Guided electromagnetic waves propagating in a plane dielectric waveguide with nonlinear permittivity

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Propagation of transverse-electric waves along a homogeneous plane dielectric waveguide is considered. The waveguide is placed between two half spaces with constant permittivities. The permittivity inside the waveguide is described by the Kerr law. The problem is to determine propagation constants of eigenmodes. It is theoretically predicted that there exists a novel type of propagation modes that does not reduce to linear modes in the limit in which the nonlinear coefficient reduces to zero. It is proved that in the presence of the Kerr effect, infinitely many new propagation constants arise. An analysis of this intriguing case is given.

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I. INTRODUCTION

The paper focuses on studying propagation of monochromatic transverse-electric (TE) electromagnetic waves along a plane dielectric waveguide Σ with the permittivity described by the Kerr law. The waveguide is placed between two half spaces with constant permittivities. We look for guided waves that decay when they move off from the boundaries of the waveguide.

Mathematical analysis of this problem, called $P_{\text{E}}(\alpha)$, implies the existence of guided waves of a novel type. These waves can be called *purely nonlinear TE guided waves*. To be more precise, let the permittivity inside Σ be $\varepsilon = \varepsilon_2 + \alpha|\mathbf{E}|^2$, where $\varepsilon_2,\alpha > 0$ are constants and $\mathbf{E}$ is the complex amplitude of an electric field. In this case, the waveguide supports two types of waves: nonlinear waves of the first type become waves of the linear problem as $\alpha \to 0$ (this case is quite expectable) and waves of the second type stay away from any linear solutions as $\alpha \to 0$. The latter case is under investigation.

The complete set of eigenmodes, which a waveguide supports, is defined by the complete set of propagation constants (PCs) of the waveguide. The finding of PCs in the case of a homogeneous or inhomogeneous waveguide with linear permittivity is usually reduced to a determination of the roots of a transcendental equation called the dispersion equation (DE). We use the *integral dispersion equation method* [1–3] in order to study an analog of the DE for the nonlinear case.

If $\alpha = 0$, we obtain the linear problem $P_{\text{E}}(0)$, which has been well studied for years [4,5]. This paper reports that the problem $P_{\text{E}}(\alpha)$ has an infinite number of PCs. As long as there is always no more than a finite number of PCs in the problem $P_{\text{E}}(0)$, the aforementioned fact implies the existence of novel guided modes—*purely nonlinear TE guided waves*. These waves cannot be determined with the help of a perturbation theory.

New PCs also arise for other types of nonlinear permittivities that take into account saturation effects [6], e.g., $\varepsilon = \varepsilon_2 + \frac{\alpha|\mathbf{E}|^2}{1+\beta|\mathbf{E}|^2}$; however, in this case, no more than a finite number of new PCs arise. It should also be noted that the Kerr nonlinearity is actively studied theoretically (see, e.g., [2,7–12] and the bibliography therein; in cited papers, some generalizations are also considered) and experimentally [13]. However, a clear theoretical explanation of the influence of the Kerr effect on the process of wave propagation has been lacking.

The result given in the paper clearly shows that in nonlinear problems, solutions can occur which cannot be considered as perturbations of solutions of corresponding linearized problems. So it is necessary to be careful when one linearizes a nonlinear problem and considers the linearized problem without proving that there are no other solutions.

If these purely nonlinear guided modes are confirmed by experiment, it will probably advance the theory of nonlinear guided wave propagation; in the case that they are not observed in experiments, then well-known and widespread formulas for nonlinear permittivities must be changed in order that mathematical analysis of these models can give results which better satisfy reality.

Up to now, researchers have not obtained similar rigorous theoretical results for the case of transverse-magnetic (TM) waves.

We also should add that problems of coupled nonlinear electromagnetic wave propagation are closely connected to the problem we consider in this paper; see [14–16] and the bibliography therein.

II. STATEMENT OF THE PROBLEM

Consider a monochromatic TE wave $\mathbf{E}e^{-i\omega t}, \mathbf{H}e^{-i\omega t}$, where ω is a circular frequency and

$$\mathbf{E} = (0,E_y,0)^T, \quad \mathbf{H} = (H_x,0,H_z)^T \tag{1}$$

are the complex amplitudes [2,17]. The TE wave propagates along the surface of the plane dielectric waveguide,

$$\Sigma := \{(x,y,z) \in \mathbb{R}^3 : 0 \le x \le h\}.$$

The half spaces $x < 0$ and $x > h$ are filled with homogeneous isotropic nonmagnetic media with constant permittivities $\varepsilon_1 \ge \varepsilon_0$ and $\varepsilon_3 \ge \varepsilon_0$, respectively; $\varepsilon_0 > 0$ is the permittivity of free space. The waveguide Σ is located in Cartesian coordinate $Oxyz$ and filled with a homogeneous isotropic nonmagnetic medium. The permittivity ε inside the layer Σ is described by the Kerr law,

$$\varepsilon = \varepsilon_2 + \alpha|\mathbf{E}|^2,$$

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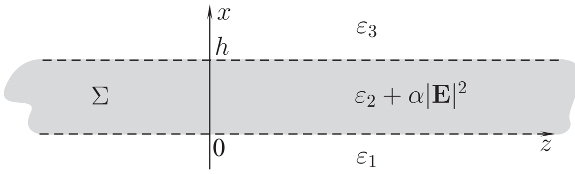

FIG. 1. Geometry of the problem.

where $\varepsilon_2 > \max(\varepsilon_1,\varepsilon_3)$ and $\alpha > 0$ are real constants [18–20]. Without loss of generality we assume that $\varepsilon_1 \geqslant \varepsilon_3$. All of the media contain no sources. Everywhere, $\mu = \mu_0 > 0$ is the permeability of free space (see Fig. 1).

Complex amplitudes (1) must satisfy Maxwell's equations in the harmonic mode,

$$\begin{aligned} \operatorname{rot} \mathbf{H} &= -i\omega\varepsilon\mathbf{E}, \\ \operatorname{rot} \mathbf{E} &= i\omega\mu\mathbf{H}; \end{aligned} \tag{2}$$

the continuity condition for the tangential components of the field on the boundaries $x = 0$ and $x = h$; and the radiation condition at infinity: the electromagnetic field decays as $O(|x|^{-1})$ when $|x| \to \infty$.

It is assumed that sought-for waves depend harmonically on z. By substituting (1) into (2), one finds that the components of (1) do not depend on y. So the components have the form

$$E_y = \mathrm{E}_y(x)e^{i\gamma z}, \quad H_x = \mathrm{H}_x(x)e^{i\gamma z}, \quad H_z = \mathrm{H}_z(x)e^{i\gamma z}, \tag{3}$$

where γ is an unknown (real) PC of a guided wave; E_y is a real function.

In the following, the explicit dependence on x or γ is omitted if it does not lead to misunderstanding.

By substituting complex amplitudes (1) with components (3) into (2), normalizing the obtained system in accordance with the formulas $\tilde{x} = k_0 x$, $\tilde{\gamma} = \gamma k_0^{-1}$, $\tilde{\varepsilon}_j = \varepsilon_j \varepsilon_0^{-1}$ ($j = 1,2,3$), and $\tilde{\alpha} = \alpha\varepsilon_0^{-1}$, where $k_0^2 = \omega^2\mu_0\varepsilon_0$, denoting $Y(\tilde{x}) := \mathrm{E}_y(\tilde{x})$, and omitting the tilde symbol, one obtains the equation

$$Y''(x) = (\gamma^2 - \varepsilon)Y(x), \tag{4}$$

where

$$\varepsilon = \begin{cases} \varepsilon_1, & x < 0, \\ \varepsilon_2 + \alpha Y^2, & 0 \leqslant x \leqslant h, \\ \varepsilon_3, & x > h. \end{cases}$$

Transmission conditions for the tangential components E_y, H_z imply transmission conditions for Y and Y':

$$\begin{aligned} Y(0-0) &= Y(0+0), \quad Y'(0-0) = Y'(0+0), \\ Y(h-0) &= Y(h+0), \quad Y'(h-0) = Y'(h+0). \end{aligned} \tag{5}$$

Problem $P_{\mathrm{E}}(\alpha)$ is to determine PCs $\widehat{\gamma}$ for which nontrivial eigenmodes $Y(x;\widehat{\gamma})$ exist; these eigenmodes must satisfy (4) and (5) and decay as $O(|x|^{-1})$ when $|x| \to \infty$.

III. LINEAR CASE

Let $k_1^2 = \gamma^2 - \varepsilon_1$, $k_2^2 = \varepsilon_2 - \gamma^2$, and $k_3^2 = \gamma^2 - \varepsilon_3$. If $\alpha = 0$, one gets the well-known linear case. The dispersion equation for the linear case has the form [2,4,5]

$$\tan k_2 h = \frac{k_2(k_1 + k_3)}{k_2^2 - k_1 k_3}. \tag{6}$$

The following result is easily derived from Eq. (6) and in similar forms can be found in the literature.

Statement 1. If $\alpha = 0$, then for any

$$h > h_* = \frac{1}{\sqrt{\varepsilon_2 - \varepsilon_1}} \arctan \frac{\sqrt{\varepsilon_1 - \varepsilon_3}}{\sqrt{\varepsilon_2 - \varepsilon_1}} \geqslant 0$$

the problem $P_{\mathrm{E}}(0)$ has a finite number (not less than one) of PCs $\widetilde{\gamma}_1,\widetilde{\gamma}_2,\ldots,\widetilde{\gamma}_p$, which are roots of (6). For any $i = \overline{1,p}$, it is true that $\widetilde{\gamma}_i^2 \in (\varepsilon_1,\varepsilon_2)$.

If $\varepsilon_1 = \varepsilon_3$, then $h_* = 0$.

IV. NONLINEAR CASE

From the mathematical standpoint, the problem $P_{\mathrm{E}}(\alpha)$ is equivalent to a nonlinear eigenvalue problem of the Sturm-Liouville type for the equation

$$Y'' = -\left(k_2^2 + \alpha Y^2\right)Y, \quad x \in [0,h], \tag{7}$$

with the third type boundary conditions

$$\begin{aligned} k_1 Y(0) - Y'(0) &= 0, \\ k_3 Y(h) + Y'(h) &= 0. \end{aligned} \tag{8}$$

It is well known that in eigenvalue problems for an equation of the second order, which depends nonlinearly on the sought-for function, one of the quantities in Eq. (8) must be prescribed. For this reason, we suppose that the value $Y(0) = A$ is known and, without loss of generality, $A > 0$.

There is a mathematical tool that allows one to study the problem given by (7) and (8) (and much more complicated ones) completely [3]. See the Appendix for all necessary mathematical details, strict mathematical formulation, and complete proofs of Statements 2 and 3.

The problem $P_{\mathrm{E}}(\alpha)$ [or problem given by (7) and (8)] is reduced to some sort of dispersion equation obtained in the form

$$\Phi(\gamma;n) \equiv \int_{-k_3}^{k_1} w\,d\eta + n\int_{-\infty}^{+\infty} w\,d\eta = h, \tag{9}$$

where $n = 0,1,2,\ldots$ is an integer, $w = \frac{1}{\sqrt{(k_2^2+\eta^2)^2 + 2\alpha C}}$, and $C = (\varepsilon_2 - \varepsilon_1)A^2 + 0.5\alpha A^4$.

DE (9) is valid for any finite $h > 0$. In fact, DE (9) is a family (but not a system) of equations for different n. In order to determine the complete set of PCs, it is necessary to solve Eq. (9) for each n.

Statement 2. The set of all real solutions to Eq. (9) coincides with the complete set of PCs of $P_{\mathrm{E}}(\alpha)$.

By definition, the DE determines all PCs; for this reason, Statement 2 looks trivial. However, Eq. (9) is derived using a special procedure and does not involve explicit solutions to Eq. (7); hence, it is necessary to prove that Eq. (9) is the DE.

It is easy to see from (9) that if γ is a solution to (9), then $-\gamma$ is also a solution to (9). In what follows, we consider only positive solutions to (9). We use two notations for the PCs of the problem $P_{\mathrm{E}}(\alpha)$: the notation $\widehat{\gamma}_i$ means that all of the PCs are arranged in the order of magnitude, and the notation $\widehat{\gamma}(m)$ means that this PC is a solution to Eq. (9) for $n = m$.

The existence of purely nonlinear guided modes results from the following statement.

Statement 3. If $\alpha > 0$ and $A \neq 0$, then for any $h > 0$ the problem $P_{\mathrm{E}}(\alpha)$ has an infinite number of PCs $\widehat{\gamma}_i$.

The PCs $\widehat{\gamma}_i$ have the following properties:

(1) $\widehat{\gamma}_1^2, \widehat{\gamma}_2^2, \ldots \in (\varepsilon_1, +\infty)$ and $\lim_{j\to\infty} \widehat{\gamma}_j^2 \to \infty$, where $\widehat{\gamma}_1, \widehat{\gamma}_2, \ldots$ are all the PCs of the problem $P_{\mathrm{E}}(\alpha)$.

(2) If there are p PCs $\widetilde{\gamma}_1 < \widetilde{\gamma}_2 < \cdots < \widetilde{\gamma}_p$ in the problem $P_{\mathrm{E}}(0)$, then there exists $\alpha_0 > 0$ such that for any $\alpha = \alpha' < \alpha_0$ it is true that

$$\widehat{\gamma}_i^2 \in (\varepsilon_1, \varepsilon_2) \text{ and } \lim_{\alpha'\to 0} \widehat{\gamma}_i = \widetilde{\gamma}_i,\ i = \overline{1,p},$$

where $\widehat{\gamma}_1, \ldots, \widehat{\gamma}_p$ are first p solutions to $P_{\mathrm{E}}(\alpha')$. For the rest of PCs $\widehat{\gamma}_q$, it is true that

$$\lim_{\alpha'\to +0} \widehat{\gamma}_q^2 = +\infty \text{ for any } q > p.$$

Remark 1. Statement 3 does not depend explicitly on the frequency ω. For this reason, the statement holds for any frequency (we do not assert that the Kerr law is valid for any frequency).

Property 1 means that there are arbitrary big PCs in the problem $P_{\mathrm{E}}(\alpha)$.

Property 2 means that the complete set of PCs of $P_{\mathrm{E}}(\alpha)$ can be split into two nonoverlapping sets σ' and σ_{nl}, where σ' contains eigenvalues $\widehat{\gamma}_i'$ such that $\lim_{\alpha\to 0} \widehat{\gamma}_i' = \widetilde{\gamma}_i$ [here, $\widetilde{\gamma}_i$ is a solution to $P_{\mathrm{E}}(0)$] and σ_{nl} contains eigenvalues $\widehat{\gamma}_i$ such that $\lim_{\alpha\to 0} \widehat{\gamma}_i = +\infty$. Set σ' can be empty for fixed α; set σ_{nl} contains an infinite number of eigenvalues for any $\alpha > 0$.

The existence of the eigenvalues from σ' is predictable without deep investigation as these eigenvalues can be considered as perturbations of solutions to $P_{\mathrm{E}}(0)$.

Eigenvalues from σ_{nl} are PCs of *purely nonlinear guided waves* and have no connections with solutions to $P_{\mathrm{E}}(0)$. The following question should be addressed to experimentalists: is it possible to observe this new propagating regime in an experiment?

Remark 2. Numerically, some new PCs in the problem $P_{\mathrm{E}}(\alpha)$ were found before. However, we should stress that Statement 3 asserts that there are infinitely many new PCs and infinitely many of them do not reduce to the corresponding linear solutions. Results of this type cannot be proved (or even demonstrated) numerically.

Figures 2 and 3 clarify Statement 3. For the dispersion curves (DCs) and eigenmodes shown below, the following parameters are used: $\varepsilon_1 = 1, \varepsilon_2 = 9, \varepsilon_3 = 4$, and $A = 1$; other parameters are specified in the captions.

In the case shown in Fig. 2, there are four solutions ($\widehat{\gamma}_1$, $\widehat{\gamma}_2$, $\widehat{\gamma}_3$, and $\widehat{\gamma}_4$) to $P_{\mathrm{E}}(\alpha)$ which tend to the aforementioned linear solutions as $\alpha \to 0$. The solutions $\widehat{\gamma}_5$, $\widehat{\gamma}_6$, and $\widehat{\gamma}_7$ (which are additionally labeled in Fig. 2) are the first of those which tend to $+\infty$ as $\alpha \to 0$. Three spikes marked with S go to the point $(h,\gamma) = (+\infty,3)$ as $\alpha \to 0$. In other words, the parts of the nonlinear DCs, which lie below S, tend to the corresponding linear DCs as $\alpha \to 0$. Any point (h,γ) that belongs to a nonlinear DC and lies above the spikes tends to the point with coordinates $(h,\gamma) = (h,+\infty)$ as $\alpha \to 0$, where h is a particular thickness corresponding to the chosen PC.

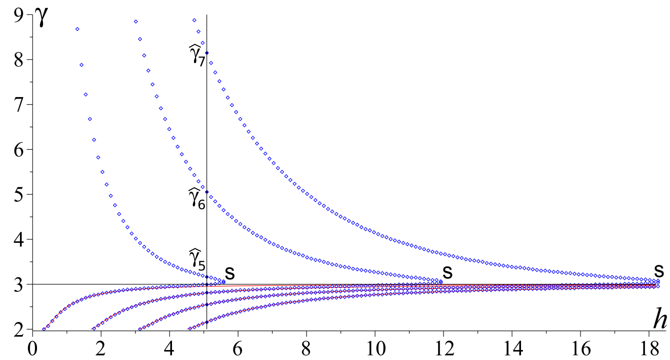

FIG. 2. (Color online) First four DCs for both nonlinear ($\alpha = 0.01$) and linear ($\alpha = 0$) cases are shown. DCs for the nonlinear case [solutions to (9)] are depicted with blue rhombuses (for the fourth curve, only the part which lies below the line $\gamma = 3$ is shown). DCs for the linear case [solutions to (6)] are depicted with solid (red) lines; all of them lie below the line $\gamma = 3$, which is their asymptote. Because of the smallness of α, solid red lines are hardly distinguishable from the curves shown with rhombuses. For $h = 5.08$, there exist four solutions to (6): $\widetilde{\gamma}_1 \approx 2.17$, $\widetilde{\gamma}_2 \approx 2.55$, $\widetilde{\gamma}_3 \approx 2.80$, $\widetilde{\gamma}_4 \approx 2.95$ (they are points of intersections of the line $h = 5.08$ with solid red lines); and there exists an infinite number of solutions (seven of them are shown and marked with filled rhombuses) to (9): $\widehat{\gamma}_1 \approx 2.17$, $\widehat{\gamma}_2 \approx 2.55$, $\widehat{\gamma}_3 \approx 2.82$, $\widehat{\gamma}_4 \approx 3.00$, $\widehat{\gamma}_5 \approx 3.17$, $\widehat{\gamma}_6 \approx 5.06$, $\widehat{\gamma}_7 \approx 8.15$ (they are points of intersections of the line $h = 5.08$ with blue rhombuses).

Figure 3 illustrates what happens with a DC (and its spike) of the problem $P_{\mathrm{E}}(\alpha)$ as $\alpha \to 0$. Due to the smallness of α, the DC of the linear problem can hardly be seen in Fig. 3.

It can be proved that $\max_{x\in(0,h)} |Y(x;\widehat{\gamma}_i)| \to \infty$ as a PC $\widehat{\gamma}_i \to \infty$. Indeed, by multiplying Eq. (7) by Y and integrating from $x = 0$ to $x = h$, one obtains

$$k_3 B^2 + k_1 A^2 + \int_0^h Y'^2 dx = k_2^2 \int_0^h Y^2 dx + \alpha \int_0^h Y^4 dx.$$

The left-hand side of this formula is positive for all possible $\widehat{\gamma}_i$ and so is the right-hand side. As the left-hand side tends to infinity as $\widehat{\gamma}_i \to \infty$, then one finds that $\lim_{\widehat{\gamma}_i\to+\infty} \int_0^h Y^4 dx = +\infty$; the previous formula results in $\lim_{\widehat{\gamma}_i\to\infty} \max_{x\in(0,h)} |Y(x;\widehat{\gamma}_i)| \to \infty$.

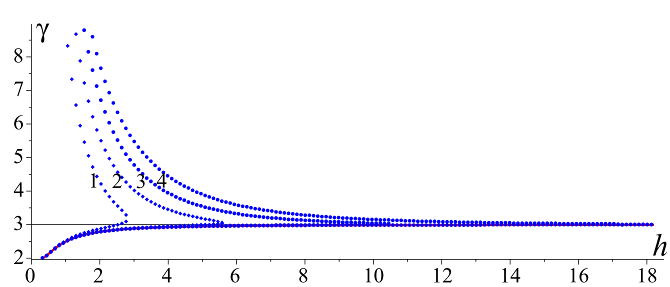

FIG. 3. (Color online) The first dispersion curve of the problem $P_{\mathrm{E}}(\alpha)$ for different α (shown with blue rhombuses and blue circles) and the first DC of the problem $P_{\mathrm{E}}(0)$ (shown in red) are plotted. All other parameters are specified above. The curves 1, 2, 3, and 4 correspond to $\alpha = 0.1$, $\alpha = 0.01$, $\alpha = 0.001$, and $\alpha = 0.0001$, respectively.

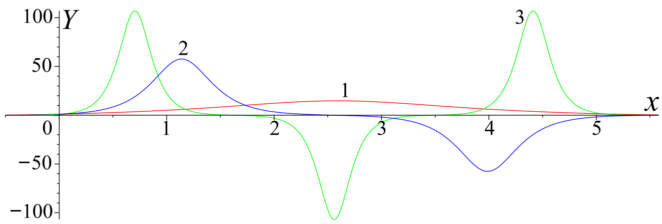

FIG. 4. (Color online) Electric components for the first three purely nonlinear guided waves are shown. The same parameters as for Fig. 2 are used. Thickness h of Σ is 5.08. The curves 1, 2, and 3 correspond to the eigenfunctions $Y(x;\widehat{\gamma}_5)$, $Y(x;\widehat{\gamma}_6)$, and $Y(x;\widehat{\gamma}_7)$, respectively.

Figure 4 demonstrates that the amplitude of novel eigenmodes increases when their PCs increase.

It is hardly reasonable to expect the existence of purely nonlinear waves for each PC from $\sigma_{\rm nl}$. However, it is possible that purely nonlinear waves can be observed in an experiment for some first PCs from the set $\sigma_{\rm nl}$. For the rest of the PCs, the value $\max_{x\in(0,h)}|Y(x;\widehat{\gamma}_i)|$ is so high that the Kerr law is not valid.

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APPENDIX

From the mathematical standpoint, the propagation constants are eigenvalues of a nonlinear transmission eigenvalue problem for Maxwell's equations. In this Appendix, we use the term "eigenvalue" instead of "propagation constant."

Statements 2′ and 3′ given below are full versions of Statements 2 and 3 and contain more facts than the statements given in the main body of the paper, where we present only the results, which are important for physical consideration.

By taking into account conditions at infinity, one obtains solutions of (4) in the half spaces in the form

$$Y(x)=\begin{cases}Ae^{k_1x}, & x<0,\\ Be^{-k_3(x-h)}, & x>h,\end{cases} \qquad \text{(A1)}$$

where the constant $A\neq 0$ is supposed to be fixed (without loss of generality $A>0$); the constant B is unknown and is determined using (5). Using solutions (A1) and conditions (5), one can easily derive conditions (8).

Equation (7) has a first integral,

$$Y'^2+k_2^2Y^2+0.5\alpha Y^4\equiv C, \qquad \text{(A2)}$$

where C is a constant of integration. Using (5), (A1), and (A2) at the point $x=0$, one calculates

$$C=(\varepsilon_2-\varepsilon_1)A^2+0.5\alpha A^4.$$

Here, C does not depend on γ; and $C>0$ if $\alpha\geq 0$.

Using (5), (A1), and (A2) at the point $x=h$, and calculated C, one gets the equation with respect to unknown B,

$$\alpha B^4+2(\varepsilon_2-\varepsilon_3)B^2-[2(\varepsilon_2-\varepsilon_1)A^2+\alpha A^4]=0. \qquad \text{(A3)}$$

Equation (A3) always has a positive solution B^2.

As said before, Eq. (9) is a family of equations for different n. In other words, let σ be a set of all eigenvalues of the problem $P_{\rm E}(\alpha)$; then the set σ can be represented in the form $\sigma=\bigcup_{i=0}^{\infty}\sigma_i$, where σ_j contains all real solutions (and only real solutions) to the equation $\Phi(\gamma;j)-h=0$. (It is also true that $\sigma_i\cap\sigma_j=\varnothing$ for any possible $i\neq j$.) To be more precise, the following result takes place.

Statement 2′. The value $\widehat{\gamma}$ is an eigenvalue of $P_{\rm E}(\alpha)$ if and only if there is an integer $n=\widehat{n}\geq 0$ such that $\widehat{\gamma}$ is a solution to $\Phi(\gamma;\widehat{n})-h=0$. In addition, let $\widehat{\gamma}$ be a solution to $\Phi(\gamma;\widehat{n})-h=0$ and $Y(x;\widehat{\gamma})$ be the corresponding eigenfunction, then $Y(x;\widehat{\gamma})$ has exactly $\widehat{n}$ zeros for $x\in(0,h)$; if x_i is the ith zero, then

$$x_i=\int_{-\infty}^{k_1}w\,d\eta+(i-1)\int_{-\infty}^{+\infty}w\,d\eta.$$

Proof. Introduce new variables

$$\tau(x)=Y^2(x),\quad \eta(x)=Y'(x)/Y(x). \qquad \text{(A4)}$$

Equation (7) can be rewritten as a normal system,

$$\tau'=2\tau\eta,\quad \eta'=-\left(k_2^2+\alpha\tau+\eta^2\right). \qquad \text{(A5)}$$

A first integral of this system can be determined directly from (A5) [or from (A2)] and has the form

$$0.5\alpha\tau^2+\left(\eta^2+k_2^2\right)\tau\equiv C. \qquad \text{(A6)}$$

Solving (A6) with respect to τ, taking into account that $\tau\geq 0$, and substituting the result into the right-hand side of the second equation (A5), one obtains

$$\eta'=-\sqrt{\left(k_2^2+\eta^2\right)^2+2\alpha C}, \qquad \text{(A7)}$$

where the radicand must be positive for all $\eta\in[0,+\infty)$. Obviously, if $\alpha C>0$, then the radicand is positive for all real γ. In other words, in contrast to the linear case where $\gamma^2\in(\varepsilon_1,\varepsilon_2)$ in the nonlinear case, it is possible to consider $\gamma^2\in(\varepsilon_1,+\infty)$.

Using (8), one finds

$$\eta(0)=k_1>0,\quad \eta(h)=-k_3<0. \qquad \text{(A8)}$$

Since $\eta'<0$, then $\eta(x)$ monotonically decreases for $x\in[0,h]$.

It follows from formula (A4) that η is continuous if and only if $Y(x)$ does not become zero for all $x\in(0,h)$. In the general case, $Y(x)$ can have zeros at some points on the interval $(0,h)$. Suppose that $Y(x)$ has n zeros $x_1,\ldots,x_n\in(0,h)$; if $n=0$, then Y does not become zero for any $x\in[0,h]$. Then $\eta(x)$ has n break points $x_1,\ldots,x_n\in(0,h)$; if $n=0$, then $\eta(x)$ is continuous for $x\in[0,h]$. It is clear that $Y'(x_i)\neq 0$ for all $i=\overline{1,n}$.

Formulas (A7) and (A8) imply that

$$\eta(x_i-0)=-\infty,\quad \eta(x_i+0)=+\infty,\quad i=\overline{1,n}. \qquad \text{(A9)}$$

Thereby, solutions to Eq. (A7) are sought on each of the intervals $[0,x_1),(x_1,x_2),\dots,(x_n,h]$:

$$\int_{\eta(x)}^{\eta(x_1-0)} w d\eta = x + c_0, \quad 0 \leqslant x < x_1,$$
$$-\int_{\eta(x_i+0)}^{\eta(x)} w d\eta = x + c_i, \quad x_i < x < x_{i+1}, \qquad (A10)$$
$$-\int_{\eta(x_n+0)}^{\eta(x)} w d\eta = x + c_n, \quad x_n < x \leqslant h,$$

where $i = \overline{1,n-1}$.

Substituting $x = 0$, $x = x_{i+1} - 0$, $x = h$ into Eqs. (A10) (into the first, second, and third, respectively), one determines the constants $c_0,c_1,\dots,c_n$:

$$c_0 = \int_{\eta(0)}^{\eta(x_1-0)} w d\eta,$$
$$c_i = -\int_{\eta(x_i+0)}^{\eta(x_{i+1}-0)} w d\eta - x_{i+1}, \quad i = \overline{1,n-1}, \qquad (A11)$$
$$c_n = -\int_{\eta(x_n+0)}^{\eta(h)} w d\eta - h.$$

With a glance at (A11), one can rewrite (A10) in the form

$$\int_{\eta(x)}^{\eta(x_1-0)} w d\eta = x + \int_{\eta(0)}^{\eta(x_1-0)} w d\eta, \quad 0 \leqslant x < x_1,$$
$$-\int_{\eta(x_i+0)}^{\eta(x)} w d\eta = x - \int_{\eta(x_i+0)}^{\eta(x_{i+1}-0)} w d\eta - x_{i+1}, \quad x_i < x < x_{i+1},$$
$$-\int_{\eta(x_n+0)}^{\eta(x)} w d\eta = x - \int_{\eta(x_n+0)}^{\eta(h)} w d\eta - h, \quad x_n < x \leqslant h, \qquad (A12)$$

where $i = \overline{1,n-1}$.

By substituting $x = x_1 - 0$, $x = x_i + 0$, $x = x_n + 0$ into Eqs. (A12) (into the first, second, and third, respectively), one obtains

$$0 = x_1 + \int_{\eta(0)}^{\eta(x_1-0)} w d\eta,$$
$$0 = x_i - \int_{\eta(x_i+0)}^{\eta(x_{i+1}-0)} w d\eta - x_{i+1}, \quad i = \overline{1,n-1}, \qquad (A13)$$
$$0 = x_n - \int_{\eta(x_n+0)}^{\eta(h)} w d\eta - h.$$

Taking into account (A8) and (A9), one finds, from (A13),

$$0 < x_1 = \int_{-\infty}^{k_1} w d\eta,$$
$$0 < x_{i+1} - x_i = \int_{-\infty}^{+\infty} w d\eta, \quad i = \overline{1,n-1}, \qquad (A14)$$
$$0 < h - x_n = \int_{-k_3}^{+\infty} w d\eta.$$

Formulas (A14) give explicit expressions for distances between zeros of Y. Indeed, if x_i is the ith zero of Y, then $x_i = \int_{-\infty}^{k_1} w d\eta + (i-1)\int_{-\infty}^{+\infty} w d\eta$. Moreover, it follows from (A14) that the improper integrals on the right-hand sides converge.

Summing up all the terms in Eq. (A14), one gets

$$x_1 + x_2 - x_1 + x_3 - x_2 + \dots + x_n - x_{n-1} + h - x_n$$
$$= \int_{-\infty}^{k_1} w d\eta + (n-1)\int_{-\infty}^{+\infty} w d\eta + \int_{-k_3}^{+\infty} w d\eta. \qquad (A15)$$

Formula (A15) can be easily transformed into DE (9).

As DE (9) results from the problem $P_E(\alpha)$, then each eigenvalue of $P_E(\alpha)$ is a solution to DE (9). It is easy to prove that each solution to DE (9) satisfies all the conditions of the problem $P_E(\alpha)$.

The assumption that η has n break points results in the statement about n zeros of an eigenfunction $Y(x;\widehat{\gamma})$.

Statement 3′. Let $\min(\varepsilon_1,\varepsilon_3) \geqslant \varepsilon_0, \max(\varepsilon_1,\varepsilon_3) < \varepsilon_2, \alpha > 0$, and $A \neq 0$. In this case, for any $h > 0$, the problem $P_E(\alpha)$ has an infinite number of eigenvalues $\widehat{\gamma}_i$ (with accumulation point at infinity).

The eigenvalues $\widehat{\gamma}_i$ have the following properties:

(1) If $\widehat{\gamma}_1,\widehat{\gamma}_2,\dots$ are all the solutions to $P_E(\alpha)$, then

$$\widehat{\gamma}_1^2,\widehat{\gamma}_2^2,\dots \in (\varepsilon_1,+\infty) \text{ and } \lim_{j\to\infty} \widehat{\gamma}_j^2 \to \infty.$$

(2) If $P_E(0)$ has p solutions $\widetilde{\gamma}_1 < \widetilde{\gamma}_2 < \dots < \widetilde{\gamma}_p$, then there exists $\alpha_0 > 0$ such that for any $\alpha = \alpha' < \alpha_0$ it is true that

$$\widehat{\gamma}_i^2 \in (\varepsilon_1,\varepsilon_2) \text{ and } \lim_{\alpha'\to 0} \widehat{\gamma}_i = \widetilde{\gamma}_i, \quad i = \overline{1,p},$$

where $\widehat{\gamma}_1,\dots,\widehat{\gamma}_p$ are first p solutions to $P_E(\alpha')$.

(2′) If $q > p$, then $\lim_{\alpha'\to+0} \widehat{\gamma}_q^2 = +\infty$.

(3) For big γ and arbitrary small $\Delta > 0$, the asymptotic two-sided inequality

$$(1-\Delta)\gamma_\bullet(m) \leqslant \widehat{\gamma}(m) \leqslant \sqrt{2}(1+\Delta)\gamma_\bullet(m+1)$$

is valid, where $\gamma_\bullet^2(m) = \varepsilon_2 + [f^{-1}(0.25h/m)]^2$, and f^{-1} is the inversion of $f(t) = t^{-1}\ln t$.

(3′) If $\sqrt{2\alpha C} < 1$, then for big γ simple asymptotic inequality $\widehat{\gamma}(m) \geq \gamma_\circ(m)$ is valid, where $\gamma_\circ^2(m) = \varepsilon_2 + [mh^{-1}\ln(2\alpha C)]^2$.

(4) If eigenvalue $\widehat{\gamma}_i \to \infty$, then $\max_{x\in(0,h)} |Y(x;\widehat{\gamma}_i)| \to \infty$.

Proof. We are going to estimate the integrals that the DE contains. It is clear that

$$nT < \Phi(\gamma;n) < (n+1)T, \qquad (A16)$$

where $n \geqslant 0$ and $T = \int_{-\infty}^{+\infty} w d\eta$. So it is necessary to estimate T. For further analysis, the easily checked inequalities $1/(a+b) \leq 1/\sqrt{a^2+b^2} \leq \sqrt{2}/(a+b)$, where $a \geq 0, b > 0$, are used. These inequalities imply

$$T^* \leqslant T \leqslant \sqrt{2}T^*, \qquad (A17)$$

where $T^* = 2\int_0^{+\infty} \frac{d\eta}{|k_2^2+\eta^2|+\sqrt{2\alpha C}}$. So, it follows from (A16) and (A17) that

$$nT^* \leqslant nT < \Phi(\gamma;n) < (n+1)T \leqslant \sqrt{2}(n+1)T^*.$$

For the integral T^*, there are three cases:

(a) If $\gamma^2 < \varepsilon_2$, then

$$T^* = \pi\theta^{-1}.$$

(b) If $\varepsilon_2 \le \gamma^2 < \varepsilon_2 + \sqrt{2\alpha C}$, then

$$T^* = -\frac{1}{\theta}\ln\frac{\sqrt{2\alpha C}}{(|k_2|+\theta)^2} + \frac{2}{\theta_1}\left(\frac{\pi}{2} - \arctan\frac{|k_2|}{\theta_1}\right).$$

(c) If $\gamma^2 \ge \varepsilon_2 + \sqrt{2\alpha C}$, then

$$T^* = -\frac{1}{\theta}\ln\frac{\sqrt{2\alpha C}}{(|k_2|+\theta)^2} - \frac{1}{\theta_2}\ln\frac{\sqrt{2\alpha C}}{(|k_2|+\theta_2)^2},$$

where $\theta = (|k_2^2| + \sqrt{2\alpha C})^{1/2}$, $\theta_1 = (-|k_2^2| + \sqrt{2\alpha C})^{1/2}$, and $\theta_2 = (|k_2^2| - \sqrt{2\alpha C})^{1/2}$.

It can be checked that T^* continuously depends on γ^2 for all $\gamma^2 \in (\varepsilon_1, +\infty)$.

It follows from (c) that $\lim_{\gamma\to\infty} T^* = 0$. This formula implies that for any prescribed $h > 0$, there exists an infinite number of positive eigenvalues $\widehat{\gamma}_i$. So, (a)–(c) prove the first property of Statement 3′.

Property 2 of Statement 3′ results from (a) and (b). Property 2′ of Statement 3′ results from (c). Property 3 of Statement 3′ results from (c). Indeed, the asymptotic formula $T^* \sim 4|k_2|^{-1}\ln|k_2|$ takes place. This formula easily results in the third property of Statement 3′.

If $\sqrt{2\alpha C} < 1$, then it follows from (c) that the asymptotic formula $T^* \sim -|k_2|^{-1}\ln 2\alpha C$ takes place. This formula easily results in property 3′ of Statement 3′.

By multiplying Eq. (4) by Y and integrating from $x = 0$ to $x = h$, one obtains

$$k_3 B^2 + k_1 A^2 + \int_0^h Y'^2 dx = k_2^2 \int_0^h Y^2 dx + \alpha \int_0^h Y^4 dx. \tag{A18}$$

It follows from (A18) that $\int_0^h Y^4 \to \infty$ if $\gamma \to \infty$. This implies the fourth property of Statement 3′.

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