Partial indistinguishability theory for multiphoton experiments in multiport devices

V. S. Shchesnovich*

Centro de Ciências Naturais e Humanas, Universidade Federal do ABC, Santo André, São Paulo, 09210-170 Brazil

(Received 6 October 2014; published 29 January 2015)

We generalize an approach for description of multiphoton experiments with multiport unitary linear optical devices, started in Phys. Rev. A **89**, 022333 (2014) with single photons in mixed spectral states, to arbitrary (multiphoton) input and arbitrary photon detectors. We show that output probabilities are always given in terms of the matrix permanents of the Hadamard product of a matrix built from the network matrix and matrices built from the spectral state of photons and spectral sensitivities of detectors. Moreover, in the case of input with up to one photon per mode, the output probabilities are given by a sum (or integral) with each term being the absolute value squared of such a matrix permanent. We conjecture that, for an arbitrary multiphoton input, zero output probability of an output configuration is *always* the result of an exact cancellation of quantum transition amplitudes of completely indistinguishable photons (a subset of all input photons) and, moreover, is *independent* of coherence between only partially indistinguishable photons. The conjecture is supported by examples. Furthermore, we propose a measure of partial indistinguishability of photons which generalizes Mandel's observation, and find the law of degradation of quantum coherence in a realistic boson-sampling device with increase of the total number of photons and/or their "classicality parameter."

DOI: 10.1103/PhysRevA.91.013844 PACS number(s): 42.50.St, 03.67.Ac, 42.50.Ar

I. INTRODUCTION

It is well known [1] that the quantum coherence of an electromagnetic field and indistinguishability of photons are intimately related to each other. The most famous quantum coherence effect of this type is the Hong-Ou-Mandel (HOM) dip [2,3], where the "dip" in the output coincidence probability of a balanced beam splitter corresponds to complete indistinguishability of single photons at its input. Many important developments in the area of multiphoton experiments with multiport optical devices have been achieved since then. A generalization of the HOM effect and a difference in behavior of bosons and fermions was analyzed for Bell multiport beam splitters [4–6]. An approach describing partial distinguishability of photons obtained from parametric down-conversion sources was developed in Refs. [7–9]. Recently, a zero-transmission law due to a symmetry of the network matrix [10] and a quantum suppression law in many-particle interferences beyond the boson and fermion statistics were found [11]. Recent advances in quantum interference experiments in linear multiport devices include characterizing temporal distinguishability of four- and six-photon states [12], experimental control over eight individual single photons [13], observation of the two-photon HOM effect on integrated three- and four-port devices [14], verification of the three-photon HOM effect and the zero-transmission law on a tritter [15], a three-photon quantum interference experiment on an integrated eight-mode optical device [16], and observation of detection-dependent multiphoton coherence times [17]. The multiphoton quantum interference is central in the boson-sampling computer [18] with indistinguishable single photons and linear optics, the output of which is hard to simulate on a classical computer. Recently the experimental realization of the boson-sampling computer was tested on a small scale [19–23]. One must also mention the well-known proposal for universal quantum computation with linear optics [24].

The advances described above with multiphoton experiments of increasing complexity (see also the review [25]) and also the recent achievements in fabrication of photon sources [26] necessitate a theoretical approach which enables one to account for the effect of partial indistinguishability of photons in a realistic general setup of a multiport device with an arbitrary multiphoton input and with account for imperfect detectors. Here such a general approach is developed by generalization of that of Ref. [27]. As in Refs. [7,9,27–29] we employ the permutation symmetry of the spectral state of photons to characterize their partial indistinguishability and further advance this relation: we derive the general output probabilities for multiphoton experiments with multiport devices for an arbitrary number of network modes and an arbitrary multiphoton input, study the physical meaning of the partial indistinguishability matrix, introduced in Ref. [27], and introduce an auxiliary Hilbert space representation of spectral states of photons, which allows one to rewrite output probabilities in a clear compact form. In view of the application to the boson-sampling experiments, we discuss in detail the case of input with at most one photon per mode, give the output probability in a simplified form, and study the effect of degradation of quantum interference on a classicality parameter and the total number of photons. Note that a different approach based on the orthonormalization of photon spectral states, used in Refs. [17,30,31], which is helpful in few-photon cases, does not have a clear physical interpretation and will not be of much help for larger N or mixed spectral states [32].

Since the symmetric (i.e., permutation) group is the key object in our approach, one might expect that usage of advanced features of the symmetric group (i.e., the group characters and the corresponding Young diagrams) is essential for understanding multiphoton experiments in multiport devices. Indeed, recently three-photon interference in a tritter was analyzed using some advanced symmetric group structures called the matrix immanants (related to the nontrivial group

*valery@ufabc.edu.br

characters) [28,29]. However, such an approach is not scalable, since Young diagrams associated with nontrivial group characters can be analyzed only case by case with no formula for the general solution. Our approach, on the other hand, *does not depend* on any such advanced group structures. Only some elementary facts about the permutation group, such as the cycle decomposition, are used. We show, for instance, that the zero-coincidence condition for partially indistinguishable photons of Refs. [28,29], involving the matrix immanants, can be restated as a zero-permanent condition of a Hadamard product of a network matrix and a matrix built from spectral states of photons and detector sensitivities. We also conjecture that zero output probability of an output configuration is *always* the result of exact cancellation of quantum transition amplitudes of completely indistinguishable photons (a subset of all input photons) when a network allows for such an exact cancellation. Moreover, in all cases, zero probability *is independent* of the degree of coherence of only partially indistinguishable photons.

Finally, it should be mentioned that the effect of partial indistinguishability of photons on probabilities at a network output has a deep relation with the duality (complementarity) between the fringe visibility and the which-way information. This duality is well understood for two-path interference experiments [33–35]. Indeed, although the output probability is related to a Glauber higher-order coherence function [36], whereas the duality pertains to the first-order coherence of a single quantum object, when all photons are detected for an input with a certain number of photons, one can reinterpret the multiphoton interference as a multipath interference experiment, where there are $N!$ paths for N photons. Such a relation was studied by Mandel [1] for $N = 2$ (see also Ref. [37]). However, following this point of view in discussion of N-photon multiport experiments for $N > 2$ meets with several obstacles and is not pursued here. One of them is that generalization of the duality to multipath coherence is not unique [38]. However, the duality supplies a clear physical interpretation of the formulas derived below. Moreover, an argument referring to the duality is used for formulation of the zero-probability conjecture.

In Sec. II we derive the general formula for the output probability in a multimode network for arbitrary multiphoton input. Some details of the derivation are placed in Appendix A. In Secs. II D and II E we compare the case of ideal (i.e., maximally efficient) detectors with that of realistic detectors for two extreme cases of input: completely indistinguishable photons and maximally distinguishable photons. In Sec. II F we express the output probability via matrix permanents of the Hadamard product of matrices, one built from the network matrix and the other from spectral states of photons and sensitivities of detectors. In Sec. II G we propose a measure for partial indistinguishability of photons generalizing Mandel's parameter for $N > 2$ photons. We focus on the input with a single photon or vacuum per input mode in Sec. III, where we give a simplified formula for the output probability and analyze its structure for single photons in pure spectral states (Sec. III A), and generalize the result to the case of single photons in mixed spectral states (Sec. III B). In Sec. III C we formulate the zero-probability conjecture and study a few examples supporting it. Some mathematical calculations of Sec. III are relegated to Appendices B and C. Finally, in Sec. III D we discuss a model of the boson-sampling computer and compute the purity of the partial indistinguishability matrix as a measure of the closeness of a realistic device with only partially indistinguishable photons to the ideal boson-sampling computer. Some final remarks are placed in the concluding Sec. IV.

II. OUTPUT PROBABILITY FORMULA FOR A FIXED NUMBER OF PHOTONS IN EACH INPUT MODE

A. Input state

Consider a linear unitary optical network of M different inputs (we consider each input to be single mode) where an n_k-photon state is injected into the kth input mode. Below we set $n_1 + \cdots + n_M = N$ (in general, the number of modes with a nonvacuum input is less than M). We are interested in the expression for the output probabilities for such an input. In view of the problem formulation, it is convenient to use a basis for photon states consisting of spatial mode k, polarization state s (where, say, $s = 0$ and $s = 1$ correspond to two orthogonal basis states of photon polarization), and frequency ω. We denote photon creation and annihilation operators in this basis by a subscript (k,s) and consider them to be functions of ω. A spatial unitary network can be defined by a unitary transformation between input $a^\dagger_{k,s}(\omega)$ and output $b^\dagger_{k,s}(\omega)$ photon creation operators, we set $a^\dagger_{k,s}(\omega) = \sum_{l=1}^{M} U_{kl} b^\dagger_{l,s}(\omega)$, where U_{kl} is the unitary matrix describing such an optical network. Below we will employ vector notation for greater convenience, e.g., $\vec{n} = (n_1,\ldots,n_M)$ for numbers of photons in spatial modes, $\vec{\omega} = (\omega_1,\ldots,\omega_N)$ for frequencies, and $\vec{s} = (s_1,\ldots,s_N)$ for polarizations. We define also $|\vec{n}| \equiv n_1 + \cdots + n_M$ and $\mu(\vec{n}) \equiv \prod_{k=1}^{M} n_k!$. The general N-photon input (a mixed state) with a certain number of photons in each input mode is given by the following expression:

$$\rho(\vec{n}) = \frac{1}{\mu(\vec{n})} \sum_{\vec{s}\,'} \sum_{\vec{s}} \int d\vec{\omega}' \int d\vec{\omega} G(\vec{s}\,',\vec{\omega}'|\vec{s},\vec{\omega}) \times \left[\prod_{\alpha=1}^{N} a^\dagger_{k_\alpha,s'_\alpha}(\omega'_\alpha)\right] |0\rangle\langle 0| \left[\prod_{\alpha=1}^{N} a_{k_\alpha,s_\alpha}(\omega_\alpha)\right], \quad (1)$$

where $k_1,\ldots,k_N$ are input modes (generally repeated where the repetition numbers are given by $\vec{n}$) and G is a function describing the spectral and polarization state (mixed, in general) of N input photons.[1] An immediate consequence of the bosonic commutation relations is that any permutation π of frequencies and polarizations associated with either creation or annihilation operators in Eq. (1), i.e., $(s_\alpha,\omega_\alpha) \to (s_{\sigma(\alpha)},\omega_{\sigma(\alpha)})$, which permutes photons from the same input mode k, leaves the function G of Eq. (1)

[1]To use the index k instead of the double index k_α would not be simpler because we have multiple k indices for $n_k > 1$. To leave just the subscript by definition $\hat{U}_{\alpha,\beta} \equiv U_{k_\alpha,l_\beta}$ is also not convenient, since some formulas have an essential dependence on the output configuration of the spatial modes due to the different detectors attached to them.

invariant. The group of such permutations, a subgroup of all permutations $\mathcal{S}_N$, is equivalent to the tensor product of groups $\mathcal{S}_{n_1} \otimes \cdots \otimes \mathcal{S}_{n_M}$ (some $\mathcal{S}_{n_\alpha}$ may be empty because $n_\alpha = 0$). Given this permutation symmetry of G, the following normalization condition can be derived from the fact that ρ of Eq. (1) is a density matrix with trace equal to 1:

$$\sum_{\vec{s}} \int d\vec{\omega}\, G(\vec{s},\vec{\omega}|\vec{s},\vec{\omega}) = 1. \tag{2}$$

The function G is also constrained by positivity of the associated density matrix ρ. Below we will frequently use two other representations of the density matrix in Eq. (1). The diagonalized form

$$\rho(\vec{n}) = \sum_i p_i |\tilde{\Phi}_i\rangle\langle\tilde{\Phi}_i|, \quad \langle\tilde{\Phi}_i|\tilde{\Phi}_j\rangle = \delta_{ij},$$
$$|\tilde{\Phi}_i\rangle = \frac{1}{\sqrt{\mu(\vec{n})}} \sum_{\vec{s}} \int d\vec{\omega}\, \Phi_i(\vec{s},\vec{\omega}) \left[\prod_{\alpha=1}^{N} a^\dagger_{k_\alpha,s_\alpha}(\omega_\alpha)\right]|0\rangle, \tag{3}$$

obtained by decomposing the function G of Eq. (1) as $G(\vec{s}\,',\vec{\omega}'|\vec{s},\vec{\omega}) = \sum_i p_i \Phi_i(\vec{s}\,',\vec{\omega}')\Phi_i^*(\vec{s},\vec{\omega})$, where $\sum_{\vec{s}} \int d\vec{\omega} |\Phi_i(\vec{s},\vec{\omega})|^2 = 1$ and $\sum_i p_i = 1$ $(p_i > 0)$, and another very important representation, which applies to sources with some fluctuating parameter(s), say x. In the latter case, the density matrix has a form similar to that of Eq. (3) but with some nonorthogonal states $|\tilde{\Phi}(x)\rangle$,

$$\rho(\vec{n}) = \int dx\, p(x)|\tilde{\Phi}(x)\rangle\langle\tilde{\Phi}(x)|, \tag{4}$$

where we assume that the state vector $|\tilde{\Phi}(x)\rangle$ is given similarly as in the second line of Eq. (3).

Typical input states encountered in experiments are covered by the input of Eq. (3) or (4). For instance, if we have N independent sources of single photons attached to modes k_α, $\alpha = 1,\ldots,N$, with source α emitting single photons in a polarized (say $s_\alpha = 1$) Gaussian state with the central frequency Ω_α, spectral width Δ_α, and arrival time t_α, then the corresponding input state is pure, $\rho = |\tilde{\Phi}\rangle\langle\tilde{\Phi}|$, where

$$|\tilde{\Phi}\rangle = \int d\vec{\omega} \left[\prod_{\alpha=1}^{N} \phi_\alpha(\omega_\alpha) a^\dagger_{k_\alpha,1}(\omega_\alpha)\right]|0\rangle, \tag{5}$$

with

$$\phi_\alpha(\omega) = \left(2\pi\Delta_\alpha^2\right)^{-1/4} \exp\left\{i\omega t_\alpha - \frac{(\omega-\Omega_\alpha)^2}{4\Delta_\alpha^2}\right\} \tag{6}$$

[note that we write $\phi_\alpha(\omega)$ and not $\phi_\alpha(\omega,t_\alpha)$ since it is a function of ω, whereas t_α is a fixed parameter, different for different indices α; we will use this rule below for the sake of simplicity]. One frequent example of this kind is of N photons in the same Gaussian state, i.e., $\Omega_\alpha = \Omega$ and $\Delta_\alpha = \Delta$, but with different arrival times. This example is, of course, only illustrative and sometimes used to model a realistic situation due to manageability of the Gaussian function and because in experiments only a few parameters, such as the central frequency and spectral width of the photon sources are known with some precision. One can contemplate a more general model of this kind: when polarized single photons have spectral states of the same shape, differing only by the delay time, the appropriate representation is $\phi_\alpha(\omega) = \int dt e^{i\omega t} f(t-t_\alpha)$ for an arbitrary function $f(t)$ with the norm equal to 1.

When spectral states of photons have fluctuating parameters, e.g., the arrival time, polarization, etc., the most appropriate representation is Eq. (4). For example, such an input gives a model of a realistic boson-sampling computer [18] (see Sec. III D for more details).

B. Output probabilities and interference of "paths"

Consider M, generally different, number-resolving detectors attached to network output modes. The probability of detecting $m_1,\ldots,m_M$ photons in network output modes $1,\ldots,M$ can be derived using photon-counting theory [36,39–41]. The result is that the probability for all photons to be detected at the network output in a configuration $\vec{m}$ is given by the expectation value of the following operator (see also Appendix A in Ref. [27]):

$$\Pi(\vec{m}) = \frac{1}{\mu(\vec{m})} \sum_{\vec{s}} \int d\vec{\omega} \prod_{\alpha=1}^{N} \Gamma_{l_\alpha}(s_\alpha,\omega_\alpha)$$
$$\times \left[\prod_{\alpha=1}^{N} b^\dagger_{l_\alpha,s_\alpha}(\omega_\alpha)\right]|0\rangle\langle 0|\left[\prod_{\alpha=1}^{N} b_{l_\alpha,s_\alpha}(\omega_\alpha)\right], \tag{7}$$

where the indices $l_1,\ldots,l_N$ comprise the sequence $1,\ldots,1,2,\ldots,2,\ldots,M,\ldots,M$, with each index j appearing m_j times, and $0 \leqslant \Gamma_l(s,\omega) \leqslant 1$ is the sensitivity function of the detector attached to the lth output mode. The output probability of a configuration $\vec{m}$ reads

$$P(\vec{m}|\vec{n}) = \text{Tr}\{\rho(\vec{n})\Pi(\vec{m})\}. \tag{8}$$

The operator $\Pi(\vec{m})$ in Eq. (8) is Hermitian and positive, but such operators generally do not sum up to the identity operator (more precisely, to the projector on the symmetric subspace corresponding to N bosons). However, for efficient detectors, when all output photons are detected, each $\Pi(\vec{m})$ becomes an element of the positive-operator valued measure realizing the detection described above. In this case the probabilities in Eq. (8) sum to 1 under the constraint $|\vec{m}| = N$.

The essence of our approach below is based on the fact that the basis variables (k,s,ω) are divided into two parts: (i) the spatial mode k, affected by a unitary network, and (ii) the spectral part (functions of polarization and frequency), not changed by the network and thus serving as a label for the partial indistinguishability of photons (by the distinguishability here and below we mean distinguishability detectable in an experiment in the setting described above). The Fock space, natural for identical particles, is not the most appropriate Hilbert space for treating partial indistinguishability, since it involves the boson creation and annihilation operators indexed by (k,s,ω), whereas only the spectral part defines the partial indistinguishability of photons. Another problem with the Fock space is that to treat partial indistinguishability it is better to employ a basis used for distinguishable particles. Below we employ such an auxiliary Hilbert space of N fictitious distinguishable particles to use for description of the spectral state of N photons. In this way a connection to the duality of the which-path information vs the interference visibility can be established: one can visualize the transitions through a unitary network as "paths" (there are $N!$ paths which can be labeled by elements of the symmetric group $\mathcal{S}_N$) whereas the spectral states serve as some internal degrees of freedom which can, in principle, be observed by the environment. Summation over the path amplitudes is affected by the indistinguishability of the spectral states of photons and also by the spectral sensitivities of detectors. For identical detectors, two permutations of the fictitious particles, one at input (σ_1) and one at output (σ_2), represent a different set of paths with respect to $\sigma_1 = \sigma_2 = I$ (identity permutation) only if they are not equal (spectral data are not changed by the network). But for different detectors even if $\sigma_2 = \sigma_1$ the output probability is generally different for different σ_1. Hence, an $(N! \times N!)$-dimensional partial indistinguishability matrix, indexed by elements of $\mathcal{S}_N$, describes all possible path interferences for general detectors, whereas at most $N!$ parameters of such a matrix are different for identical detectors.

Now let us give the output probability for an arbitrary input Eq. (1). Due to the relation $a^\dagger_{k,s}(\omega) = \sum_{l=1}^M U_{kl} b^\dagger_{l,s}(\omega)$ between input and output modes, Eq. (8) is a nonnegative quadratic form with complex arguments equal to products of N matrix elements of a network matrix U, where the spectral part defines the matrix of this quadratic form. We have from Eqs. (1), (7), and (8) (the details can be found in Appendix A)

$$P(\vec{m}|\vec{n}) = \frac{1}{\mu(\vec{m})\mu(\vec{n})} \sum_{\sigma_1} \sum_{\sigma_2} J(\sigma_1,\sigma_2) \times \prod_{\alpha=1}^{N} U^*_{k_{\sigma_1(\alpha)},l_\alpha} U_{k_{\sigma_2(\alpha)},l_\alpha}, \quad (9)$$

where the matrix J, the partial indistinguishability matrix, indexed by two permutations σ_1 and σ_2 of N elements, reads

$$J(\sigma_1,\sigma_2) = \sum_{\vec{s}} \int d\vec{\omega} \prod_{\alpha=1}^{N} \Gamma_{l_\alpha}(s_\alpha,\omega_\alpha) \times G\big(\{s_{\sigma_1^{-1}(\alpha)},\omega_{\sigma_1^{-1}(\alpha)}\}\big|\{s_{\sigma_2^{-1}(\alpha)},\omega_{\sigma_2^{-1}(\alpha)}\}\big). \quad (10)$$

Here we note that for different detector sensitivities $\Gamma_{l_1},\ldots,\Gamma_{l_N}$, the matrix elements $J(\sigma_1,\sigma_2)$ also depend on the chosen output modes, and thus a subscript $\vec{m}$ must be attached to them. However, for simplicity of notation we omit it. The matrix J is Hermitian, $J^*(\sigma_1,\sigma_2) = J(\sigma_2,\sigma_1)$, and nonnegative definite.

The J-matrix expansion of the output probability was first introduced in Ref. [27] for N single photons in mixed spectral states to study a model of a boson-sampling computer with realistic sources. It is also equivalent to the rate matrix used in the more recent Ref. [29]. Our J matrix generalizes an old observation [1] that there is a deep relation between indistinguishability of photons and fringe visibility at the output of a beam splitter (see the details in Sec. II G 1). For two photons in mixed spectral states, a similar approach based on identifying a partial indistinguishability parameter was also used in Ref. [37].

There is a continuous family of spectral states of photons which correspond to the same J matrix (see also Sec. II D below) and, therefore, to the same probability distribution at the output of a given unitary network. Let us unite all possible output probability distributions for all possible unitary networks U (for all M) with the same J matrix in a single class. The question is whether a unique J matrix corresponds to each such class of output probability distributions. In other words, are two different setups corresponding to two different J matrices distinguishable by experiments with unitary linear networks? On first sight, there seem to be more parameters in a J matrix than one can recover from such a class of output probability distributions. Indeed, the quadratic form of Eq. (9) depends on $N!$ complex variables, but is evaluated at $X_\sigma \equiv \prod_{\alpha=1}^N U_{k_{\sigma(\alpha)},l_\alpha}$, i.e., involving at most N^2 independent elements of a network matrix U. Thus it seems that for sufficiently large N the information contained in the J matrix cannot be deduced from a given class of output probabilities (which would require independently varying X_σ for different $\sigma \in \mathcal{S}_N$). However, note also that not every positive definite Hermitian matrix can be a J matrix of a photonic input, since it must be given according to Eq. (10) which imposes some conditions, making the above reasoning not conclusive. We will not discuss this question any further in this work, relegating it to a future investigation.

The output probability of Eq. (9) can be also thought of as a multinomial, of total power N^2, in $2N^2$ matrix elements U_{k_α,l_β} and $U^*_{k_\alpha,l_\beta}$. But this approach, although reducing the number of variables used, loses the attractive simplicity of our approach with the J matrix with a clear physical interpretation, given above, where X_σ serves as a "path amplitude" of fictitious particles (this interpretation is employed in Sec. III C below for formulation of the zero-probability conjecture).

C. Auxiliary Hilbert space for spectral states

To clarify the mathematical structure of the expressions in Eqs. (9) and (10) let us introduce an auxiliary Hilbert space $\mathcal{H}$ for description of the spectral state of photons (a similar method was employed in Ref. [27]). Let us denote by $|s,\omega\rangle$ a basis vector for expansion of the spectral state of a single particle; then

$$\sum_s \int d\omega |s,\omega\rangle\langle s,\omega| = I. \quad (11)$$

A spectral state of N particles belongs to the tensor product space $\mathcal{H}^{\otimes N}$ (the auxiliary particles are distinguishable objects). A basis vector in $\mathcal{H}^{\otimes N}$ will be denoted by $|\vec{s},\vec{\omega}\rangle \equiv |s_1,\omega_1\rangle \otimes \cdots \otimes |s_N,\omega_N\rangle$. With these definitions, a density matrix describing the spectral state of photons is obtained by simply replacing the Fock basis states in the expansion of the input density matrix $\rho(\vec{n})$ of Eq. (1) by the respective tensor product states, i.e.,

$$\hat{\rho} \equiv \sum_{\vec{s}\,'} \sum_{\vec{s}} \int d\vec{\omega}' \int d\vec{\omega} G(\vec{s}\,',\vec{\omega}'|\vec{s},\vec{\omega}) |\vec{s}\,',\vec{\omega}'\rangle\langle\vec{s},\vec{\omega}|, \quad (12)$$

the normalization condition of Eq. (2) ensures that $\hat{\rho}$ has trace equal to 1 [positivity of $\hat{\rho}$ also follows from that of ρ in Eq. (1)]. Permutation operations in the auxiliary space $\mathcal{H}^{\otimes N}$ play an essential role below. A permutation operator P_σ, corresponding to a permutation σ of N elements, acts in $\mathcal{H}^{\otimes N}$ as follows:

$$P_\sigma |j_1\rangle \otimes \cdots \otimes |j_N\rangle \equiv |j_{\sigma^{-1}(1)}\rangle \otimes \cdots \otimes |j_{\sigma^{-1}(N)}\rangle \quad (13)$$

[by this definition the vector from the kth Hilbert space $\mathcal{H}$ in the tensor product goes to the $\sigma(k)$th space]. The set of operators P_σ is a representation of the symmetric (permutation) group $\mathcal{S}_N$, i.e., we have $P_{\sigma_1}P_{\sigma_2} = P_{\sigma_1\sigma_2}$ (note that $P^\dagger_\sigma = P_{\sigma^{-1}}$). Below we will frequently refer to permutations π exchanging spectral states of photons in each input mode between themselves; thus we associate with the Hilbert space in position α in the tensor product $\mathcal{H}^{\otimes N}$ an input mode index k_α of naturally ordered set $(k_1,\ldots,k_N)$, and therefore we can identify such permutations with subgroup $\mathcal{S}_{n_1}\otimes\cdots\otimes\mathcal{S}_{n_M}$ acting on $\mathcal{H}^{\otimes N}$.

Due to the symmetry property of the G function of Eq. (1), we have for any permutation $\pi\in\mathcal{S}_{n_1}\otimes\cdots\otimes\mathcal{S}_{n_M}$

$$P_\pi\hat{\rho} = \hat{\rho}P_\pi = \hat{\rho}. \tag{14}$$

For instance, in the case of diagonal representation, Eq. (3), and in the fluctuating parameter case, Eq. (4) with respective basis states $|\tilde{\Phi}(x)\rangle$ being linearly independent (e.g., photons in spectral states of a Gaussian shape with fluctuating arrival times), the property (14) implies that the respective basis functions $\Phi(x;\vec{s},\vec{\omega})$ are invariant under permutations $\pi\in\mathcal{S}_{n_1}\otimes\cdots\otimes\mathcal{S}_{n_M}$ of (s_α,ω_α).

Let us also introduce a detector operator which has a diagonal representation in the above-defined auxiliary Hilbert space, i.e.,

$$\hat{\Gamma}_l \equiv \sum_{\vec{s}}\int d\vec{\omega}\,\Gamma_l(s,\omega)|s,\omega\rangle\langle s,\omega|. \tag{15}$$

Then the matrix J defined in Eq. (10) assumes the following compact form:

$$\begin{aligned} J(\sigma_1,\sigma_2) &= \sum_{\vec{s}}\int d\vec{\omega}\,\langle\vec{s},\vec{\omega}|\hat{\Gamma}_{l_1}\otimes\cdots\otimes\hat{\Gamma}_{l_N}P^\dagger_{\sigma_2}\hat{\rho}P_{\sigma_1}|\vec{s},\vec{\omega}\rangle \\ &= \mathrm{Tr}\{\hat{\Gamma}_{l_1}\otimes\cdots\otimes\hat{\Gamma}_{l_N}P^\dagger_{\sigma_2}\hat{\rho}P_{\sigma_1}\}, \end{aligned} \tag{16}$$

where the trace is taken in $\mathcal{H}^{\otimes^N}$. In its turn, the output probability of Eq. (9) becomes

$$P(\vec{m}|\vec{n}) = \frac{1}{\mu(\vec{m})\mu(\vec{n})}\mathrm{Tr}\{\hat{\Gamma}_{l_1}\otimes\cdots\otimes\hat{\Gamma}_{l_N}\mathcal{U}_N\hat{\rho}\,\mathcal{U}^\dagger_N\}, \tag{17}$$

where we have introduced an operator $\mathcal{U}_N$ acting in $\mathcal{H}^{\otimes^N}$ and given by

$$\mathcal{U}_N \equiv \sum_\sigma\left[\prod_{\alpha=1}^N U_{k_{\sigma(\alpha)},l_\alpha}\right]P^\dagger_\sigma. \tag{18}$$

Although Eqs. (16)–(18) are an equivalent representation of Eqs. (9) and (10), the former set of equations makes it clearer how to analyze the results by application of the methods of linear algebra in the Hilbert space.

By definition, in the case of a general (e.g., entangled) input, the J matrix involves a trace in the tensor product space $\mathcal{H}^{\otimes N}$. However, one can easily show that in the case of a factorized input (e.g., for independent photon sources),

$$\hat{\rho} = \prod_{\alpha=1}^{N}{}^{\otimes}\hat{\rho}_\alpha, \tag{19}$$

or for an input that is a convex combination of such factorized states the corresponding J matrix is expressed through some traces only in $\mathcal{H}$. Indeed, for an arbitrary permutation σ, by using Eq. (13), we obtain the following identity between a trace in $\mathcal{H}^{\otimes N}$ and that in $\mathcal{H}$:

$$\mathrm{Tr}\left\{P^\dagger_\sigma\prod_{\alpha=1}^{N}{}^{\otimes}A_\alpha\right\} = \prod_{j=1}^q\mathrm{Tr}\{A_{\alpha_{j1}}\cdots A_{\alpha_{j\ell_j}}\}, \tag{20}$$

where $c_1,\ldots,c_q$ is the set of disjoint cycles in the decomposition $\sigma = c_1\cdots c_q$, the cycle c_i is assumed to be given by $\alpha_{j1}\to\alpha_{j2}\to\cdots\to\alpha_{j\ell_j}\to\alpha_{j1}$, and ℓ_j is the cycle length. Therefore, assuming the above cycle structure of $\sigma_R\equiv\sigma_2\sigma_1^{-1}$, for an input of Eq. (19) we obtain from Eq. (16)

$$J(\sigma_1,\sigma_2) = \prod_{j=1}^q\mathrm{Tr}\left\{\hat{\Gamma}_{l_{\sigma_2^{-1}(\alpha_{j1})}}\hat{\rho}_{\alpha_{j1}}\cdots\hat{\Gamma}_{l_{\sigma_2^{-1}(\alpha_{j\ell_j})}}\hat{\rho}_{\alpha_{j\ell_j}}\right\}. \tag{21}$$

From Eq. (21) it is seen that for identical detectors and input (19) $J(\sigma_1,\sigma_2)$ depends only on the cycle decomposition of the relative permutation $\sigma_2\sigma_1^{-1}$.

D. Completely indistinguishable and maximally distinguishable photons with ideal detectors: *J* matrices and corresponding inputs

First of all, one can easily verify that for maximally efficient detectors, $\Gamma_l(s,\omega)=1$, the output probabilities sum to 1, as they should. Indeed, in this case Eqs. (16)–(18) give

$$\begin{aligned} \sum_{|\vec{m}|=N}P(\vec{m}|\vec{n}) &= \sum_{\vec{l}}\frac{\mu(\vec{m})}{N!}\frac{1}{\mu(\vec{m})\mu(\vec{n})}\mathrm{Tr}\{\mathcal{U}_N\hat{\rho}\,\mathcal{U}^\dagger_N\} \\ &= \frac{1}{N!\mu(\vec{n})}\sum_{\sigma_1}\sum_\pi\mathrm{Tr}\{P^\dagger_{\sigma_1}P^\dagger_\pi\hat{\rho}P_{\sigma_1}\} \\ &= \mathrm{Tr}\{\hat{\rho}\} = 1, \end{aligned} \tag{22}$$

where we have used an identity due to unitarity of the network matrix U,

$$\prod_{\alpha=1}^N\sum_{l_\alpha=1}^M U_{k_{\sigma_2(\alpha)},l_\alpha}U^*_{k_{\sigma_1(\alpha)},l_\alpha} = \prod_{\alpha=1}^N\delta_{k_{\sigma_1(\alpha)},k_{\sigma_2(\alpha)}} = \sum_\pi\delta_{\sigma_2\sigma_1^{-1},\pi},$$

with $\pi\in\mathcal{S}_{n_1}\otimes\cdots\otimes\mathcal{S}_{n_M}$ [thus $\sum_\pi 1 = \mu(\vec{n})$].

Equations (17) and (18) generalize the well-known fact [42,43] that in the ideal case of completely indistinguishable photons and ideal detectors the bosonic output probability in a unitary linear network is expressed through the absolute value squared of the matrix permanent of an $(N\times N)$-dimensional matrix $U[\vec{n}|\vec{m}]$, built from the network matrix by selecting, with repetitions, rows (columns) corresponding to the input $\vec{n}$ (the output $\vec{m}$) of a considered transition, i.e.,

$$P^{(\mathrm{ind})}(\vec{m}|\vec{n}) = \frac{\left|\sum_\sigma\prod_{\alpha=1}^N U_{k_\sigma(\alpha),l_\alpha}\right|^2}{\mu(\vec{m})\mu(\vec{n})} = \frac{|\mathrm{per}(U[\vec{n}|\vec{m}])|^2}{\mu(\vec{m})\mu(\vec{n})}, \tag{23}$$

where the permanent of an$(N\times N)$-dimensional matrix A is defined as $\mathrm{per}(A)=\sum_\sigma\prod_{\alpha=1}^N A_{\sigma(\alpha),\alpha}$ (for a discussion of properties of the matrix permanents, see Ref. [44]). In this case

$$J^{(\mathrm{ind})}(\sigma_1,\sigma_2) = 1 \tag{24}$$

for all permutations σ_1 and σ_2, i.e., the matrix $J^{(\mathrm{ind})}$ (24) is pure (has rank 1)

$$J^{(\mathrm{ind})} = v^\dagger v, \quad v \equiv (1,\ldots,1), \quad |\vec{v}| = N!, \tag{25}$$

where $|v| \equiv \sum_j |v_j|$. It has only one nonzero eigenvalue equal to $N!$. Now let us see what input states give the J matrix of Eq. (24). Using that $\mathrm{Tr}\{P^\dagger_{\sigma_2}\hat{\rho}^{(\mathrm{ind})}P_{\sigma_1}\} = \mathrm{Tr}\{P_{\sigma_1\sigma_2^{-1}}\hat{\rho}^{(\mathrm{ind})}\} = 1$ one can establish that in the diagonal representation following from Eq. (3), i.e.,

$$\hat{\rho} = \sum_i p_i|\Phi_i\rangle\langle\Phi_i|, \quad |\Phi_i\rangle = \sum_{\vec{s}}\int d\vec{\omega}\,\Phi_i(\vec{s},\vec{\omega})|\vec{s},\vec{\omega}\rangle, \tag{26}$$

applied to $\hat{\rho}^{(\mathrm{ind})}$, the basis states are symmetric: $P_\sigma|\Phi_i\rangle = |\Phi_i\rangle$ for any $\sigma \in \mathcal{S}_N$. A similar conclusion applies to an expansion over a basis of nonorthogonal linearly independent states, following from Eq. (4). The corresponding functions $\Phi_i(\vec{s},\vec{\omega})$ and, hence, $G^{(\mathrm{ind})}(\vec{s}\,',\vec{\omega}'|\vec{s},\vec{\omega})$ are symmetric with respect to any permutation of their arguments. We note that a similar condition was first established in Ref. [7]. For completely indistinguishable single photons each basis state $|\Phi_i\rangle$ in the expansion of $\hat{\rho}^{(\mathrm{ind})}$ is of the form

$$|\Phi_i\rangle = \frac{c_i}{N!}\sum_\sigma \prod_{\alpha=1}^{N}{}^{\otimes}|\phi^{(i)}_{\sigma(\alpha)}\rangle, \tag{27}$$

where the normalization coefficient is given by $c_i^2 = N!/\mathrm{per}(\mathcal{G}^{(i)})$ with $\mathcal{G}^{(i)}_{\alpha\beta} = \langle\phi^{(i)}_\alpha|\phi^{(i)}_\beta\rangle$. A similar observation was first employed in Ref. [45] for engineering complete indistinguishability by coherently overlapping two processes for creation of a pair of photons.

Guided by the above, we will say that the photons are maximally distinguishable if the respective matrix J is maximally mixed as allowed by Eq. (14). From Eqs. (14) and (16) we have for $\pi_{1,2} \in \mathcal{S}_{n_1}\otimes\cdots\otimes\mathcal{S}_{n_M}$

$$J(\pi_1\sigma_1,\pi_2\sigma_2) = J(\sigma_1,\sigma_2); \tag{28}$$

hence, the most mixed J reads

$$J^{(cl)}(\sigma_1,\sigma_2) = \sum_\pi \delta_{\sigma_2,\pi\sigma_1} = \frac{1}{\mu(\vec{n})}\sum_\pi\sum_{\pi'}\delta_{\pi'\sigma_2,\pi\sigma_1}, \tag{29}$$

where the second form manifests compliance with the required symmetry of Eq. (28). Note that the matrix $J^{(cl)}$ has a block-diagonal form

$$J^{(cl)} = \sum_q{}^{\oplus} v_q^\dagger v_q, \quad v_q \equiv (1,\ldots,1), \quad |v_q| = \mu(\vec{n}), \tag{30}$$

where there are $\frac{N!}{\mu(\vec{n})}$ blocks (terms in the direct sum). The states in the diagonal representation (26) of $\hat{\rho}^{(cl)}$ satisfy the property

$$\langle\Phi_i|P_\sigma|\Phi_i\rangle = 0 \tag{31}$$

for all permutations $\sigma \notin \mathcal{S}_{n_1}\otimes\cdots\otimes\mathcal{S}_{n_M}$. The same property applies to expansion as in Eq. (26) over a basis of nonorthogonal but linearly independent states. In an equivalent form, this condition can be formulated for the corresponding function $G^{(cl)}$ as the following orthogonality condition:

$$\sum_{\vec{s}}\int d\vec{\omega}\,G^{(cl)}(\{s_{\sigma(\alpha)},\omega_{\sigma(\alpha)}\}|\{s_\alpha,\omega_\alpha\}) = 0 \tag{32}$$

for $\sigma \notin \mathcal{S}_{n_1}\otimes\cdots\otimes\mathcal{S}_{n_M}$. A similar condition was first discussed in Ref. [7]. The output probability corresponding to the $J^{(cl)}$ of Eq. (29) reads

$$P^{(cl)}(\vec{m}|\vec{n}) = \frac{\sum_\sigma\sum_\pi\prod_{\alpha=1}^N U^*_{k_{\sigma(\alpha)},l_\alpha}U_{k_{\pi\sigma(\alpha)},l_\alpha}}{\mu(\vec{m})\mu(\vec{n})} = \frac{\sum_\sigma\prod_{\alpha=1}^N\left|U_{k_{\sigma(\alpha)},l_\alpha}\right|^2}{\mu(\vec{m})}, \tag{33}$$

since for $\pi \in \mathcal{S}_{n_1}\otimes\cdots\otimes\mathcal{S}_{n_M}$ we have $U_{k_{\pi(\alpha)},l_\alpha} = U_{k_\alpha,l_\alpha}$.

Let us note the following feature. The trace of matrix J, i.e., $\mathrm{Tr}\{J\} = \sum_\sigma J(\sigma,\sigma)$, for ideal detectors, coincides with the number $N!$ of different paths. For completely indistinguishable photons, Eq. (25), all paths interfere with equal weights [see Eq. (23)], whereas when photons in different input modes are maximally distinguishable, Eq. (30), there is no path interference contribution to the output probability. The output probability in the latter case has a natural classical interpretation, if one assumes that classical particles are classically indistinguishable, i.e., if their paths through the network are not traced. In this case, Eq. (33) describes the transition probability of N indistinguishable classical particles through a Markovian network whose transition matrix element A_{kl} is defined by $A_{kl} = |U_{kl}|^2$.

E. Completely indistinguishable and maximally distinguishable photons with realistic detectors

Let us see what changes occur in the above two extreme cases when realistic detectors with generally different efficiencies $\Gamma_l(s,\omega)$ are used. In this case the probability formula (17) applies to a postselected case, when all input photons are detected. The trace of the J matrix in this case is less than $N!$. We have

$$J(\sigma,\sigma) = \mathrm{Tr}\{\hat{\Gamma}_{l_1}\otimes\cdots\otimes\hat{\Gamma}_{l_N}P^\dagger_\sigma\hat{\rho}P_\sigma\} = \sum_{\vec{s}}\int d\vec{\omega}\,G(\vec{s},\vec{\omega}|\vec{s},\vec{\omega})\prod_{\alpha=1}^N\Gamma_{l_\alpha}(s_{\sigma(\alpha)},\omega_{\sigma(\alpha)}). \tag{34}$$

For completely indistinguishable photons $J(\sigma,\sigma)$ is independent of σ since G is completely symmetric under $\mathcal{S}_N$. Therefore, to reduce this case with realistic detectors to that of ideal detectors, a single additional parameter, the detection probability D,

$$D^{(\mathrm{ind})} = \sum_{\vec{s}}\int d\vec{\omega}\,G^{(\mathrm{ind})}(\vec{s},\vec{\omega}|\vec{s},\vec{\omega})\prod_{\alpha=1}^N\Gamma_{l_\alpha}(s_\alpha,\omega_\alpha), \tag{35}$$

independent of the considered network, must be defined. We obtain a J matrix of the form [compare with Eq. (25)]

$$J^{(\mathrm{ind})} = D^{(\mathrm{ind})}v^\dagger v, \quad v \equiv (1,\ldots,1), \quad |v| = N!. \tag{36}$$

The output probability is thus multiplied by $D^{(\mathrm{ind})}$.

For maximally distinguishable photons one can use the diagonal form (26) and note that by definition in the maximally distinguishable case $J(\sigma_1,\sigma_2) \neq 0$ only for $\sigma_2\sigma_1^{-1} \in \mathcal{S}_{n_1}\otimes\cdots\otimes\mathcal{S}_{n_M}$. This occurs under a condition involving detector sensitivities [replacing Eq. (31)]

$$\langle\Phi_i|\left[\prod_{\alpha=1}^{N}{}^{\otimes}\hat{\Gamma}_{l_{\sigma_1^{-1}(\alpha)}}\right]P^{\dagger}_{\sigma_2\sigma_1^{-1}}|\Phi_i\rangle = 0 \quad (37)$$

for all permutations satisfying $\sigma_2\sigma_1^{-1} \notin \mathcal{S}_{n_1}\otimes\cdots\otimes\mathcal{S}_{n_M}$ and l_α of the considered transition. Equation (37), thanks to the dependence also on σ_1, places more conditions on the spectral states of photons than Eq. (31) for ideal detectors. Moreover, for general dissimilar detectors, the corresponding $J(\sigma,\sigma)$ depends on σ. In matrix form [compare with Eq. (30)]

$$J^{(cl)} = \sum_{\tau}^{\oplus} D^{(cl)}(\tau)v_\tau^{\dagger}v_\tau, \quad v_\tau \equiv (1,\ldots,1),$$
$$|v_\tau| = \mu(\vec{n}), \quad (38)$$

where τ is the permutation of indices α belonging to different input modes in the decomposition $\sigma_2\sigma_1^{-1} = \tau\pi$ with $\pi \in \mathcal{S}_{n_1}\otimes\cdots\otimes\mathcal{S}_{n_M}$ and

$$D^{(cl)}(\tau) = \sum_{\vec{s}}\int d\vec{\omega}\, G^{(cl)}(\vec{s},\vec{\omega}|\vec{s},\vec{\omega})\prod_{\alpha=1}^{N}\Gamma_{l_\alpha}(s_{\tau(\alpha)},\omega_{\tau(\alpha)}). \quad (39)$$

The above two examples imply that one has to be careful in attributing a nearly zero output probability to quantum interference (for nonzero probability of a single-particle transition), since it may well happen that the zero probability is due to some generalization of the above-defined detection factors $J(\sigma,\sigma) \ll 1$, present in the maximally distinguishable (classical) case as well. A specific case of Gaussian-shaped single photons with different arrival times is considered in Appendix C. For arbitrary detectors and arbitrary input Eq. (1) we introduce a reduced J matrix in Sec. II G below.

F. Output probability in terms of the matrix permanents

Let us establish the form of output probability in the general case of arbitrary input of Eq. (1). We employ the diagonal representation (26). The output probability Eq. (17) can also be cast as

$$P(\vec{m}|\vec{n}) = \frac{1}{\mu(\vec{m})\mu(\vec{n})}\sum_i p_i\langle\Psi^{(i)}|\Psi^{(i)}\rangle, \quad (40)$$

where we have introduced $|\Psi^{(i)}\rangle \in \mathcal{H}^{\otimes N}$ as follows:

$$|\Psi^{(i)}\rangle \equiv \sum_{\sigma}\left[\prod_{\alpha=1}^{N}{}^{\otimes}U_{k_{\sigma(\alpha)},l_\alpha}\sqrt{\hat{\Gamma}_{l_\alpha}}\right]P^{\dagger}_{\sigma}|\Phi_i\rangle. \quad (41)$$

Let us use an orthogonal basis $|j\rangle$ in the Hilbert space $\mathcal{H}$ and expand

$$|\Phi_i\rangle = \sum_{\vec{j}} C^{(i)}_{\vec{j}}|\vec{j}\rangle, \quad (42)$$

where $|\vec{j}\rangle = |j_1\rangle\otimes\cdots\otimes|j_N\rangle \in \mathcal{H}^{\otimes N}$. From Eqs. (41) and (42) we obtain

$$\langle\vec{j}|\Psi^{(i)}\rangle = \sum_{\vec{j}'}C^{(i)}_{\vec{j}'}\sum_{\sigma}\prod_{\alpha=1}^{N}U_{k_{\sigma(\alpha)},l_\alpha}\langle j_\alpha|\sqrt{\hat{\Gamma}_{l_\alpha}}|j'_{\sigma(\alpha)}\rangle$$
$$= \sum_{\vec{j}'}C^{(i)}_{\vec{j}'}\mathrm{per}[U[\vec{n}|\vec{m}]\cdot B(\vec{j},\vec{j}')], \quad (43)$$

Here (and throughout the text) the central dot in a product of two matrices denotes the Hadamard (entrywise) product, in this case of the matrix $U[\vec{n}|\vec{m}]$ (built, as described above, from the network matrix U) and the matrix $B(\vec{j},\vec{j}')$ defined as follows:

$$B_{\beta,\alpha}(\vec{j},\vec{j}') \equiv \langle j_\alpha|\sqrt{\hat{\Gamma}_{l_\alpha}}|j'_\beta\rangle. \quad (44)$$

Using Eq. (43) in Eq. (40) we obtain the result

$$P(\vec{m}|\vec{n}) = \frac{1}{\mu(\vec{m})\mu(\vec{n})}\sum_i p_i\sum_{\vec{j}}\left|\sum_{\vec{j}'}C^{(i)}_{\vec{j}'}\mathrm{per}[V(\vec{j},\vec{j}')]\right|^2 \quad (45)$$

with $V(\vec{j},\vec{j}') \equiv U[\vec{n}|\vec{m}]\cdot B(\vec{j},\vec{j}')$.

One can use any basis of tensor product states for expansion in Eq. (42); for instance, in the standard spectral basis $|\vec{s},\vec{\omega}\rangle$ we have

$$P(\vec{m}|\vec{n}) = \frac{1}{\mu(\vec{m})\mu(\vec{n})}\sum_i p_i\sum_{\vec{s}}\int d\vec{\omega}$$
$$\times\left|\sum_{\vec{s}\,'}\int d\vec{\omega}'\Phi_i(\vec{s}\,',\vec{\omega}')\mathrm{per}[V(\vec{s},\vec{\omega},\vec{s}\,',\vec{\omega}')]\right|^2, \quad (46)$$

where $V(\vec{s},\vec{\omega},\vec{s}\,',\vec{\omega}') = U[\vec{n}|\vec{m}]\cdot B(\vec{s},\vec{\omega},\vec{s}\,',\vec{\omega}')$ with

$$B_{\beta,\alpha}(\vec{s},\vec{\omega},\vec{s}\,',\vec{\omega}') \equiv \delta_{s'_\beta,s_\alpha}\delta(\omega'_\beta-\omega_\alpha)\left[\Gamma_{l_\alpha}(s_\alpha,\omega_\alpha)\right]^{1/2}. \quad (47)$$

For example, Eq. (46) simplifies to Eq. (33) for ideal detectors if, using the definition of the B matrix (47), one first integrates (sums) over $\vec{\omega}$ ($\vec{s}$) in Eq. (46) by using the orthogonality condition (31), i.e., $\sum_{\vec{s}}\int d\vec{\omega}\Phi_i{}^*(\vec{s},\vec{\omega})\Phi_i(\{s_{\sigma(\alpha)},\omega_{\sigma(\alpha)}\}) = \delta_{\sigma,\pi}$ where $\pi \in \mathcal{S}_{n_1}\otimes\cdots\otimes\mathcal{S}_{n_M}$. The result is nothing but the J-matrix representation (9) with J of Eq. (29) which can be evaluated further according to the calculation of Sec. II D; see Eqs. (29) and (33).

One final observation is in order. In Eq. (45) or (46) the squared absolute value is taken of a coherent sum of the matrix permanents. In the case of single photons from independent sources, i.e., when the input density matrix is given by Eq. (19) with each $\hat{\rho}_\alpha$ being a density matrix in $\mathcal{H}$, one can also express the output probability as a sum (or integral) over the absolute values squared of the matrix permanents by using a different matrix for the spectral data in the Hadamard product (see Secs. III A and III B below).

G. J-matrix-based measure of the quantum coherence of photons

We have found above the form of the J matrix in the extreme cases of completely indistinguishable and maximally distinguishable photons for arbitrary detectors. Taking into account these results, it is suggestive to look for a J-matrix-based measure of quantum coherence of a multiphoton input for a given set of detectors. Note that quantum coherence of photon paths is reflected in the J matrix in a way very similar as in the usual density matrix of a quantum system (with the exception of the normalization). Using this observation, below we propose to use the purity as a measure of coherence for photons, which generalizes Mandel's parameter [1] for $N > 2$.

This measure is also a measure of partial indistinguishability, as it is in Mandel's case of two photons. We consider an arbitrary M-mode network given by some unitary matrix U.

1. Mandel's degree of indistinguishability for two photons

To begin with, let us first consider the two-photon case studied in Ref. [1] (and after that also in Ref. [37]) where it was found that a single parameter is both a degree of indistinguishability and a degree of quantum coherence (how the degree of indistinguishability depends on different parameters in spectral states of photons was recently studied in Ref. [46]). For two single photons in spectral states $\hat{\rho}_1$ and $\hat{\rho}_2$ at input modes $k_1 \neq k_2$ we have only two permutations $\sigma = I$ (trivial) and $\sigma = T$ (transposition of two photons). From Eq. (16), by using the properties $P_t \hat{\Gamma}_{l_1} \otimes \hat{\Gamma}_{l_2} P_t = \hat{\Gamma}_{l_2} \otimes \hat{\Gamma}_{l_1}$ and $\mathrm{Tr}(A \otimes B P_t) = \mathrm{Tr}(AB)$ (where the latter trace is in $\mathcal{H}$, whereas the former is in $\mathcal{H} \otimes \mathcal{H}$), we obtain

$$
\begin{aligned}
J(I,I) &= \mathrm{Tr}\big(\hat{\Gamma}_{l_1}\hat{\rho}_1\big)\mathrm{Tr}\big(\hat{\Gamma}_{l_2}\hat{\rho}_2\big),\\
J(T,T) &= \mathrm{Tr}\big(\hat{\Gamma}_{l_2}\hat{\rho}_1\big)\mathrm{Tr}\big(\hat{\Gamma}_{l_1}\hat{\rho}_2\big),\\
J(T,I) &= \mathrm{Tr}\big(\hat{\Gamma}_{l_1}\hat{\rho}_1\hat{\Gamma}_{l_2}\hat{\rho}_2\big) = J^*(I,T).
\end{aligned} \tag{48}
$$

Detectors reduce the total probability of detection. Let us first try to distinguish this effect of detectors from their influence on the quantum coherence of photons. By introducing a diagonal matrix $D(\sigma_1,\sigma_2) = \delta_{\sigma_1,\sigma_2} J(\sigma_1,\sigma_1)$ let us define a reduced $\hat{J}$ matrix as follows:

$$
\hat{J} \equiv D^{-1/2} J D^{-1/2} = \begin{pmatrix} 1 & \mathcal{V}^* \\ \mathcal{V} & 1 \end{pmatrix}, \tag{49}
$$

where

$$
\mathcal{V} \equiv \frac{J(T,I)}{\sqrt{J(I,I)J(T,T)}}. \tag{50}
$$

Now, it is easy to see that $\mathcal{V}$ is exactly Mandel's indistinguishability parameter [1], whose absolute value gives the strength of coherence for two photons. Indeed, if both photons are detected, then the defined $\hat{J}$ matrix describes their indistinguishability. It has the correct trace and, since the original matrix J is Hermitian and positive definite, $|\mathcal{V}| \leqslant 1$. Following [1] we expand (setting $\mathcal{V} = |\mathcal{V}|e^{i\theta}$)

$$
\hat{J} = P_{ID} J_{ID} + P_D \mathrm{diag}(1,1), \quad J_{ID} = \begin{pmatrix} 1 & e^{-i\theta} \\ e^{i\theta} & 1 \end{pmatrix}, \tag{51}
$$

where J_{ID} is a J matrix corresponding to completely indistinguishable photons and arbitrary detectors (if the detectors are identical $\mathcal{V}$ is real) with probability $P_{ID} = |\mathcal{V}|$ and the identity matrix corresponds to maximally distinguishable photons. Moreover, from Eqs. (48) and (50) we obviously get $\mathcal{V} = 1$ for $\hat{\rho}_1 = \hat{\rho}_2 = |\phi\rangle\langle\phi|$, for arbitrary $|\phi\rangle$.

2. Degree of indistinguishability for $N \geqslant 2$

Guided by the examples of Secs. II D, II E, and II G 1, we propose to use a normalized purity $0 \leqslant \mathcal{P} \leqslant 1$ of the reduced $\hat{J}$ matrix as a measure of partial indistinguishability of photons. We define the normalized purity as

$$
\mathcal{P} \equiv \frac{N!}{N!-1}\left[\mathrm{Tr}\left\{\left(\frac{\hat{J}}{N!}\right)^2\right\} - \frac{1}{N!}\right], \tag{52}
$$

since $\mathrm{Tr}\{\hat{J}\} = N!$ and the matrix $\hat{J}$ is $(N! \times N!)$ dimensional. In Mandel's case Eq. (49) we obtain $\mathcal{P} = |\mathcal{V}|^2$.

As in the two-photon case, for N photons we define a $\hat{J}$ matrix by rescaling the J matrix by its diagonal part

$$
\hat{J}(\sigma_1,\sigma_2) = \frac{J(\sigma_1,\sigma_2)}{\sqrt{J(\sigma_1,\sigma_1)}\sqrt{J(\sigma_2,\sigma_2)}}. \tag{53}
$$

The necessary property $|\hat{J}(\sigma_1,\sigma_2)| \leqslant 1$ follows from positivity of the J matrix by using the Sylvester criterion. The output probability becomes

$$
P(\vec{m}|\vec{n}) = \frac{1}{\mu(\vec{m})\mu(\vec{n})} X^\dagger \hat{J} X, \tag{54}
$$

where the column vector X has elements, indexed by $\sigma \in \mathcal{S}_N$, equal to the path amplitudes reduced by the detectors

$$
X_\sigma = \sqrt{J(\sigma,\sigma)} \prod_{\alpha=1}^{N} U_{k_{\sigma(\alpha)},l_\alpha}. \tag{55}
$$

This transformation can be easily understood by referring to the classical case, where $|X_\sigma|^2$ is the probability of a transition of *distinguishable* classical particles in a Markovian network with losses of particles due to imperfect detections.

Although, in general, there is no density matrix resulting in the $\hat{J}$ matrix (53) by Eq. (16) with ideal detectors, it is possible to sometimes consider the effect of general detectors in a way mathematically equivalent to the case of ideal detectors by adopting a generalized inner product in the auxiliary Hilbert space $\mathcal{H}^{\otimes N}$ in the trace definition of the J matrix Eq. (16) with the detector-dependent kernel

$$
\hat{K}_{\vec{l}} \equiv \prod_{\alpha=1}^{N}{}^{\otimes} \hat{\Gamma}_{l_\alpha}, \tag{56}
$$

specific to a considered output configuration. For instance, this approach is employed in discussion of the zero-probability conjecture in Sec. III C below.

In Sec. III D we analytically compute the purity (52) for a model of a realistic boson-sampling computer with partially distinguishable single photons.

III. INPUT CONSISTING OF ONE PHOTON OR VACUUM PER INPUT MODE

The case of input consisting of a photon or vacuum per input mode can be analyzed in considerable detail in the most general form, i.e., for arbitrary detector efficiencies and photonic spectral states. Moreover, in this case a considerable simplification of the resulting formulas is possible, which elucidates the effect of partial indistinguishability of photons on output probabilities. This case is also of much importance in view of the recent proposal of the boson-sampling computer [18].

A. Single photons in pure spectral states

Consider an input (12) corresponding to single photons in pure spectral states. In this case the density matrix factorizes

$$
\hat{\rho} = \hat{\rho}_1 \otimes \cdots \otimes \hat{\rho}_N, \quad \hat{\rho}_\alpha = |\phi_\alpha\rangle\langle\phi_\alpha|, \tag{57}
$$

where

$$|\phi_\alpha\rangle = \sum_s \int d\omega\, \phi_\alpha(s,\omega)|s,\omega\rangle. \quad (58)$$

The partial indistinguishability matrix J (16) becomes

$$J(\sigma_1,\sigma_2) = \prod_{\alpha=1}^{N} \langle \phi_{\sigma_1(\alpha)} | \hat{\Gamma}_{l_\alpha} | \phi_{\sigma_2(\alpha)} \rangle \quad (59)$$

where we have used Eq. (13). One feature of Eq. (59) should be noted: Since the detector operator $\hat{\Gamma}_l$ enters between two spectral states in Eq. (59), one can simply project it on the subspace spanned by the spectral states of photons, i.e., use instead the operator $\hat{\Gamma}'_l \equiv \mathrm{Pr}\hat{\Gamma}_l\mathrm{Pr}$, where a minimum rank projector Pr is such that $\mathrm{Pr}|\phi_\alpha\rangle = |\phi_\alpha\rangle$ for each spectral state $|\phi_\alpha\rangle$ at the network input. Below this is implicitly assumed. This observation simply restates our physical intuition that detectors do not increase the dimension of the linear subspace required to describe spectral states of photons.

For identical detectors $\hat{\Gamma}_l = \hat{\Gamma}$, from Sec. II [see Eq. (21)] we know that a J matrix corresponding to the input of Eq. (57) actually depends only on the cycle decomposition of the relative permutation $\sigma_R \equiv \sigma_2\sigma_1^{-1}$. We get

$$J(\sigma_1,\sigma_2) = \prod_{j=1}^{q}\prod_{i=1}^{\ell_j} \langle \phi_{\alpha_{j,i}} | \hat{\Gamma} | \phi_{\alpha_{j,i+1}} \rangle, \quad (60)$$

where the relative permutation is decomposed into disjoint cycles, $\sigma_R = c_1 \cdots c_q$, and it is assumed that cycle c_j is $\alpha_{j,1} \to \alpha_{j,2} \to \cdots \to \alpha_{j,\ell_j} \to \alpha_{j,1}$ (i.e., $\ell_j + 1 \equiv 1$).

Let us give a reduced form of the output probability. From Eqs. (41) and (45) we obtain (we omit the input argument $\vec{n}$, $n_k \leqslant 1$, for simplicity)

$$P(\vec{m}) = \frac{\langle \Psi|\Psi\rangle}{\mu(\vec{m})}, \quad (61)$$

where

$$|\Psi\rangle \equiv \sum_\sigma \prod_{\alpha=1}^{N}{}^{\otimes} U_{k_{\sigma(\alpha)},l_\alpha} \sqrt{\hat{\Gamma}_{l_\alpha}} |\phi_{\sigma(\alpha)}\rangle. \quad (62)$$

The components of $|\Psi\rangle$ in the basis $|\vec{s},\vec{\omega}\rangle$ are given as the matrix permanents of an $(N \times N)$-dimensional matrix $V(\vec{s},\vec{\omega})$ with elements

$$V_{\beta,\alpha}(\vec{s},\vec{\omega}) = U_{k_\beta,l_\alpha}\phi_\beta(s_\alpha,\omega_\alpha)\left[\Gamma_{l_\alpha}(s_\alpha,\omega_\alpha)\right]^{1/2}. \quad (63)$$

Indeed, we have

$$\begin{aligned}\langle \vec{s},\vec{\omega}|\Psi\rangle &= \sum_\sigma \prod_{\alpha=1}^{N} U_{k_{\sigma(\alpha)},l_\alpha}\left[\Gamma_{l_\alpha}(s_\alpha,\omega_\alpha)\right]^{1/2}\langle s_\alpha,\omega_\alpha|\phi_{\sigma(\alpha)}\rangle \\ &= \mathrm{per}[V(\vec{s},\vec{\omega})]. \end{aligned} \quad (64)$$

The matrix V is a Hadamard product $V(\vec{s},\vec{\omega}) = U[\vec{n}|\vec{m}] \cdot S(\vec{s},\vec{\omega})$ [instead of using the above B matrix (47), in the case of input with at most one photon per mode we can incorporate spectral states of photons into a new matrix S], where the matrix S reads

$$S_{\beta,\alpha}(\vec{s},\vec{\omega}) \equiv \phi_\beta(s_\alpha,\omega_\alpha)\left[\Gamma_{l_\alpha}(s_\alpha,\omega_\alpha)\right]^{1/2} \quad (65)$$

[column α of S depends on the spectral data (s_α,ω_α), where each entry is equal to the spectral state of a photon multiplied by the square root of the spectral sensitivity of a detector taken at (s_α,ω_α)]. In terms of the matrix function $V(\vec{s},\vec{\omega})$ Eq. (61) becomes

$$P(\vec{m}) = \frac{1}{\mu(\vec{m})}\sum_{\vec{s}}\int d\vec{\omega}\, |\mathrm{per}[V(\vec{s},\vec{\omega})]|^2. \quad (66)$$

Instead of using the natural spectral basis (s,ω) for expansion of the spectral state of a photon, one can employ any other basis, which is judged more suitable for some reason. Indeed, given N spectral states of photons (for arbitrary detectors) one needs at most N basis states (but different basis states for different setups). Let $|1\rangle, \ldots, |r\rangle$, with $r \leqslant N$, be the required basis set. Denoting $|\vec{j}\rangle = |j_1\rangle \otimes \cdots \otimes |j_N\rangle$ we get

$$\begin{aligned}\langle \vec{j}|\Psi_{\vec{m}}\rangle &= \sum_\sigma \prod_{\alpha=1}^{N} U_{k_{\sigma(\alpha)},l_\alpha}\langle j_\alpha|\sqrt{\hat{\Gamma}_{l_\alpha}}|\phi_{\sigma(\alpha)}\rangle \\ &= \mathrm{per}\{U[\vec{n}|\vec{m}]\cdot S(\vec{j})\} \equiv \mathrm{per}[V(\vec{j})], \end{aligned} \quad (67)$$

where $V(\vec{j}\,) = U[\vec{n}|\vec{m}] \cdot S(\vec{j}\,)$ with the following matrix $S(\vec{j}\,)$:

$$S_{\beta,\alpha}(\vec{j}\,) \equiv \langle j_\alpha|\sqrt{\hat{\Gamma}_{l_\alpha}}|\phi_\beta\rangle. \quad (68)$$

In this case, the integral of Eq. (66) becomes a finite sum of at most $\frac{(N+r-1)!}{N!(r-1)!}$ terms (recall that r is the rank of a given set of spectral states of N bosons),

$$P(\vec{m}) = \frac{1}{\mu(\vec{m})}\sum_{\vec{j}} |\mathrm{per}[V(\vec{j}\,)]|^2. \quad (69)$$

One observation is in order. The matrix form to represent spectral data $S_{\beta,\alpha}$, $\alpha,\beta = 1,\ldots,N$, is visually attractive; however, one should keep in mind that the probability is given by a matrix permanent which does not change under permutation of rows or columns of U_{k_β,l_α} and $S_{\beta,\alpha}$, i.e., when such a permutation is applied simultaneously to both matrices. For instance, the permutation σ, applied to input states of S and U (row indices), can be transferred to basis states of S and output states in U (column indices). This is used below for physical interpretation of the results.

From Eq. (66) or (69) it follows that the zero-output-probability condition of Refs. [28,29], given as a linear combination of the matrix permanent, the determinant, and a generalization to nontrivial group characters, called the matrix immanants, can be replaced by a condition involving only permanents: $\mathrm{per}[V(\vec{s},\vec{\omega})] = 0$ or $\mathrm{per}[V(\vec{j}\,)] = 0$ (in the latter case the basis is arbitrary).

When photons are completely indistinguishable, the detectors being identical, the matrix S of Eq. (65) has all elements equal to some function $f(\vec{s},\vec{\omega})$ and $S(\vec{j}\,)$ of Eq. (68) has all its elements equal to 1 (we have $r = 1$ and set $|1\rangle = |\phi\rangle$). In this case Eq. (66) or (69) reduce to a single matrix permanent of U_{k_α,l_β}. Single photons with slightly different spectral states or slightly dissimilar detectors destroy this trivial factorization. However, it turns out that zero output probability can occur in some cases when the input contains, besides a subset of completely indistinguishable, also only partially indistinguishable photons. One possibility is when $N-1$ photons are completely indistinguishable in some spectral state $|\varphi_1\rangle$

and the Nth photon is in any other spectral state $|\varphi_2\rangle$. The output probability is zero for some configurations of input and output when the network matrix is a Fourier matrix [32]. Understanding such cases is important for generalization of the HOM effect [2] to multiphoton interference, which could serve also for a conditional verification of the boson-sampling computer [47]. We will study such cases in detail in Sec. III C, where we formulate a conjecture about zero output probability.

B. Single photons in mixed spectral states

We have considered single photons with pure spectral states; however, this is an unrealistic idealization. Let us therefore generalize the above results to single photons in arbitrary mixed spectral states. In this case the input state $\hat{\rho}$ of Eq. (19) consists of

$$\hat{\rho}_\alpha = \int dx\, p_\alpha(x)|\phi_\alpha(x)\rangle\langle\phi_\alpha(x)|, \quad p_\alpha(x) \geqslant 0, \tag{70}$$

where $\langle s,\omega|\phi_\alpha(x)\rangle = \phi_\alpha(s,\omega;x)$ and $\int dx\, p_\alpha(x) = 1$. One can interpret the state (70) as given by a source with parameter x fluctuating according to the probability $p_k(x)$ (in general, no orthogonality condition on vectors is imposed). It is trivial to extend Eq. (70) to several fluctuating parameters. The corresponding partial indistinguishability matrix J is a generalization of that in Eq. (59):

$$J(\sigma_1,\sigma_2) = \int dx_1 p_1(x_1)\cdots\int dx_N p_N(x_N) \times \prod_{\alpha=1}^{N}\langle\phi_{\sigma_1(\alpha)}(x_{\sigma_1(\alpha)})|\hat{\Gamma}_{l_\alpha}|\phi_{\sigma_2(\alpha)}(x_{\sigma_2(\alpha)})\rangle. \tag{71}$$

Therefore, the corresponding output probability is an obvious generalization of that in Eq. (61),

$$P(\vec{m}) = \frac{1}{\mu(\vec{m})}\int dx_1 p_1(x_1)\cdots\int dx_N p_N(x_N)\langle\Psi(\vec{x})|\Psi(\vec{x})\rangle, \tag{72}$$

with $\vec{x} \equiv (x_1,\ldots,x_N)$ and

$$|\Psi(\vec{x})\rangle \equiv \sum_\sigma \prod_{\alpha=1}^{N}{}^{\otimes} U_{k_{\sigma(\alpha)},l_\alpha}\sqrt{\hat{\Gamma}_{l_\alpha}}|\phi_{\sigma(\alpha)}(x_{\sigma(\alpha)})\rangle. \tag{73}$$

In this case the corresponding matrix V, the Hadamard product of spectral data and network matrix, also depends on the fluctuating parameters $x_1,\ldots,x_N$ and an expression for the output probability similar to that of Eq. (66) or (69), depending on the chosen basis, involves also an averaging over these fluctuating parameters. We note here that the above formulas can be generalized in a similar way to account for detectors with fluctuating spectral sensitivities.

C. Zero output probability

Now let us analyze the zero output probability which occurs in some cases of only partially indistinguishable photons, when the network matrix is a Fourier matrix [32]. The physical meaning of a zero output probability with only partially indistinguishable photons can be established by answering the following question: Is there an *exact* cancellation of path amplitudes of not completely indistinguishable photons? In view of the connection with the duality of the which-way information and the interference visibility, noted in Sec. II, one would rule out such a possibility (recall that the exact HOM dip [2] with two photons is used for asserting their complete indistinguishability). Let us consider a few examples below.

1. N photons with each photon pair in linearly independent or coinciding spectral states

With the aim of answering the above question, let us analyze the examples of Ref. [32] in more detail using our approach (we consider photons in pure spectral states and ideal detectors, $\Gamma_l = 1$, for a while). Let us first consider $N-1$ photons in a spectral state $|\varphi_1\rangle$ and a photon in a different spectral state $|\varphi_2\rangle$ (not necessarily orthogonal to $|\varphi_1\rangle$). It is convenient to employ the dual basis of nonorthogonal states $\langle 1|,\langle 2|$, i.e., $\langle j|\varphi_i\rangle = \delta_{ij}$. One can easily verify that in the linear span of spectral states of photons, the subspace of $\mathcal{H}$,

$$\sum_{j,l=1,2}|j\rangle\langle\varphi_j|\varphi_l\rangle\langle l| = I; \tag{74}$$

thus an expansion similar to that of Eq. (69) will contain a nondiagonal quadratic form with the Gram matrix $\langle\varphi_j|\varphi_l\rangle$.

We first employ the approach based on the S matrix (68) and then show that the same result rather naturally follows from the form of the J matrix (59). Setting the row order for the S matrix of Eq. (68) by arranging the basis vectors as $(\langle 1|,\ldots,\langle 1|,\langle 2|)$ we get the result that $S(\vec{j})$ matrices which result in a nonzero contribution to the probability in Eq. (69) correspond to $\vec{j}$ being a permutation of $(1,\ldots,1,2)$. Such an S matrix reads

$$S(\vec{j}) = \mathcal{M}(\vec{j})\begin{pmatrix}1 & \ldots & 1 & 0\\ \vdots & & \vdots & \vdots\\ 1 & \ldots & 1 & 0\\ 0 & \ldots & 0 & 1\end{pmatrix} = \mathcal{M}(\vec{j})(v^\dagger v \oplus 1), \tag{75}$$

where $v = (1,\ldots,1)$, $|v| = N-1$, whereas $\mathcal{M}(\vec{j})$ is the matrix representation of a permutation τ induced by a choice of basis vector $\langle\vec{j}| = [\langle 1|\otimes\cdots\otimes\langle 1|\otimes\langle 2|]P_\tau$, i.e., $\mathcal{M}_{kl} = \delta_{l,\tau(k)}$. Note that permutations between indistinguishable photons do not induce any change in the matrix S; thus distinct matrices S_α correspond to $N-1$ transpositions $\tau_\alpha = (\alpha,N)$, $\alpha = 1,\ldots,N-1$, between each pair of photons in states $|\varphi_1\rangle$ and $|\varphi_2\rangle$ and one for the identity permutation. Due to the block-matrix structure of $(v^\dagger v \oplus 1)$, for each such matrix S_α the matrix permanent $\mathrm{per}(U[\vec{n}|\vec{m}]\cdot S_\alpha)$ factorizes into a product of two amplitudes, one corresponding to the $N-1$ indistinguishable photons and an amplitude corresponding to the Nth photon. To get a clear physical interpretation of the result, we will transfer permutations to column indices, i.e., to l_α in U and to j_α in S. Due to the nonorthogonality of the dual basis the output probability is given as a quadratic form of such matrix permanents. With these observations, setting also $|\vec{\varphi}\rangle = [|\varphi_1\rangle]^{\otimes(N-1)}\otimes|\varphi_2\rangle$, we obtain [see Eq. (74)]

$$P(\vec{m}) = \frac{1}{\mu(\vec{m})}\sum_{\alpha,\beta=1}^{N}\langle\vec{\varphi}|P^\dagger_{(\alpha,N)}P_{(\beta,N)}|\vec{\varphi}\rangle Y^*_\alpha Y_\beta, \tag{76}$$

where $Y_\alpha = U_{k_N,l_\alpha}\mathrm{per}(U[\vec{n}-\vec{1}_N|\vec{m}-\vec{1}_{l_\alpha}])$. Here we have defined a vector $\vec{1}_j$ with only one nonzero entry, equal to 1, in a row (column) with index j [we subtract one particle in a mode with index j from the corresponding input (output) configuration]. Now, due to linear independence of the vectors $P_{(\alpha,N)}|\vec{\varphi}\rangle$, for $\alpha = 1,\dots,N$, a zero output probability in Eq. (76) occurs only if $Y_\alpha = 0$ for all $\alpha = 1,\dots,N$. This condition (besides the trivial case of some $U_{kl} = 0$) obviously does not involve interference of paths of distinguishable photons (i.e., it does not depend on such interference). We summarize: A zero output probability in the case described by (76) is formulated as *an exact cancellation of paths of only completely indistinguishable photons*. We note here that, in the considered example, zero probability requires that N different quantum amplitudes of indistinguishable photons are equal to zero, which occurs with the Fourier matrices and special input modes [11,32].

One can generalize the above result (on which path interference is responsible for an exactly zero probability) to $Q \geqslant 2$ groups of photons, where group q consists of photons in the spectral state $|\varphi_q\rangle$, the spectral states $|\varphi_1\rangle,\dots,|\varphi_Q\rangle$ being linearly independent. In this case the corresponding matrix $S(\vec{j})$, resulting in a nonzero output probability (see more details in Appendix B), is a product of a permutation matrix $\mathcal{M}(\vec{j})$ and a matrix equal to a direct sum of matrices with each entry being equal to 1:

$$S(\vec{j}) = \mathcal{M}(\vec{j})\left(\sum_{q=1}^{Q}{}^{\oplus} v_q^\dagger v_q\right), \quad v_q \equiv (1,\dots,1), \quad |v_q| = c_q, \tag{77}$$

where c_q is the number of photons in spectral state $|\varphi_q\rangle$. A notable feature of this case is that path interference of photons within each group is maximally possible. Note that photons in linear independent nonorthogonal pure spectral states can be discriminated, but only with a nonzero probability of an inconclusive result [48]. This agrees with path interference in our case also between different groups. Only when the spectral states of different groups become orthogonal does the cross-group coherence disappear.

The above conclusions on path interference can be seen directly from the J matrix (which is also unique for a given set of spectral states in contrast to the basis-dependent S matrix). Indeed, let us take the $Q \geqslant 2$ groups of photons as in the above example. Since permutations of photons in each group between themselves do not change the spectral states, the corresponding J matrix (59) factorizes into a tensor product. Indeed, let us decompose a permutation $\sigma = \tau\pi$, where τ exchanges photons between different groups (without exchanging the order within each group) and π exchanges photons within each group. We then have a property $J(\sigma_1,\sigma_2) = J_R(\tau_1,\tau_2)$, which in matrix form reads [compare with Eqs. (28) and (30)]

$$J = J_R \otimes \left(\sum_{q=1}^{Q}{}^{\oplus} v_q^\dagger v_q\right), \quad J_R(\tau_1,\tau_2) = \langle\vec{\varphi}|P_{\tau_1}P_{\tau_2}^\dagger|\vec{\varphi}\rangle, \tag{78}$$

where v_q is defined in Eq. (77), $|\vec{\varphi}\rangle = \prod_{q=1}^{Q}{}^{\otimes}(|\varphi_q\rangle^{\otimes c_q})$, and the reduced J_R matrix accounts for interference between photons from different groups [$J(\sigma_1,\sigma_2)$ with the above property is indeed a matrix tensor product: if $C = A\otimes B$ the double index notation reads $C_{ij,kl} = A_{ik}B_{jl}$; in our case $\sigma_i = \tau_i\pi_i$, $i = 1,2$, with $\tau_{1,2}$ being the indices of J_R and $\pi_{1,2}$ the indices of $\sum_q{}^{\oplus} v_q^\dagger v_q$]. Observing that summation over in-group permutations π in the product $\prod_{\alpha=1}^{N} U_{k_{\pi(\alpha)},l_\alpha}$ of Eq. (9) gives the product of Q quantum amplitudes, one from each group of photons, we can pass directly to the argument below Eq. (76) now generalized to Q groups of photons.

2. General case: Zero-probability conjecture

Now let us consider a general (single photon per mode) input and nonideal (generally dissimilar) detectors. It is clear that nonideal detectors can result in effective linear dependence of the spectral states of photons that are otherwise linearly independent. Consider the above example of Q groups of photons, with c_q photons in the qth group having a spectral state $|\varphi_q\rangle$. For nonideal detectors, if permuted vectors $P_\tau[\prod_{q=1}^{Q}{}^{\otimes}(|\varphi_q\rangle^{\otimes c_q})]$ for different τ (permuting vectors between the groups without changing the order within each group) are still linearly independent now under the generalized inner product in $\mathcal{H}^{\otimes N}$ with the kernel $\hat{K}_{\vec{l}} \equiv \prod_{\alpha=1}^{N}{}^{\otimes}\hat{\Gamma}_{l_\alpha}$, the above consideration still applies, with the same conclusion about the zero output probability. The above condition is equivalent to $\det(\mathcal{G}^{(\alpha)}) \neq 0$ for all $\alpha = 1,\dots,N$, where $\mathcal{G}^{(\alpha)}_{ij} = \langle\varphi_i|\hat{\Gamma}_{l_\alpha}|\varphi_j\rangle$.

From the above consideration it is clear that although general detectors modify the linear dependence of spectral states, they still can be effectively accounted for (after scaling out their effect on the detection probability, as in Sec. II G) by considering another input case with different linear dependence properties of the spectral states of photons. Can an output probability for only partially indistinguishable photons vanish exactly for more general linearly dependent spectral states? In Ref. [28], where a three-photon coincidence probability was analyzed, it was found that dissimilar detectors strongly influence the coincidence probability for single photons: it can be numerically close to zero for a nonzero difference of photon arrival times, if the sensitivities of detectors are strongly different. However, this cannot be an exact zero probability. Indeed, in the example considered in Sec. III C 1 an exact cancellation is possible (for a special network) and, by the above change of kernel in an inner product, now is extended to detector sensitivities resulting in a nonsingular kernel, but the relevant condition is still formulated for *completely indistinguishable* photons (e.g., does not depend on nonzero time delays). In a more general case, when detectors result in a singular kernel, this is still true. Let us analyze the example of three photons with only two linearly independent spectral states. Indeed, in this case we have $|\varphi_3\rangle = c_1|\varphi_1\rangle + c_2|\varphi_2\rangle$ for some $c_{1,2}$ and linearly independent $|\varphi_{1,2}\rangle$. We will employ the S-matrix approach with the dual basis $\langle j|$, $j = 1,2$. In this case there are two sets of S matrices contributing to output probabilities. They correspond to two choices of three indices (j_1,j_2,j_3): (i) $(1,2,1)$ and permutations $\tau \in \{I,(1,2),(2,3)\}$ of this set; or (ii) $(1,2,2)$ and permutations $\{I,(1,2),(1,3)\}$ of this set. The respective S matrices read [compare with Eq. (75)]

$$S^{(i)} = \mathcal{M}(\tau)\begin{pmatrix}1 & 0 & c_1\\ 0 & 1 & c_2\\ 1 & 0 & c_1\end{pmatrix}, \quad S^{(ii)} = \mathcal{M}(\tau)\begin{pmatrix}1 & 0 & c_1\\ 0 & 1 & c_2\\ 0 & 1 & c_2\end{pmatrix}. \tag{79}$$

In these two cases an exactly zero output probability corresponds to a set of equations for the respective quantum amplitudes. Dividing the amplitudes of case (i) of Eq. (79) by c_i, $i = 1,2$ (thus we assume $c_i \neq 0$ otherwise we are in the already considered case) and setting $\hat{U}_{\alpha,\beta} \equiv U_{k_\alpha,l_\beta}$ we obtain the two sets as follows. For $\tau = I,(1,2),(2,3)$ in Eq. (79) set (i) reads

$$\begin{aligned}\hat{U}_{11}\hat{U}_{22}\hat{U}_{33} + \hat{U}_{13}\hat{U}_{22}\hat{U}_{31} = 0,\\ \hat{U}_{21}\hat{U}_{12}\hat{U}_{33} + \hat{U}_{23}\hat{U}_{12}\hat{U}_{31} = 0,\\ \hat{U}_{11}\hat{U}_{32}\hat{U}_{23} + \hat{U}_{13}\hat{U}_{32}\hat{U}_{21} = 0,\end{aligned} \tag{80}$$

whereas set (ii) for $\tau = I,(1,2),(1,3)$ in Eq. (79) reads

$$\begin{aligned}\hat{U}_{11}\hat{U}_{22}\hat{U}_{33} + \hat{U}_{11}\hat{U}_{23}\hat{U}_{32} = 0,\\ \hat{U}_{21}\hat{U}_{12}\hat{U}_{33} + \hat{U}_{21}\hat{U}_{13}\hat{U}_{32} = 0,\\ \hat{U}_{31}\hat{U}_{22}\hat{U}_{13} + \hat{U}_{31}\hat{U}_{23}\hat{U}_{12} = 0.\end{aligned} \tag{81}$$

(In each set the second and third equations are obtained by transposition of row indices, as dictated by τ, of the first equation.) There are six different terms in Eqs. (80) and (81), each being a product of three different single-particle amplitudes. Moreover $\hat{U}_{ii} \neq 0$ for $i = 1,2,3$; otherwise $\hat{U} = 0$. We simplify Eqs. (80) and (81) by introducing $\gamma_{ij} \equiv \hat{U}_{ij}/\hat{U}_{ii}$ and dividing all equations by $\hat{U}_{11}\hat{U}_{22}\hat{U}_{33}$. From the first equation in each system we get

$$\gamma_{12}\gamma_{23}\gamma_{31} = 1, \quad \gamma_{13}\gamma_{21}\gamma_{32} = 1, \tag{82}$$

but the second and third equations in each system result in

$$\gamma_{12}\gamma_{21} = -1, \ \gamma_{23}\gamma_{32} = -1, \ \gamma_{13}\gamma_{31} = -1. \tag{83}$$

Equations (83) and (82) are obviously incompatible (as seen by multiplying them in each case).

The above analysis reveals that in the examples involving dissimilar detectors in Ref. [28] there is only a nearly zero output probability, since it occurs for a certain set of nonzero time delays, and thus cannot be a generalization of the HOM effect [2]. What happens is that strongly dissimilar detectors significantly decrease the probability of detection, as discussed in Sec. II E. For reference, in Appendix C we also consider the output probability for Gaussian spectral states of photons in the (s,ω) basis. Generalizing, let us formulate the following zero-probability conjecture for an arbitrary multiphoton input $\vec{n}$ with mixed spectral states of photons.

Zero probability. The condition for exactly zero output probability of some output configuration is an exact cancellation of path amplitudes of completely indistinguishable photons (generally, a subset of all input photons). Moreover, in such cases the output probability remains equal to zero when the degree of distinguishability (for instance, difference in the arrival times) between partially indistinguishable photons is changed.

By the above, zero output probability generally corresponds to various continuously varying degrees of indistinguishability for $N > 2$, as was first established in Ref. [32] and generalized above to groups of completely indistinguishable photons. In the case of two photons there is no possibility of exact cancellation of the output amplitude if the photons are not completely indistinguishable, which is a restatement of the HOM effect [1,2]. In the case of three photons with linearly dependent spectral states, with the photons being only partially indistinguishable pairwise, an exactly zero probability is not possible at all as shown above. We conjecture the zero-probability result to hold for any input of the type given in Eq. (1), general detectors, and an arbitrary unitary network.

D. A model of a realistic boson-sampling device

Consider identical photon sources and identical detectors (this case was first considered in Ref. [27]). This is a basic model of input for an optical realization of the boson-sampling computer [18] which requires single photons to be as indistinguishable as possible. Single photons from realistic sources [26], as well as realistic detectors, have fluctuating parameters which cannot be compensated for (a postselection is the only way to deal with such fluctuations at the expense of increasing the number of runs of the boson-sampling device, which decreases its advantage over classical computers). Note that, in contrast, any bias between sources or between detectors can be detected and thus corrected for, without resorting to postselection in a boson-sampling experiment. Hence, we assume that the main error of a realistic boson-sampling device comes from fluctuations due to mixed spectral states of photons and unstable detector sensitivities, neglecting any bias error. We focus on the original proposal of Ref. [18], although it is easy to generalize the results to boson sampling with variable input [49] or to another proposal with time-bin modes replacing spatial modes [50] (in this case spatial indices are replaced with time-bin indices).

One can incorporate fluctuating sensitivities of unstable detectors into spectral states of photons (see below) or, alternatively, use the generalizer kernel for the inner product in $\mathcal{H}^{\otimes N}$ and reduced $\hat{J}$ matrix as discussed in Sec. II G. Consider the corresponding partial indistinguishability matrix J. From Eq. (71) we obtain

$$J(\sigma_1,\sigma_2) = \int dx_1\, p(x_1)\cdots\int dx_N\, p(x_N) \times \prod_{\alpha=1}^{N}\langle\phi(x_{\sigma_1(\alpha)})|\hat{\Gamma}|\phi(x_{\sigma_2(\alpha)})\rangle. \tag{84}$$

The crucial point (see also Ref. [27]) is that the matrix element $J(\sigma_1,\sigma_2)$ of Eq. (84) actually depends only on the cycle structure of the relative permutation $\sigma_R \equiv \sigma_2\sigma_1^{-1}$, where the cycle structure is $(C_1,\ldots,C_N)$ with C_k being the number of occurrences in the cycle decomposition of a cycle of length k [51]. Indeed, due to identical detectors, $J(\sigma_1,\sigma_2)$ of Eq. (84) depends only on the cycle decomposition of the relative permutation σ_R, as is shown in Sec. II [see Eq. (21)]. The cycle decomposition factorizes the product $\prod_{\alpha=1}^{N}\langle\phi(x_\alpha)|\hat{\Gamma}|\phi(x_{\sigma_R(\alpha)})\rangle$ into similar products for each cycle. Thanks to the same probability function $p(x)$ for all single photons the indices of the integration variables x_α are not important; thus two cycles of the same length (number of elements) contribute the same factor. Each factor corresponding to a k cycle of the relative permutation (equivalent to $x_j \to x_{j+1}$, for $j = 1,\ldots,k$ with $k+1 = 1$, by some relabeling of the integration variables) can be cast as follows:

$$\int dx_1 p(x_1)\cdots\int dx_k p(x_k)\prod_{j=1}^{k}\langle\phi(x_j)|\hat{\Gamma}|\phi(x_{j+1})\rangle$$
$$= \mathrm{Tr}\left\{\left(\sqrt{\hat{\Gamma}}\hat{\rho}\sqrt{\hat{\Gamma}}\right)^k\right\}.$$

Therefore, we get the following formula for the partial indistinguishability matrix:

$$J(\sigma_1,\sigma_2)=\prod_{k=1}^{N} g_k^{C_k(\sigma_2\sigma_1^{-1})},\quad g_k\equiv \mathrm{Tr}\left\{\left(\sqrt{\hat{\Gamma}}\hat{\rho}\sqrt{\hat{\Gamma}}\right)^k\right\}. \quad (85)$$

It is easy to see from the definition that the parameters $0 \leqslant g_k \leqslant 1$, describing the partial indistinguishability of single photons from identical sources, satisfy the constraint $g_{k+m} \leqslant g_k g_m$ which indicates that generally one will have decrease of indistinguishability of photons with increase of the number of sources (see also Fig. 1 below).

Equation (85) implies that detector sensitivities can be dealt with by introducing an (unnormalized) spectral state of a photon visible to a detector as follows:

$$\Phi(s,\omega;x,y)\equiv\phi(s,\omega;x)\sqrt{\Gamma(s,\omega;y)}, \quad (86)$$

where y is some fluctuating parameter(s) of the detector. One can easily see that in this case the corresponding reduced $\hat{J}$ matrix is given as J matrix (84) with $\hat{\Gamma}=I$ and the spectral states of Eq. (86).

Let us consider in some detail the case of single photons with a fixed polarization and random arrival times, when their spectral function (augmented by detector sensitivities) is a Gaussian,

$$\Phi(\omega;\tau)=(2\pi\Delta\omega^2)^{-1/4}\exp\left(i\omega\tau-\frac{\omega^2}{4\Delta\omega^2}\right), \quad (87)$$

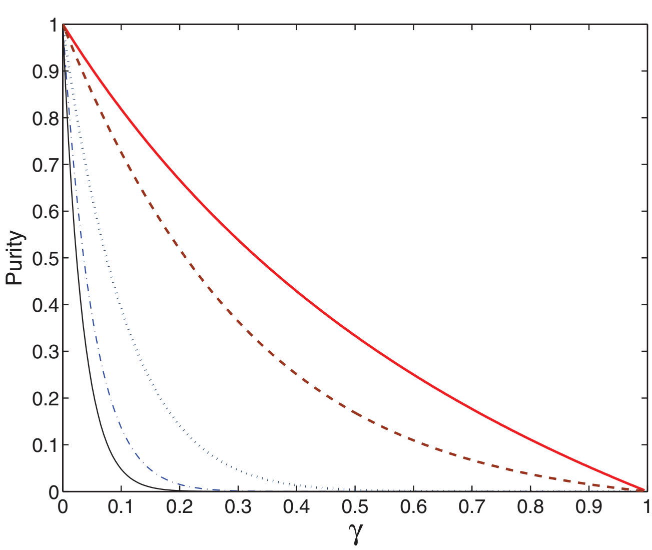

FIG. 1. (Color online) Purity $\mathcal{P}$ (52) of the partial indistinguishability matrix J vs the parameter γ. Here, $N=2$ (thick solid line), $N=4$ (dashed line), $N=10$ (dotted line), $N=20$ (dash-dotted line), and $N=30$ (thin solid line).

as well as the distribution of their arrival times,

$$p(\tau)=\frac{1}{\sqrt{2\pi}\Delta\tau}\exp\left(-\frac{\tau^2}{2\Delta\tau^2}\right). \quad (88)$$

We have (see also Ref. [27]) $g_k=(1-\gamma)^{k/2}(1-\gamma^k)^{-1/2}$ where $\gamma=2\eta^2/(1+2\eta^2)$ with $\eta=\Delta\omega\Delta\tau$ being the classicality parameter (the case of completely indistinguishable photons corresponds to $\eta=0$, whereas for maximally distinguishable photons $\eta=\infty$). The partial indistinguishability matrix reads [27]

$$J(\sigma_1,\sigma_2)=(1-\gamma)^{N/2}\prod_{k=1}^{N}(1-\gamma^k)^{-C_k/2}, \quad (89)$$

where $(C_1,\ldots,C_N)$ is the cycle structure of $\sigma_2\sigma_1^{-1}$.

To measure how close is the matrix J of Eq. (89) to the case of completely indistinguishable photons, let us study its purity defined in Eq. (52) of Sec. II G 2. We have

$$\mathrm{Tr}\left\{\left(\frac{J}{N!}\right)^2\right\}=\frac{(1-\gamma)^N}{N!}\sum_{\sigma}\prod_{k=1}^{N}(1-\gamma^k)^{-C_k(\sigma)}$$
$$=(1-\gamma)^N Z_N(1/(1-\gamma),\ldots,1/(1-\gamma^N)), \quad (90)$$

where $Z_N=Z_N(a_1,\ldots,a_N)$, the sum of powers $\prod_{k=1}a_k^{C_k}$ over all permutations divided by $N!$, is known as the cycle index for which there is a generating function [51]

$$F(x)\equiv\sum_{N\geqslant 0}Z_N(a_1,\ldots,a_N)x^N=\exp\left(\sum_{k=1}^{\infty}\frac{a_k x^k}{k}\right). \quad (91)$$

In our case $a_k=1/(1-\gamma^k)$ and we obtain

$$\sum_{k=1}^{\infty}\frac{x^k}{k(1-\gamma^k)}=\sum_{l=0}^{\infty}\sum_{k=1}^{\infty}\frac{(\gamma^l x)^k}{k}=-\sum_{l=0}^{\infty}\ln(1-\gamma^l x). \quad (92)$$

Using the following identity involving the q-Pochhammer symbol $(x;q)_N\equiv\prod_{k=0}^{N-1}(1-xq^k)$:

$$\prod_{k\geqslant 0}(1-x\gamma^k)=\sum_{N\geqslant 0}\frac{x^N}{(\gamma;\gamma)_N},$$

from Eqs. (90)–(92) we obtain

$$\mathrm{Tr}\left\{\left(\frac{J}{N!}\right)^2\right\}=\frac{(1-\gamma)^N}{\prod_{k=1}^{N}(1-\gamma^k)}. \quad (93)$$

Equation (93) is the law of purity; thus photon indistinguishability decreases with increase of the number of sources N and/or the classicality parameter γ of each source. For small $\gamma \ll 1$ (i.e., $\eta^2 \ll 1$) we obtain $\mathrm{Tr}\{(\frac{J}{N!})^2\}\approx 1-2(N-1)\eta^2$. The behavior of $\mathcal{P}$ with γ for various N is illustrated in Fig. 1. Finally, small bias errors can be considered similarly as in Ref. [52].

IV. CONCLUSION

We have developed a theory of partial indistinguishability of photons for multiphoton experiments in multiport devices. The key object is the partial indistinguishability matrix, a nonnegative definite Hermitian matrix built from spectral states of photons and detector sensitivities. Although only a fraction of information in the partial indistinguishability matrix seems to be derivable from the corresponding output probabilities, using an expression for output probability as a quadratic form and a clear physical interpretation of its arguments as path amplitudes is quite appealing; moreover, it allows physical insights. For instance, a connection with the complementarity of the which-way information vs the interference visibility is used in formulation of the zero-probability conjecture. The permutation (symmetric) group is the key object of the theory; the partial indistinguishability matrix is indexed by permutations of photonic spectral states and has the dimension $N! \times N!$ for N photons. It is interesting to note that the advanced features of the group, such as nontrivial group characters and the matrix immanants related to them, do not play any role in our approach. For instance, we have shown that the output probability is always expressed in terms of the matrix permanents only (the matrix permanent is related to the trivial character of the permutation group). In special cases the partial indistinguishability matrix reduces to much simpler forms, amenable for even an analytical analysis. We have also found that a possible generalization of Mandel's indistinguishability parameter for $N > 2$ photons is given by the purity of a reduced partial indistinguishability matrix, where only the effect of detectors on partial indistinguishability is retained, whereas their effect on the total probability is scaled out. We have found an analytical expression giving the purity measure of quantum coherence for a model of a realistic boson-sampling computer. Besides experiments with optical multiports, the theory can be applied also to quantum walks with several photons [53–55] where indistinguishability of photons is essential for such multiparticle walks to show quantum correlations of a many-boson system. The approach developed here was already used for derivation of very interesting results in Ref. [32].

ACKNOWLEDGMENTS

This work was supported by the CNPq (Brazil). A part of this work was done during a visit to the B. I. Stepanov Institute of Physics, National Academy of Sciences of Belarus. The author is grateful to Dmitri Mogilevtsev for many invaluable discussions. The author is indebted to Malte C. Tichy for pointing out a loophole in the initial formulation of the zero-output-probability result.

APPENDIX A: DERIVATION OF THE PROBABILITY FORMULA

We will use the identity

$$\langle 0| \left[\prod_{\alpha=1}^{N} b_{l_\alpha,s_\alpha}(\omega_\alpha)\right]\left[\prod_{\alpha=1}^{N} b^\dagger_{l'_\alpha,s'_\alpha}(\omega'_\alpha)\right]|0\rangle = \sum_\sigma \prod_{\alpha=1}^{N} \delta_{l'_\alpha,l_{\sigma(\alpha)}}\delta_{s'_\alpha,s_{\sigma(\alpha)}}\delta(\omega'_\alpha - \omega_{\sigma(\alpha)}), \quad \text{(A1)}$$

where the summation is over all permutations σ of N elements. Inserting Eqs. (1) and (7) into Eq. (8) we obtain

$$P(\vec{m}|\vec{n}) = \frac{1}{\mu(\vec{m})\mu(\vec{n})}\sum_{\vec{s}}\sum_{\vec{s}\,'}\sum_{\vec{s}\,''}\int d\vec{\omega}\int d\vec{\omega}'\int d\vec{\omega}'' \times \left[\prod_{\alpha=1}^{N}\Gamma_{l_\alpha}(s_\alpha,\omega_\alpha)\right] G(\vec{s}\,',\vec{\omega}'|\vec{s}\,'',\vec{\omega}'') \times \langle 0|\left[\prod_{\alpha=1}^{N} a_{k_\alpha,s''_\alpha}(\omega''_\alpha)\right]\left[\prod_{\alpha=1}^{N} b^\dagger_{l_\alpha,s_\alpha}(\omega_\alpha)\right]|0\rangle \times \langle 0|\left[\prod_{\alpha=1}^{N} b_{l_\alpha,s_\alpha}(\omega_\alpha)\right]\left[\prod_{\alpha=1}^{N} a^\dagger_{k_\alpha,s'_\alpha}(\omega'_\alpha)\right]|0\rangle. \quad \text{(A2)}$$

By using the network transformation $a^\dagger_{k,s}(\omega) = \sum_{l=1}^{M} U_{kl} b^\dagger_{l,s}(\omega)$ and Eq. (A1) we get, for instance,

$$\langle 0|\left[\prod_{\alpha=1}^{N} a_{k_\alpha,s''_\alpha}(\omega''_\alpha)\right]\left[\prod_{\alpha=1}^{N} b^\dagger_{l_\alpha,s_\alpha}(\omega_\alpha)\right]|0\rangle = \sum_{\vec{l}''}\left[\prod_{\alpha=1}^{N} U^*_{k_\alpha,l''_\alpha}\right]\sum_\sigma\prod_{\alpha=1}^{N}\delta_{l''_\alpha,l_{\sigma(\alpha)}}\delta_{s''_\alpha,s_{\sigma(\alpha)}}\delta(\omega''_\alpha-\omega_{\sigma(\alpha)}) = \sum_\sigma\left[\prod_{\alpha=1}^{N} U^*_{k_\alpha,l_{\sigma^{-1}(\alpha)}}\right]\prod_{\alpha=1}^{N}\delta_{s''_\alpha,s_{\sigma(\alpha)}}\delta(\omega''_\alpha-\omega_{\sigma(\alpha)}).$$

This identity and a similar relation for the second inner product in Eq. (A2) transform Eq. (A2) to a resulting expression equivalent to Eq. (9) of Sec. II. The final step is to transfer permutations from the l indices to the k indices in the two products of network matrix elements by using the following general identity for any two permutations σ and τ:

$$\prod_\alpha A_{\alpha,\tau\sigma(\alpha)} = \prod_\alpha A_{\sigma^{-1}(\alpha),\tau(\alpha)}, \quad \text{(A3)}$$

which easily follows from the independence of a product of scalars of their order and the fact that a permutation is just a bijection between two sets of indices.

APPENDIX B: S MATRICES NOT CONTRIBUTING TO OUTPUT PROBABILITY

Consider $Q \geqslant 2$ groups of photons, where group q consists of photons in a spectral state $|\varphi_q\rangle$, the spectral states $|\varphi_1\rangle,\ldots,|\varphi_Q\rangle$ being linearly independent. What choice of $\vec{j}$ in the matrix $S(\vec{j})$ trivially results in zero output probability in Eq. (69) (i.e., irrespective of U)? Let c_q be the number of photons in the spectral state $|\varphi_q\rangle$. If $|\vec{j}\rangle$ is a tensor product of vectors which do not represent a permutation of the dual basis set $|1\rangle \otimes \cdots \otimes |1\rangle \otimes |2\rangle$ then the corresponding matrix $S(\vec{j})$ consists of nonsquare (rectangular) blocks of entries equal to 1, whereas the complementary blocks have zeros in each entry. Then, irrespective of the network matrix U, the matrix permanent of the Hadamard product of matrices S and $U[\vec{n}|\vec{m}]$ can be expanded by using the analog of the Laplace formula for a permanent of an $(N \times N)$-dimensional matrix [44]

$$\mathrm{per}(A) = \sum_{1\leqslant i_1<\cdots<i_k\leqslant N} \mathrm{per}(A[1,\ldots,k|i_1,\ldots,i_k]) \times \mathrm{per}(A[k+1,\ldots,N|i_{k+1},\ldots,i_N]), \tag{B1}$$

where $(i_1,\ldots,i_N)$ is a permutation of $(1,\ldots,N)$ and we have divided the matrix A into two square blocks of dimension k and $N-k$. Now, by the structure of matrix $S(\vec{j})$ for $\vec{j}$ not a permutation of the dual basis, the permanent of one of the blocks of S in each term in a sum similar to that of Eq. (B1) is always equal to zero, since there are sets of k rows (or columns) of such a matrix S containing *strictly fewer* than k columns (rows) which are nonzero.

APPENDIX C: PHOTONS IN PURE GAUSSIAN STATES

For strongly dissimilar detectors the output probabilities approach zero (for some or even all output configurations), if the product of detector sensitivities approaches zero, simply due to the fact that there are also detection probabilities in this case, i.e., given by the matrix $D_{\vec{m}}$ of Sec. II G. Following Ref. [28], let us consider single photons of the same polarization and with Gaussian spectral functions of center frequencies Ω_α and arrival times t_α. Thus

$$\phi_\alpha(\omega) = \left(2\pi\epsilon\Delta_\alpha^2\right)^{-1/4} \exp\left\{it_\alpha\omega - \frac{(\omega-\Omega_\alpha)^2}{4\epsilon\Delta_\alpha^2}\right\}, \tag{C1}$$

where we have inserted $\epsilon > 0$ to study the limit of monochromatic photons (see below). We have from Eq. (59) of Sec. III A

$$J(\sigma_1,\sigma_2) = \int d\vec{\omega}\left[\prod_{\alpha=1}^N \left(2\pi\epsilon\Delta_\alpha^2\right)^{-1/2}\Gamma_{l_\alpha}(\omega_\alpha)\right] \times \exp\left\{-\sum_{\alpha=1}^N\sum_{i=1,2}\frac{(\omega_\alpha-\Omega_{\sigma_i(\alpha)})^2}{4\epsilon\Delta_{\sigma_i(\alpha)}^2} + i\sum_{\alpha=1}^N \omega_\alpha(t_{\sigma_2(\alpha)}-t_{\sigma_1(\alpha)})\right\}. \tag{C2}$$

Let us consider the output probability for $J(\sigma_1,\sigma_2)$ of Eq. (C2), for arbitrary detector sensitivities. Indeed, the output probability in this case can be easily rewritten as follows (setting $\epsilon = 1$):

$$P(\vec{m}) = \int d\vec{\omega}\left|\sum_\sigma Z_\sigma(\vec{\omega})\right|^2, \quad Z_\sigma(\vec{\omega}) \equiv \prod_{\alpha=1}^N \left(2\pi\Delta_\alpha^2\right)^{-1/4}\sqrt{\Gamma_{l_\alpha}(\omega_\alpha)}X_{\sigma(\alpha)}(\omega_\alpha), \tag{C3}$$

where $X_\beta(\omega_\alpha) = \exp\{i\omega_\alpha t_\beta - \frac{(\omega_\alpha-\Omega_\beta)^2}{4\Delta_\beta^2}\}U_{k_\beta,l_\alpha}$. The sum in Eq. (C3) is nothing but the matrix permanent; we have

$$P(\vec{m}) = \int d\vec{\omega}\,|\mathrm{per}[V(\vec{\omega})]|^2, \quad V_{\beta,\alpha}(\vec{\omega}) \equiv \left(2\pi\Delta_\alpha^2\right)^{-1/4}\sqrt{\Gamma_{l_\alpha}(\omega_\alpha)}X_\beta(\omega_\alpha). \tag{C4}$$

For $P(\vec{m})$ of Eq. (C3) to be zero requires that $\sum_\sigma Z_\sigma(\vec{\omega}) = 0$ at any point $\vec{\omega}$. We note that the sum $\sum_\sigma Z_\sigma(\vec{\omega})$ can be rather close to zero, when the detectors have strongly dissimilar sensitivities. Precisely this happens in the examples of Ref. [28].

In the limit of monochromatic single photons, $\epsilon \to 0$, the above expressions reduce to those of the two extreme cases discussed in Sec. II E. In this limit one does not need to specify detector sensitivities as only some point values will be needed. Using the following expansion in powers of ϵ:

$$\frac{1}{\sqrt{2\pi\epsilon}}\exp\left\{-\frac{1}{2\epsilon^2}\sum_{i=1,2}\frac{(\omega-\Omega_i)^2}{2\Delta_i^2}\right\} = \delta_{\Omega_1,\Omega_2}\delta(\omega-\Omega_1)\frac{\sqrt{2}\Delta_1\Delta_2}{\sqrt{\Delta_1^2+\Delta_2^2}} + O(\epsilon), \tag{C5}$$

we easily obtain from Eq. (C2)

$$J(\sigma_1,\sigma_2) = F(\sigma_1,\sigma_2)\left[\prod_{\alpha=1}^N \delta_{\Omega_{\sigma_1(\alpha)},\Omega_{\sigma_2(\alpha)}}\right] \times \exp\left\{i\sum_{\alpha=1}^N \Omega_{\sigma_1(\alpha)}(t_{\sigma_2(\alpha)}-t_{\sigma_1(\alpha)})\right\} + O(\epsilon), \tag{C6}$$

where we have set

$$F(\sigma_1,\sigma_2) \equiv \prod_{\alpha=1}^N \Gamma_{l_\alpha}\left(\Omega_{\sigma_1(\alpha)}\right)\left[\frac{2\Delta_{\sigma_1(\alpha)}\Delta_{\sigma_2(\alpha)}}{\Delta_{\sigma_1(\alpha)}^2+\Delta_{\sigma_2(\alpha)}^2}\right]^{1/2}. \tag{C7}$$

It immediately follows that if the frequencies Ω_α of monochromatic single photons are pairwise different then the corresponding partial indistinguishability matrix J (C6) is diagonal (i.e., maximally mixed) $J(\sigma_1,\sigma_2) = D(\sigma_1)\delta_{\sigma_1,\sigma_2}$ with

$$D(\sigma_1) \equiv \prod_{\alpha=1}^N \Gamma_{l_\alpha}\left(\Omega_{\sigma_1(\alpha)}\right) \tag{C8}$$

[compare with Eq. (38) of Sec. II E]. In this case monochromatic photons behave in a way similar to classical particles.

In the opposite extreme case, when single photons have equal frequencies, $\Omega_\alpha = \Omega$, assuming also the same spectral width, $\Delta_\alpha = \Delta$, we get from Eq. (C6) $J(\sigma_1,\sigma_2) = D$, where D comes from Eq. (C8) with $\Omega_\alpha = \Omega$ [compare with Eq. (36) of Sec. II E]. The output probability in this case is the same as for completely indistinguishable photons, i.e.,

$$P(\vec{m}) = \frac{D}{\mu(\vec{m})}|\mathrm{per}(U[\vec{n}|\vec{m}])|^2. \tag{C9}$$

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