Josephson relation for disordered superfluids

Cord A. Müller
Fachbereich Physik, Universität Konstanz, 78457 Konstanz, Germany and
Institut Non Linéaire de Nice, CNRS and Université Nice–Sophia Antipolis, 06560 Valbonne, France

(Received 2 October 2014; published 3 February 2015)

The Josephson sum rule relates the superfluid density to the condensate order parameter via the infrared residue of the single-particle Green's function. We establish an effective Josephson relation for disordered condensates valid upon ensemble averaging. This relation has the merit of showing explicitly how superfluidity links to the coherent density, i.e., the density of condensed particles with zero momentum. Detailed agreement is reached with perturbation theory for weak disorder.

DOI: 10.1103/PhysRevA.91.023602 PACS number(s): 03.75.Kk, 67.85.Hj

I. INTRODUCTION

Bose-Einstein condensation (BEC) and superfluidity are certainly linked, and yet this link is difficult to state with precision in situations that involve, e.g., strong interactions, low dimensions, external potentials or temperatures close to critical. Josephson [1] has derived a relation between the superfluid mass density ρ_s and the BEC order parameter ψ that provides such a link:

$$\frac{m|\psi|^2}{\rho_s} = -\lim_{k\to 0}\frac{\boldsymbol{k}^2}{m}G(\boldsymbol{k},0). \tag{1}$$

Here, m is the mass of the individual bosonic particle, and $G(\boldsymbol{k},0)$ is the single-particle Green's function at momentum $\boldsymbol{k}$ and zero (Matsubara) frequency. Because Green's function can be written as a frequency integral over its imaginary part, the spectral function, this relation is also referred to as the Josephson sum rule [2,3]. Only within mean-field theory, neglecting quantum and thermal fluctuations, one finds that $\rho_s = m|\psi|^2$ [see Eq. (22) below], and there is no need for subtle distinctions between condensate and superfluid. But especially under critical conditions, the Josephson relation is precious because it connects the scaling properties of condensate and superfluid order parameters through the Josephson (hyper-)scaling law [1,4,5].

Because of its conceptual and practical importance, the Josephson relation has been rederived over the years using various methods [2,3,5–7]. These derivations all make use of translational invariance and thus are only valid, strictly speaking, in clean systems. Although the Josephson scaling law has been occasionally applied [8,9] (and questioned [10]) in disordered systems, it is not immediately clear how to read the relation (1) in that case. Indeed, the BEC order parameter $\psi(\boldsymbol{r})$ acquires a spatial dependence on each realization of disorder, and also Green's function is no longer diagonal in momentum. Since one can take ρ_s to be a self-averaging quantity in a bulk system of linear size L, one may be tempted to think that Eq. (1) should hold under the ensemble average, noted by the overbar $\overline{(\dots)}$:

$$\frac{m\overline{|\psi|^2}}{\rho_s} = -\lim_{k\to 0}\frac{\boldsymbol{k}^2}{m}\overline{G(\boldsymbol{k},0)}. \tag{2}$$

If this were true, the Josephson relation would constrain the ratio of superfluid density to the average condensate density [11],

$$\overline{|\psi|^2} = L^{-d}\int d\boldsymbol{r}|\psi(\boldsymbol{r})|^2 =: n_c. \tag{3}$$

The purpose of this paper is to show that this is *not* the case. In the following, the correct Josephson relation is first stated and briefly discussed, then derived, and finally analytically checked in the simplest accessible regime of low temperatures, weak interactions, and weak disorder.

II. INHOMOGENEOUS JOSEPHSON RELATION

Our main result is the following Josephson relation for inhomogeneous systems valid upon ensemble averaging:

$$\frac{m|\overline{\psi}|^2}{\rho_s} = -\lim_{k\to 0}\frac{\boldsymbol{k}^2}{m}\overline{G(\boldsymbol{k},0)}. \tag{4}$$

Here, instead of the average condensate density of Eq. (3), it is the *coherent density*

$$|\overline{\psi}|^2 = \left|L^{-d}\int d\boldsymbol{r}\,\psi(\boldsymbol{r})\right|^2 =: n_{coh} \tag{5}$$

of condensed particles with $\boldsymbol{k}=0$ that is linked with the peculiar long-range, phase-coherent transport properties that we call superfluid stiffness. The coherent density can be defined equivalently by $n_{coh} = \lim_{|\boldsymbol{r}|\to\infty}\overline{\langle\hat{\psi}^\dagger(\boldsymbol{r})\hat{\psi}(0)\rangle}$ as the component with off-diagonal long-range order of the *ensemble-averaged* one-body density matrix. As recognized already by Penrose and Onsager [12], in systems that are not fully translation invariant, the condensate properly speaking comprises all particles in the maximally populated eigenmode $\psi(\boldsymbol{r})$ [13,14] and thus contains the coherent component with $\boldsymbol{k}=0$ plus the "glassy" component with $\boldsymbol{k}\neq 0$ [15,16].

Qualitatively, this strong link between superfluid and coherent density may not surprise us much, and indeed it has been observed in numerical calculations [17,18] that superfluid and coherent fractions vanish together at (one and the same) superfluid-insulator critical point, as implied by a finite right-hand side of Eq. (4) at criticality. The preference of Eq. (4) over Eq. (2) is also consistent with the view that the insulating Bose glass close to the transition is a collection of locally condensed puddles with finite mean density [Eq. (3)], which fail to connect phase-coherently over the full system size [19–22]. However, Eq. (4) provides a general, fully quantitative, and testable relation. Moreover, recent numerical results in $d = 2$ [23] seem to suggest that superfluid and coherent density do not vanish together. Therefore, we believe it worthwhile to derive Eq. (4) by a microscopic calculation and check its validity in an analytically tractable limit.

III. DERIVATION

We consider a single-component Bose-condensed fluid with repulsive interactions in its kinematic ground state, at inverse temperature β, confined to a d-dimensional volume of linear size L and subject to an external one-body potential $V(\boldsymbol{r})$. The total average density $n = L^{-d}\int d\boldsymbol{r}\langle\hat\psi^\dagger(\boldsymbol{r})\hat\psi(\boldsymbol{r})\rangle$ is fixed by the chemical potential μ and splits into the sum of the condensate density (3) and the noncondensed density. The latter comprises quantum-depleted and, at $T > 0$, thermally excited particles. The condensate is described by a scalar, stationary BEC order parameter $\psi(\boldsymbol{r})$. Such an order parameter may be defined as the macroscopically populated eigenmode of the one-body density matrix [12]. In the U(1) symmetry-breaking picture of BEC [24], one rather defines $\psi(\boldsymbol{r}) = \langle\hat\psi(\boldsymbol{r})\rangle$ as the expectation value of the bosonic field operator; we use the latter definition for its technical simplicity.

Following Baym [2] (see also [3]), we calculate via linear response how much adding a particle with momentum $\boldsymbol{k}$ changes the order parameter on the one hand and the current density on the other. We assume that the external potential is an ergodic random process, and reach translation invariance by ensemble-averaging. Comparing the changes in order parameter and current density then leads to the generalized Josephson relation (4).

To this end, let

$$\delta\hat H_{\boldsymbol{k}} = \delta\,\hat a_{\boldsymbol{k}}^\dagger = \frac{\delta}{L^{d/2}}\int d\boldsymbol{r}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\hat\psi^\dagger(\boldsymbol{r}) \tag{6}$$

be the small perturbation ($|\delta| \ll \mu$) that adds a particle with momentum $\boldsymbol{k}$ to the system [25]. The linear response of the condensate amplitude on average is

$$\overline{\delta\psi(\boldsymbol{r})} = -\int_0^\beta d\tau\,\overline{\langle\hat\psi(\boldsymbol{r},\tau)\delta\hat H_{\boldsymbol{k}}(0)\rangle} = \frac{\delta}{L^{d/2}}e^{i\boldsymbol{k}\boldsymbol{r}}\overline{G(\boldsymbol{k},0)}, \tag{7}$$

which brings about the zero-frequency component of the ensemble-averaged Matsubara-Green function

$$\overline{G(\boldsymbol{k},i\omega_n)} = -\int d\boldsymbol{r}\ e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\int_0^\beta d\tau\, e^{i\omega_n\tau}\overline{\langle\hat\psi(\boldsymbol{r},\tau)\hat\psi^\dagger(0,0)\rangle}. \tag{8}$$

(If the ensemble average were not taken at this stage, one would face a Green's function that is not diagonal in $\boldsymbol{k}$, which would compromise the following derivation.) The average condensate amplitude,

$$\overline{\psi(\boldsymbol{r})} + \overline{\delta\psi(\boldsymbol{r})} =: \overline{\psi(\boldsymbol{r})}[1 + i\,\delta\varphi], \tag{9}$$

is changed by a pure phase factor when

$$\delta\varphi = -i\frac{\overline{\delta\psi(\boldsymbol{r})}}{\overline{\psi(\boldsymbol{r})}} = \frac{-i\delta}{L^{d/2}\overline{\psi(\boldsymbol{r})}}e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\overline{G(\boldsymbol{k},0)} \tag{10}$$

is real, which can be realized by choosing the phase of δ appropriately and in the limit $\boldsymbol{k} \to 0$ (this is the step where taking the limit is required). This phase's gradient then induces on average the superfluid mass current

$$m\,\overline{\delta\boldsymbol{j}(\boldsymbol{r})} = \frac{\rho_s}{m}\nabla\delta\varphi\Big|_{\boldsymbol{k}\to0} = \delta\frac{\rho_s\boldsymbol{k}e^{i\boldsymbol{k}\cdot\boldsymbol{r}}}{L^{d/2}m\overline{\psi(\boldsymbol{r})}}\,\overline{G(\boldsymbol{k},0)}\Big|_{\boldsymbol{k}\to0}. \tag{11}$$

Now we calculate the current directly via linear response,

$$\delta\boldsymbol{j}(\boldsymbol{r}) = -\int_0^\beta d\tau\langle\hat{\boldsymbol{j}}(\boldsymbol{r},\tau)\delta\hat H_{\boldsymbol{k}}(0)\rangle. \tag{12}$$

Yet, even for a perturbation as simple as Eq. (6), this is in general impossible, for one cannot compute the full time dependence of the current in the presence of interactions. But we can invoke particle number conservation, as expressed by the continuity equation, in imaginary time:

$$i\partial_\tau\hat n(\boldsymbol{r},\tau) + \nabla\cdot\hat{\boldsymbol{j}}(\boldsymbol{r},\tau) = 0. \tag{13}$$

(Its proof is elementary: Given the Hamiltonian $\hat H[\hat\psi,\hat\psi^\dagger] = \hat K + \hat U$ with kinetic energy $\hat K = \frac{1}{2m}\int d\boldsymbol{r}\nabla\hat\psi^\dagger\cdot\nabla\hat\psi$, and an interaction $\hat U = U[\hat n]$ that is a functional of the density only, Eq. (13) is equivalent to the equation of motion $\partial_\tau\hat n = [\hat K,\hat n]$.) In the momentum representation, the continuity equation (13) becomes

$$\partial_\tau\hat n_{\boldsymbol{p}}(\tau) + \boldsymbol{p}\cdot\hat{\boldsymbol{j}}_{\boldsymbol{p}}(\tau) = 0, \tag{14}$$

and thus permits us to replace the longitudinal current by the density variation according to $|\boldsymbol{p}|\hat j_{\boldsymbol{p}}^\parallel = -\partial_\tau\hat n_{\boldsymbol{p}}$. This allows us to evaluate the Matsubara-time integral

$$\delta j_{\boldsymbol{p}}^\parallel = |\boldsymbol{p}|^{-1}\int_0^\beta d\tau\langle\partial_\tau\hat n_{\boldsymbol{p}}(\tau)\delta\hat H_{\boldsymbol{k}}(0)\rangle = -|\boldsymbol{p}|^{-1}\langle[\hat n_{\boldsymbol{p}},\delta\hat H_{\boldsymbol{k}}]\rangle, \tag{15}$$

and we are left with the simple equal-time commutator

$$[\hat n_{\boldsymbol{p}},\delta\hat H_{\boldsymbol{k}}] = \delta\,\hat a_{\boldsymbol{k}-\boldsymbol{p}}^\dagger. \tag{16}$$

Thus we find after ensemble-averaging

$$\overline{\delta j^\parallel(\boldsymbol{r})} = -\frac{\delta}{L^{d/2}|\boldsymbol{k}|}\overline{\psi^*(\boldsymbol{r})}e^{i\boldsymbol{k}\cdot\boldsymbol{r}}. \tag{17}$$

Comparing this result with Eq. (11), whose leading contribution in the limit $\boldsymbol{k} \to 0$ is also purely longitudinal, then establishes Eq. (4). We remark that the zero-frequency Green's function appearing here contains the full dynamical single-particle correlations and can in general not be reduced to the equal-time momentum distribution that enters, for instance, the one-body density matrix [26].

IV. CONSISTENCY CHECK IN PERTURBATION THEORY

Exact analytical results are hard to obtain, but we can evaluate the factors entering Eq. (4) perturbatively for weak disorder using inhomogeneous quadratic Bogoliubov theory [14,27] and check whether they match. First, the coherent density is given by Eq. (11) in [14],

$$n_{\mathrm{coh}} = |\overline{\psi}|^2 = n_c[1 - V_2 + O(V^3)], \tag{18}$$

and it is thus smaller than the total condensate density, Eq. (3), by a factor that is determined by the glassy fraction [15]

$$V_2 := \sum_{p} \frac{\overline{|V_p|^2}}{(\epsilon_p^0 + 2gn_c)^2}, \tag{19}$$

with $\epsilon_p^0 = p^2/2m$ the free dispersion. Furthermore, using Eqs. (18)–(20) of [14], the single-particle Green's function can be expressed in terms of quasiparticle normal and anomalous Green's functions,

$$\overline{G(k,0)} = \sum_{p,q} [\overline{(u_{kp}u_{kq}^* + v_{k,-p}v_{k,-q}^*)G_{pq}(0)} - \overline{(v_{kp}u_{kq}^* + u_{k,-p}v_{k,-q}^*)F_{pq}(0)}]. \tag{20}$$

The matrix coefficients u_{kp} and v_{kp} generalize the usual Bogoliubov factors u_k, v_k to the case where the condensate, or Bogoliubov vacuum, is inhomogeneous. They encode the condensate deformation by the external potential $V(r)$ on the mean-field level. All these factors can be Taylor-expanded to the desired order in V (see Sec. 3.4 in [14] and Sec. III.B. in [27]).

To zeroth order in V, for the clean system, one has

$$\overline{G^{(0)}(k,0)} = -(u_k^2 + v_k^2)\epsilon_k^{-1}, \tag{21}$$

where $\epsilon_k = [\epsilon_k^0(\epsilon_k^0 + 2gn_c)]^{1/2}$ is the Bogoliubov dispersion. Multiplication by k^2 and taking the limit $k \to 0$ as required by Eq. (4) selects the most divergent contribution, which reduces the number of terms quite substantially. Eq. (21) diverges like $1/(2a_k^2\epsilon_k) = 1/2\epsilon_k^0 = m/k^2$, such that from Eq. (4) one finds

$$\rho_s = m|\psi|^2 = mn_c =: \rho_c. \tag{22}$$

As expected, in a clean system and to the quadratic order of the Bogoliubov Hamiltonian considered, the whole condensate is superfluid.

At order V^2 in disorder strength, two types of contributions survive in Eq. (20): (i) products like

$$u_{kp}^{(2)}u_{kq}^{(0)}G_{pq}^{(0)}(0) \propto V_2\delta_{kp}\delta_{kq}u_k(u_k - 2v_k)\epsilon_k^{-1}, \tag{23}$$

with the clean, normal propagator $G_{pq}^{(0)}(0) = -\delta_{pq}\epsilon_p^{-1}$, but no anomalous terms since $F^{(0)} = 0$, and (ii) products like

$$u_{kp}^{(0)}u_{kq}^{(0)}G_{pq}^{(2)}(0) \propto \delta_{kp}\delta_{kq}u_k^2 G_k^{(2)}(0) \tag{24}$$

and similar with $u_k v_k F_k^{(2)}(0)$. Mixed terms of the type $u^{(1)}u^{(0)}G^{(1)}$ and the like do not survive the limit $k \to 0$.

Type (i) terms yield, after taking the limit $k \to 0$, a correction $(1 - V_2)$ on the right-hand side of Eq. (4) that cancels exactly the same factor introduced on the left-hand side by the coherent fraction (18). Type (ii) terms after a bit of algebra finally yield a correction of the form

$$\lim_{k\to 0} \sum_p \frac{(k\cdot p)^2}{\epsilon_k^0\epsilon_p^0} \frac{\overline{|V_p|^2}}{(\epsilon_p^0 + 2gn_c)^2} = \frac{4m^2}{d}V_2. \tag{25}$$

All in all, Eq. (4) predicts to order V^2 the correction

$$\rho_s = \rho_c\left(1 - \frac{4}{d}V_2\right), \tag{26}$$

which is already well documented in the literature, see Eq. (12) in [28], Eq. (19) in [29], Eq. (20) in [30], and Eq. (6) in [23]. This then explicitly validates the inhomogeneous Josephson relation (4) to order V^2 and at the same time rules out Eq. (2).

Note, though, that one cannot obtain a temperature dependence from Eq. (20) with the quadratic quasiparticle Hamiltonian of [14,27] that contains only elastic impurity scattering. In order to recover Landau's celebrated finite-temperature superfluid depletion [31] microscopically, one would have to introduce interactions between the quasiparticles.

A different method of calculating the superfluid density is to compute the normal density $\rho_n = \rho_c - \rho_s$ directly from the transverse current-current correlation [2,31]. Inhomogeneous Bogoliubov theory [14,27] then predicts, at $T = 0$,

$$\rho_n = \frac{1}{4n_c}\sum_{p,q}\frac{p_z q_z}{a_p a_q}\overline{n_{ck-p}n_{cq-k}[F_{pq}(0) - G_{pq}(0)]}\big|_{k=k_\perp\to 0}. \tag{27}$$

Here, $n_{ck} = L^{-d}\int dr\, e^{-ikr}n_c(r)$ are the Fourier components of the deformed condensate density, and $k_\perp$ lies in the xy plane transverse to the z axis. In the clean case, to zeroth order in V, the condensate is homogeneous, $n_{cq} = n_c\delta_{q,0}$, and since $k_z = 0$, the normal density vanishes. To order V^2, only a single type of term survives the limit $k_\perp \to 0$, namely, $n^{(1)}n^{(1)}G^{(0)}$. Using Eq. (11) of [27], this expression evaluates rather immediately to $(4/d)\rho_c V_2$ and thus agrees with Eq. (26). Clearly, to this order it is much simpler to evaluate Eq. (27) than to find ρ_s from the Josephson relation, since there are no common terms that cancel, like on the two sides of Eq. (4), and only the clean quasiparticle propagator $G_{pq}^{(0)}(0)$ enters together with the condensate deformation. Lastly, we remark that this approach can be generalized to finite temperature and thus permits us to derive disorder corrections to Landau's superfluid depletion [32].

V. SUMMARY

A Josephson-type relation has been established for disordered Bose fluids between the superfluid density, the infrared residue of the single-particle Green's function, and the coherent density, i.e., density of condensed particles with zero momentum. Its validity for weak interactions and disorder has been checked in detail by a perturbative calculation using inhomogeneous Bogoliubov theory. The numerical results of [17,18] agree qualitatively with its prediction at the superfluid-insulator transition where coherent and superfluid fraction vanish together. Although it may not be evident to extract the infrared residue of the average zero-frequency Green's function with precision in the numerics, it would be interesting to investigate the quantitative validity of the sum rule (4) near the critical point in different dimensions.

ACKNOWLEDGMENTS

The author is indebted to K. Krutitsky, S. Pilati, and M. Ueda for helpful discussions and to C. Gaul for constructive comments on the manuscript.

[1] B. Josephson, Phys. Lett. **21**, 608 (1966).
[2] G. Baym, in *Mathematical Methods in Solid State and Superfluid Theory*, edited by R. C. Clark and G. H. Derrick (Oliver and Boyd, Edinburgh, 1969), p. 121.
[3] M. Ueda, *Fundamentals and New Frontiers of Bose-Einstein Condensation* (World Scientific, Singapore, 2010).
[4] M. Holzmann and G. Baym, Phys. Rev. Lett. **90**, 040402 (2003).
[5] M. Holzmann and G. Baym, Phys. Rev. B **76**, 092502 (2007).
[6] A. Griffin, Phys. Rev. B **30**, 5057 (1984).
[7] J. F. Dawson, B. Mihaila, and F. Cooper, Phys. Rev. A **86**, 013603 (2012).
[8] P. A. Crowell, F. W. Van Keuls, and J. D. Reppy, Phys. Rev. B **55**, 12620 (1997).
[9] S. Balibar and F. Caupin, J. Phys. C: Cond. Matter **20**, 173201 (2008).
[10] J. D. Reppy, J. Low Temp. Phys. **87**, 205 (1992).
[11] We assume statistical homogeneity and trade ensemble for spatial averages where appropriate.
[12] O. Penrose and L. Onsager, Phys. Rev. **104**, 576 (1956).
[13] G. E. Astrakharchik and K. V. Krutitsky, Phys. Rev. A **84**, 031604 (2011).
[14] C. A. Müller and C. Gaul, New J. Phys. **14**, 075025 (2012).
[15] V. I. Yukalov and R. Graham, Phys. Rev. A **75**, 023619 (2007).
[16] C. Krumnow and A. Pelster, Phys. Rev. A **84**, 021608 (2011).
[17] S. Pilati, S. Giorgini, and N. Prokof'ev, Phys. Rev. Lett. **102**, 150402 (2009).
[18] G. Carleo, G. Boéris, M. Holzmann, and L. Sanchez-Palencia, Phys. Rev. Lett. **111**, 050406 (2013).
[19] K. Yamamoto, Y. Shibayama, and K. Shirahama, Phys. Rev. Lett. **100**, 195301 (2008).
[20] G. M. Falco, T. Nattermann, and V. L. Pokrovsky, Phys. Rev. B **80**, 104515 (2009).
[21] G. M. Falco, T. Nattermann, and V. L. Pokrovsky, Europhys. Lett. **85**, 30002 (2009).
[22] S. O. Diallo, R. T. Azuah, D. L. Abernathy, J. Taniguchi, M. Suzuki, J. Bossy, N. Mulders, and H. R. Glyde, Phys. Rev. Lett. **113**, 215302 (2014).
[23] G. E. Astrakharchik, K. V. Krutitsky, and P. Navez, Phys. Rev. A **87**, 061601 (2013).
[24] P. Hohenberg and P. Martin, Ann. Phys. (NY) **34**, 291 (1965).
[25] In a number-conserving setting, one could consider $\delta\hat{H}_{\boldsymbol{k}} = \delta\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_0$, i.e., the promotion of a particle from zero to finite momentum.
[26] L. Pitaevskii and S. Stringari, *Bose-Einstein Condensation* (Clarendon, Oxford, 2003).
[27] C. Gaul and C. A. Müller, Phys. Rev. A **83**, 063629 (2011).
[28] K. Huang and H.-F. Meng, Phys. Rev. Lett. **69**, 644 (1992).
[29] S. Giorgini, L. Pitaevskii, and S. Stringari, Phys. Rev. B **49**, 12938 (1994).
[30] A. V. Lopatin and V. M. Vinokur, Phys. Rev. Lett. **88**, 235503 (2002).
[31] L. Landau and E. Lifshitz, *Statistical Physics Part 2*, Course of Theoretical Physics, Vol. 9 (Pergamon, New York, 1980).
[32] C. A. Müller (unpublished).