{\dagger} = -e^{i\phi/3}J(a_0^{\dagger}a_{-1} + a_1^{\dagger}a_0 + a_{-1}^{\dagger}a_1) + \text{H.c.} - Je^{-i\gamma}a_0^{\dagger}a_0, \tag{31}$$

which is the Hermitian conjugation of H. According to pseudo-Hermitian quantum mechanics [48], the eigenfunctions of H and $H^{\dagger}$ can construct the biorthogonal basis except in the case of spectral singularity, at which the biorthonormal set is spoiled. By the same procedure, the scattering wave functions can be obtained in the forms

$$\overline{\psi}_{\rm L}^{k}(j) = \begin{cases} e^{ikj} + \overline{r}_{\rm L}^{k}e^{-ikj}, & (j \ll 0) \\ \overline{t}_{\rm L}^{k}e^{ikl}, & (j \gg 0) \end{cases}, \tag{32}$$

and

$$\overline{\psi}_{\rm R}^{k}(j) = \begin{cases} \overline{t}_{\rm R}^{k}e^{-ikj}, & (j \ll 0) \\ e^{-ikj} + \overline{r}_{\rm R}^{k}e^{ikj}, & (j \gg 0) \end{cases}, \tag{33}$$

where

$$\overline{r}_{\rm L}^{k} = \overline{r}_{\rm R}^{k} = r_{\rm L}^{k}(\gamma \to -\gamma), \tag{34}$$

$$\overline{t}_{\rm L,R}^{k} = t_{\rm L,R}^{k}(\gamma \to -\gamma). \tag{35}$$

However, it becomes a little complicated when we consider the solution for $k = k_c$. We find that the eigenfunctions with k_c do not exist when the flux ϕ takes the value $\phi_c = \pi - \gamma$. We investigate the limit of $\overline{r}_{\rm L,R}^{k}$ and $\overline{t}_{\rm L,R}^{k}$ as $(k,\phi) \to (\gamma,\pi-\gamma)$ along the following two paths: (I) $\phi = \pi - k$, $k \downarrow \gamma$ and (II) $\phi = \pi - k$, $k \uparrow \gamma$, respectively. A straightforward calculation shows that two different paths give unequal limits, i.e.,

$$\text{(I)} \quad \begin{cases} \lim_{k\downarrow\gamma} \overline{r}_{\rm L,R}^{k} = 0 \\ \lim_{k\downarrow\gamma} \overline{t}_{\rm L}^{k} = -\infty + i\infty \\ \lim_{k\downarrow\gamma} \overline{t}_{\rm R}^{k} = e^{i\pi/3}e^{-i4\gamma/3} \end{cases}, \tag{36}$$

$$\text{(II)} \quad \begin{cases} \lim_{k\uparrow\gamma} \overline{r}_{\rm L,R}^{k} = 0 \\ \lim_{k\uparrow\gamma} \overline{t}_{\rm L}^{k} = +\infty - i\infty \\ \lim_{k\uparrow\gamma} \overline{t}_{\rm R}^{k} = e^{i\pi/3}e^{-i4\gamma/3} \end{cases}, \tag{37}$$

We see that the transmission amplitude $\overline{t}_{\rm L}^{k}$ has a singularity at the point $k = k_c = \gamma$, which indicates that the Bathe ansatz solutions in the form of Eqs. (32) and (33) do not exist. We can also investigate this point from another way, taking the limits of $\overline{r}_{\rm L,R}^{k}$ and $\overline{t}_{\rm L,R}^{k}$ as $(k,\phi) \to (\gamma,\pi-\gamma)$ along the following two paths: (I) $k = \gamma, \phi \uparrow \phi_c$ and (II) $k = \gamma, \phi \downarrow \phi_c$, respectively . To demonstrate the singularity, we plot $t_{\rm L,R}^{k_c}$ as functions of ϕ for $k_c = \gamma = \pi/6$ in Fig. 3. The profiles of the plots show clearly that the one-sided limits from the left and from the right for the real and imaginary parts are discontinuous and divergent at $\phi = 5\pi/6$ and $\phi = 7\pi/6$, respectively. These observations imply that the biorthogonality of the eigenfunctions may be defective at this point.

A good way to confirm this is to take the Bethe ansatz solutions of H and $H^{\dagger}$ in a general form as

$$\varphi^{k}(j) = \begin{cases} A_{-}e^{ikj} + B_{-}e^{-ikj}, & (j \ll 0) \\ A_{+}e^{ikj} + B_{+}e^{-ikj}, & (j \gg 0) \end{cases}, \tag{38}$$

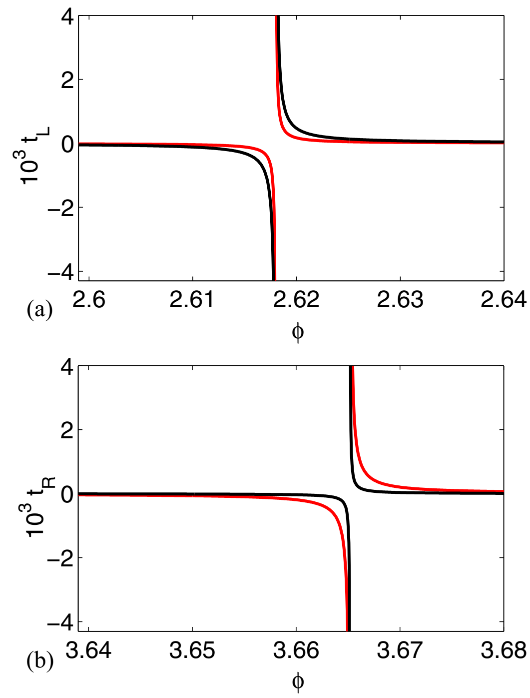

FIG. 3. (Color online) Transmission amplitudes from Eq. (35) with $k = \gamma = \pi/6$. The unit of ϕ is radian (rad). It shows that real and imaginary parts of the amplitudes are discontinuous and divergent at $\phi = 5\pi/6$ and $\phi = 7\pi/6$, respectively.

and

$$\overline{\varphi}^{k}(j) = \begin{cases} \overline{A}_{-}e^{ikj} + \overline{B}_{-}e^{-ikj}, & (j \ll 0) \\ \overline{A}_{+}e^{ikj} + \overline{B}_{+}e^{-ikj}, & (j \gg 0) \end{cases}, \tag{39}$$

respectively. It is easy to check that at the point $k = k_c = \gamma$, we have $A_{+} = 0$, $e^{i(\pi-4k)/3}B_{+} = B_{-}$ and $\overline{A}_{-} = 0$, $e^{i(\pi-4k)/3}\overline{B}_{+} = \overline{B}_{-}$. Also, to satisfy the Dirac normalization, we take $B_{\pm} = 0$, $A_{-} = 1/\sqrt{N}$ and $B_{-} = 1/\sqrt{2N}e^{i(\pi-4k)/3}B_{+} = 1/\sqrt{2N}e^{i(\pi-4k)/3}$, $A_{-} = 0$. Hence, the eigenfunctions of H are

$$\varphi_{s,1}^{k}(j) = \frac{1}{\sqrt{N}} \begin{cases} e^{ikj}, & (j \ll 0) \\ 0, & (j \gg 0) \end{cases}, \tag{40}$$

$$\varphi_{s,2}^{k}(j) = \frac{1}{\sqrt{2N}} \begin{cases} e^{i(\pi-4k)/3}e^{-ikj}, & (j \ll 0) \\ e^{-ikj}, & (j \gg 0) \end{cases}, \tag{41}$$

where s represents spectral singularity, a special situation, and N is the system size. Similarly, the eigenfunctions of $H^{\dagger}$ are

$$\overline{\varphi}_{s,1}^{k}(j) = \frac{1}{\sqrt{N}} \begin{cases} 0, & (j \ll 0) \\ e^{ikj}, & (j \gg 0) \end{cases}, \tag{42}$$

$$\overline{\varphi}_{s,2}^{k}(j) = \frac{1}{\sqrt{2N}} \begin{cases} e^{i(\pi-4k)/3}e^{-ikj}, & (j \ll 0) \\ e^{-ikj}, & (j \gg 0) \end{cases}. \tag{43}$$

The physics of the solutions is clear that $\varphi_{s,2}^{k}(j)$ and $\overline{\varphi}_{s,2}^{k}(j)$ describe the reflectionless transmission from the right to the left side, while $\varphi_{s,1}^{k}(j)$ and $\overline{\varphi}_{s,1}^{k}(j)$ represent unilateral reflectionless absorption and self-sustained emission from the scattering center, respectively. We can readily check that

$$\langle \varphi^k_{s,1}(j)|\overline{\varphi}^k_{s,1}(j)\rangle = 0, \quad \langle \varphi^k_{s,1}(j)|\overline{\varphi}^k_{s,2}(j)\rangle = 0, \tag{44}$$

$$\langle \varphi^k_{s,2}(j)|\overline{\varphi}^k_{s,1}(j)\rangle = 0, \quad \langle \varphi^k_{s,2}(j)|\overline{\varphi}^k_{s,2}(j)\rangle = 1, \tag{45}$$

which indicates that the biorthogonality of the eigenfunctions of H and $H^\dagger$ is destroyed. Therefore, we conclude that the perfect asymmetric transmission corresponds to the existence of the spectral singularity of the non-Hermitian diode model.

It is noted that the theory of spectral singularity for a non-Hermitian scattering center arising from pure complex potential has been well established [21,29]. It is shown that the transfer matrix can be employed to identify the spectral singularity.

We believe that this formalism is applicable to the discrete system. Similarly, the eigenvalue equation $H\psi = E\psi$ yields the following asymptotic expressions for the eigenfunctions of H:

$$\varphi^k(j) \to A_\pm e^{ikj} + B_\pm e^{-ikj} \quad \text{for } j \to \pm\infty. \tag{46}$$

$A_\pm$ and $B_\pm$ are possibly k-dependent complex coefficients that are related by the so-called transfer matrix M according to

$$\begin{pmatrix} A_+ \\ B_+ \end{pmatrix} = M \begin{pmatrix} A_- \\ B_- \end{pmatrix}. \tag{47}$$

The transfer matrix reveals almost complete features of the scattering center. Next, the Jost solution of H can be readily constructed from Eq. (26) as the form

$$\varphi^k_+(j) = \begin{cases} e^{ikj}/t^k_{\rm L} + r^k_{\rm L} e^{-ikj}/t^k_{\rm L}, & (j \ll 0) \\ e^{ikj}, & (j \gg 0) \end{cases}, \tag{48}$$

$$\varphi^k_-(j) = \begin{cases} e^{-ikj}, & (j \ll 0) \\ e^{-ikj}/t^k_{\rm R} + r^k_{\rm R} e^{ikj}/t^k_{\rm R}, & (j \gg 0) \end{cases}, \tag{49}$$

which satisfies the asymptotic boundary conditions

$$\varphi^k_\pm(j) \to e^{\pm ikj} \text{ as } j \to \pm\infty. \tag{50}$$

Then the corresponding transfer matrix can be written as

$$M_k = \begin{pmatrix} \frac{t^k_{\rm L} t^k_{\rm R} - (r^k_{\rm R})^2}{t^k_{\rm R}} & \frac{r^k_{\rm R}}{t^k_{\rm R}} \\ -\frac{r^k_{\rm R}}{t^k_{\rm R}} & \frac{1}{t^k_{\rm R}} \end{pmatrix}, \tag{51}$$

which connects the asymptotic scattering wave functions at $\pm\infty$.

Now we consider the case with $\phi = 0$, which represents the non-Hermitian scattering center arising from complex on-site potentials. Straightforward derivation shows that when the momentum $k = k_c$, with

$$\sin k_c = -\frac{\cos\gamma + 1}{2\sin\gamma}, \tag{52}$$

we have

$$(M_{k_c})_{22} = 1/t^{k_c}_{\rm R} = 0. \tag{53}$$

It identifies a spectral singularity at k_c, which also corresponds to $|t^{k_c}_{\rm R,L}| = |r^{k_c}_{\rm R,L}| = \infty$. This result accords with the theorems proposed in Ref. [29]. Furthermore, the corresponding eigenfunctions can be written as

$$\varphi^{k_c}_\pm(j) = \begin{cases} e^{-ik_c j}, & (j \ll 0) \\ e^{ik_c j}, & (j \gg 0) \end{cases}, \tag{54}$$

because

$$\lim_{k\to k_c} r^k_{\rm L}/t^k_{\rm L} = \lim_{k\to k_c} r^k_{\rm R}/t^k_{\rm R} = 1. \tag{55}$$

Obviously, the physics of the solution in Eq. (54) corresponds to the unidirectional plane wave, which has been proposed in Refs. [24,31]. It can be seen from the following analysis. The group velocities of the incident plane waves from left and right are denoted as $\upsilon_{\rm L} = \upsilon^-_{k_c}$ and $\upsilon_{\rm R} = \upsilon^+_{k_c}$, where

$$\upsilon^\pm_{k_c} = \left(\frac{\partial E_k}{\partial k}\right)_{\pm k_c} = \pm 2J \sin k_c. \tag{56}$$

For the center loss potential $V = -Je^{i\gamma}$ with $\gamma \in (0,\pi)$, we have $\sin k_c < 0$ according to Eq. (52) and then yields $\upsilon_{\rm L} > 0$, $\upsilon_{\rm R} < 0$. It indicates that the solution in Eq. (54) represents the current flow from both sides to the center. It corresponds to reflectionless absorbtion with $\mathcal{P}$ symmetry, which is schematically illustrated in Fig. 1(c).

On the other hand, when we consider the nonzero ϕ case, we will find that such a criterion is invalid for the spectral singularity under the condition of the perfect asymmetric transmission. From Eq. (26), the corresponding transfer matrix can be written as

$$M_{k_c} = \begin{pmatrix} 0 & 0 \\ 0 & e^{i(4k_c - \pi)/3} \end{pmatrix}, \tag{57}$$

which indicates $M_{22} \neq 0$. It implies that the criterion for the existence of spectral singularity is not necessary when the magnetic field is involved.

Finally, we would like to discuss the asymmetric transmission from another perspective. It turns out that there is another peculiar phenomenon, reflectionless absorption, in the semi-infinite non-Hermitian system [24,31]. We have shown above that such a phenomenon in its symmetrized version can occur in the present system with zero ϕ. Now we will show the connection between the perfect unidirectionality and

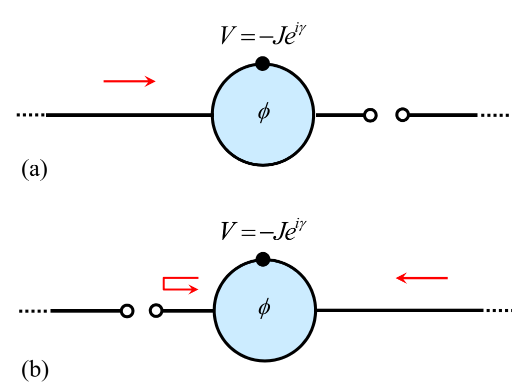

FIG. 4. (Color online) Sketch of semi-infinite systems possessing the reflectionless absorption characteristic, which can be constructed by disconnecting the (a) right or (b) left lead from a diode configuration.

reflectionless absorption. In contrast to the previous study, we consider the system in the presence of a magnetic flux. One can see this connection simply by disconnecting one of two leads in the system with the Hamiltonian in Eq. (16). Figures 4(a) and 4(b), which are obtained by cutting off the right and left leads, respectively, schematically illustrate this geometry. Then we can conclude that a perfect diode scattering center can always be reduced to a setup of reflectionless absorption. However, the latter is not sufficient to construct a diode device.

V. SUMMARY AND DISCUSSION

In summary, we have studied the possibility of asymmetric transmission induced by a non-Hermitian scattering center embedded in a one-dimensional waveguide. We have shown that the non-Hermiticity of a scattering center is not sufficient for the asymmetric transmission, while it is forbidden for a Hermitian scattering center. We have constructed a concrete setup possessing the perfect unidirectionality of the transmission, which comprises a non-Hermitian triangular ring threaded by an Aharonov-Bohm flux. It seems to imply that a magnetic flux is crucial for such a phenomenon. Furthermore, the analytical solution shows the connection between the perfect unidirectionality and spectral singularity. We have also showed that the criterion for spectral singularity associated with the transfer matrix is invalid when a magnetic field is involved.

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