Parametrically coupled fermionic oscillators: Correlation functions and phase-space description

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(Received 21 October 2014; published 23 January 2015)

A fermionic analog of a parametric amplifier is used to describe the joint quantum state of the two interacting fermionic modes. Based on a two-mode generalization of the time-dependent density operator, time evolution of the fermionic density operator is determined in terms of its two-mode Wigner and P function. It is shown that the equation of motion of the Wigner function corresponds to a fermionic analog of Liouville's equation. The equilibrium density operator for fermionic fields developed by Cahill and Glauber is thus extended to a dynamical context to show that the mathematical structures of both the correlation functions and the weight factors closely resemble their bosonic counterpart. It has been shown that the fermionic correlation functions are marked by a characteristic upper bound due to Fermi statistics, which can be verified in the matter wave counterpart of photon down-conversion experiments.

DOI: 10.1103/PhysRevA.91.013835 PACS number(s): 42.65.Yj, 42.65.Lm, 03.75.−b

I. INTRODUCTION

The fundamental process of parametric amplification [1] to amplify an oscillating signal by means of a particular coupling through an idler mode has attracted wide attention over the last few decades. The major motivation relies on its relevance to several physical phenomena in quantum optics and laser physics, such as low-noise amplifier in radio-frequency region, frequency splitting of light beams in nonlinear media, coherent Raman effect, Brillouin scattering, and so on [2–9]. Traditionally, the two modes of the parametric amplifier are represented by harmonic oscillators, which are bosonic in nature [10]. However, recent experimental advances in the field of fermionic quantum atom optics [11–15] have opened up the possibility that the fermionic counterpart of parametric amplifier could be a promising candidate for describing the behavior of fermionic four-wave mixing [16], association of fermionic atoms into molecules [17–19], or phase sensitivity of fermionic interferometer [20]. Although differences leading to distinctive behavior of a fermionic oscillator in contrast to a traditional harmonic oscillator have been emphasized in several earlier issues, particularly, in connection with dissipative quantum coherence [21–24], full understanding of their implications to other areas is rather new. Experimental control over degenerate quantum gases of neutral atoms in this regard sets up a new stage where the atomic correlations and the quantum statistics of the constituent atoms can be directly probed by analyzing the time-of-flight (TOF) absorption images of the atomic gases [11–14]. While the most direct analogy with quantum optics corresponds to the case of bosonic statistics of parametric down-conversion realized through dissociation of a Bose-Einstein condensate (BEC) of molecular dimers $^{23}Na_2$ and $^{87}Rb_2$ [25,26], the dissociation of $^{40}K_2$ [27] molecules with a characteristic upper bound in the atomic correlation functions reveals a nontrivial twist. This provides a major motivation for the study of full interacting systems of parametrically correlated fermionic oscillators as undertaken here.

The basis of this analysis for the description of the statistical behavior of the two interacting fermionic modes is based on a time-dependent density operator, which is well known for more familiar bosonic fields over many decades [28,29]. However, the main reason for which the straightforward extension of the scheme to their fermionic counterpart remained problematic for a long time is the anticommuting nature of fermionic operators [30]. To overcome this difficulty, Cahill and Glauber have shown in their seminal work on the density operator for fermions [31] using a practical calculus of anticommuting numbers that the mathematical methods that have been used to analyze the statistical properties of boson fields have their counterpart for fermionic fields. This, in particular, indicates that the density operator and the quasiprobability functions for boson have interesting fermionic analogs and thus allow us to calculate correlation functions and counting distributions for a general system of fermions.

The fundamental development as outlined above [31] is however centered around equilibrium domain. Very recently, we proposed a fermionic analog of parametric amplifier and based on a time-dependent reduced density operator for fermionic fields the behavior of the output of one of the two modes of the system has been examined [32]. In view of quantum mesoscopic systems and nanoscale devices, which are implemented as parametric amplifiers [33,34], the possibility of vacuum amplification, population trapping, and quantum control for a single mode of the coupled system have been analyzed [32]. However, the coupling between the modes leads to correlations between the mode amplitudes, which may be readily detected in experiments. To predict the results of this type of experiments and the others that correspond to the state of both the interacting modes, we need to develop the full statistical description of the two-mode system. With this in mind, we explore in this paper the joint density operator $\hat{\rho}(t)$ for the two modes and discuss the dynamics of the amplifier system in terms of a two-mode fermionic Wigner and P-distribution function. The present analysis reveals that the two-mode Wigner function for the fermionic parametric amplifier evolves in a surprisingly similar fashion as their bosonic counterpart and satisfies fermionic Liouville's equation. A detailed calculation is carried out to correlate the upper bounds of the fermionic correlation

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functions, which corroborates quantum optical experiments with fermionic atoms.

The outline of the paper is as follows: In Sec. II we introduce the model and discuss the general aspects of the operator equations of motion for the system. With a brief overview of the density operator for fermionic fields, a formal solution of the joint density operator is derived in Sec. III in terms of its characteristic function. The properties of the characteristic function is then used in Sec. IV to find out the time evolution of the density operator in terms of its two-mode Wigner and P function. Finally, the behavior of the fermionic atom correlation functions are analyzed in Sec. V, in connection with experiments. The paper is concluded in Sec. VI.

II. PARAMETRIC COUPLING OF TWO FERMIONIC OSCILLATORS

We begin with the model of a fermionic analog of parametric amplifier described by the following Hamiltonian:

$$\hat{H} = \hbar\omega_a \hat{a}^\dagger \hat{a} + \hbar\omega_b \hat{b}^\dagger \hat{b} - \hbar\kappa[\hat{a}^\dagger \hat{b}^\dagger e^{-i\omega t} + \hat{a}\hat{b}e^{i\omega t}], \quad (2.1)$$

where we have introduced fermionic oscillators to represent fermionic field modes in analogy with harmonic oscillators that are used to represent bosonic field modes of traditional parametric amplifier. Thus the two uncoupled modes of the fermionic oscillators (A and B modes respectively) are described by the annihilation ($\hat{a}$ and $\hat{b}$) and creation ($\hat{a}^\dagger$ and $\hat{b}^\dagger$) operators, which obey fermionic anticommutation relations

$$\{\hat{a},\hat{a}^\dagger\} = \{\hat{b},\hat{b}^\dagger\} = 1 \quad (2.2)$$

$$\{\hat{a},\hat{b}\} = \{\hat{a},\hat{b}^\dagger\} = 0 \quad (2.3)$$

instead of commutation relations as obeyed by the usual bosonic creation/annihilation operators. These two modes are further assumed to be coupled by a parameter κ, which oscillates at a frequency ω satisfying the resonance condition

$$\omega = \omega_a + \omega_b. \quad (2.4)$$

A typical physical situation to keep in mind for the realization of the present model Hamiltonian is the atom optics counterpart of parametric down-conversion that has been realized through dissociation of a BEC of $^{40}K_2$ molecular dimers [27]. Using the above Hamiltonian we have explained the average mode occupancy of any one of the resonant modes that undergo characteristic oscillation due to Fermi statistics [32] and we are able to justify the upper bounds for the atomic correlation functions that can be directly accessed via experimental measurements of atom shot noise and atom counting techniques [11–15].

The solutions to the Heisenberg equations of motion, which follow from the Hamiltonian [Eq. (2.1)], have been recently derived [32] and may be written in terms of their initial conditions as follows:

$$\hat{a}(t) = \hat{a}(0)\hat{C}_a(t) + \hat{b}^\dagger(0)\hat{S}_a(t) \quad (2.5)$$

$$\hat{b}^\dagger(t) = \hat{b}^\dagger(0)\hat{C}_b^*(t) + \hat{a}(0)\hat{S}_b^*(t). \quad (2.6)$$

In Eqs. (2.5)–(2.6) the operator functions are given by

$$\hat{C}_a(t) \equiv e^{-i\omega_a t}\cos(\Delta\hat{\phi}), \quad \hat{C}_b^*(t) \equiv e^{i\omega_b t}\cos(\Delta\hat{\phi}), \quad (2.7)$$

$$\hat{S}_a(t) \equiv -ie^{-i\omega_a t}\sin(\Delta\hat{\phi}), \quad \hat{S}_b^*(t) \equiv ie^{i\omega_b t}\sin(\Delta\hat{\phi}), \quad (2.8)$$

where $\Delta\hat{\phi}$ corresponds to a relative phase difference operator between the A and B modes of the system, which in turn is related to a time-independent population difference operator $\Delta\hat{N}$ of the respective modes by a relation $\Delta\hat{\phi} = \kappa(1 + 2\Delta\hat{N})t$. The constant of motion $\Delta\hat{N}$ in that relation may be expressed as

$$\hat{N}_a(t) - \hat{N}_b(t) = \hat{N}_a(0) - \hat{N}_b(0) = \Delta\hat{N}, \quad (2.9)$$

which specifies a conservation law between the number of quanta present in the two modes. Now by denoting the initial values of the operators in Eqs. (2.5)–(2.6) as

$$\hat{a}(0) \equiv \hat{a} \quad \text{and} \quad \hat{b}^\dagger(0) \equiv \hat{b}^\dagger, \quad (2.10)$$

the formal solutions of the Heisenberg operators $\hat{a}(t)$ and $\hat{b}(t)$ [Eqs. (2.5)–(2.6)] can be rewritten in the following form

$$\hat{a}(t) = \hat{U}^{-1}(t)\hat{a}\hat{U}(t) = \hat{a}\hat{C}_a(t) + \hat{b}^\dagger\hat{S}_a(t) \quad (2.11)$$

$$\hat{b}^\dagger(t) = \hat{U}^{-1}(t)\hat{b}^\dagger\hat{U}(t) = \hat{b}^\dagger\hat{C}_b^*(t) + \hat{a}\hat{S}_b^*(t). \quad (2.12)$$

Here $\hat{U}(t)$ refers to the unitary time translation operator, which connects the equations of motion of the system between the Heisenberg and Schrödinger picture. Since the two representations coincide at $t = 0$, Eq. (2.10) is used to recast Eqs. (2.5)–(2.6) in the form of Eqs. (2.11)–(2.12) and from now on this notation will be used for all future purposes. Moreover, in order to represent the above transformations to be canonical, all the algebraic anticommutation relations for fermionic operators, in particular $\{\hat{a},\hat{a}^\dagger\} = \{\hat{b},\hat{b}^\dagger\} = 1$, have to be preserved. Unlike the bosonic field operators whose algebraic properties are preserved under hyperbolic transformation, fermionic anticommutation relations are invariant under rotation. The above transformation therefore rotates the fermionic field variables into each other and thereby preserve their anticommutation properties [41].

To find out the various time-dependent expectation values or correlations of the respective field modes, explicit solutions of the Heisenberg operators [Eqs.(2.11)–(2.12)] can be used. On the other hand, the Schrödinger picture provides a more compact way of evaluating such averages by describing the total system in terms of a time-dependent density operator $\hat{\rho}(t)$. Although approaches based on density operator method and its several variants are well known for bosonic parametric amplifier for a long time [29] and form the basis for understanding the phase-space properties of electromagnetic fields [28], an extension of the scheme to its fermionic counterpart is difficult. The main reason as pointed out by Schwinger [30] is the anticommuting nature of fermionic field operators, which take their values in anticommuting Grassmann algebra. While such anticommuting numbers are well studied in mathematics [35] and field theory [36], other applications of anticommuting numbers are much less popular. However, in the last few years a number of investigations on fermionic systems by adopting Grassmann algebra have appeared in the literature. Among them are the characterization of quantum qubit channels [37], non-Markovian stochastic Schrödinger equation [38], study of decoherence and dissipation in a fermionic bath [39], counting of strongly correlated fermions in and out of thermal equilibrium [40], to name just a few.

In recent times, the statistical description of the fermionic counterpart of a standard parametric amplifier has been formulated in terms of a reduced density operator for fermionic fields, which describes any one of the two modes of the total system [32]. Since the coupling between the modes leads to correlations between them, which can be measured in experiments, a complete description requires specification of the state of both the interacting modes. In what follows in the next section we first briefly review the relevant parts of the density operator of fermionic fields as developed by Cahill and Glauber [31] and in the process discuss the properties of two-mode characteristic function and construct the time-dependent density operator $\hat{\rho}(t)$ for the field amplitudes of the two modes. One may find that analogous to bosonic fields, one of the useful ways of expressing a single-mode fermionic density operator is by means of its P representation. The weight function $P(\boldsymbol{\alpha},\boldsymbol{\beta},t)$, which appears in the P representation of $\hat{\rho}(t)$ is in this sense a fermionic analog of the bosonic joint-probability distribution for finding the A and B modes of the system with Grassmann amplitudes $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, respectively. From the time evolution of this function it is shown that the solution of the density operator may be expressed in a form that points directly to the experimental measurement of correlation functions that are made using absorption images of spatial or momentum space column densities. The dynamics of the joint system in terms of fermionic Wigner function are also discussed, a quantum mechanical analog of phase-space distribution for fermionic fields.

III. TWO-MODE FERMIONIC DENSITY OPERATOR AND THE CHARACTERISTIC FUNCTION

A. Density operator for fermionic field

In the spirit of quasiprobability functions for bosonic or electromagnetic fields, Cahill and Glauber have shown that density operator for fermionic field may also be expressed as a statistical mixture of pure coherent states and a suitable weight factor of P distribution as follows [31]:

$$\hat{\rho} = \int d^2\boldsymbol{\alpha}\, P(\alpha,\alpha^*)|\alpha\rangle\langle -\alpha|. \tag{3.1}$$

Similar to the bosonic field, the fermionic coherent state $|\alpha\rangle$ also acts as an eigenstate of the fermionic annihilation operator $\hat{a}$ [29] as $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, with an eigenvalue α. Since fermionic operators anticommute with each other, these eigenvalues are anticommuting numbers, which can be treated by the rules of Grassmann algebra [35,36]. These numbers satisfy very unusual properties, i.e., for a set of anticommuting numbers $\{\alpha_i\}$, $i = 1,2,\ldots,n$ we have

$$\alpha_i\alpha_j + \alpha_j\alpha_i \equiv \{\alpha_i,\alpha_j\} = 0 \quad \forall i,j. \tag{3.2}$$

Equation (3.2) implies that for any given i, we have $\alpha_i^2 = 0$. In other words, Grassamnn numbers are nilpotent, which is an important property for the treatment of fermions. Secondly, since the square of every Grassmann monomial vanishes identically, no Grassmann monomial can be an ordinary real, imaginary, or complex number. This fundamental difference between the Grassmann variables and the ordinary variables has far-reaching consequences as, for example, differentiation is identical to integration for Grassmann numbers. They also anticommute with their fermionic operators as follows:

$$\{\alpha_i,\hat{a}\} = 0; \quad \{\alpha_i,\hat{a}^\dagger\} = 0. \tag{3.3}$$

Lastly, Hermitian conjugation (H.c.) reverses the order of all fermionic quantities, i.e., both the operators and the anticommuting numbers. For instance, we have

$$(\hat{a}\alpha\hat{b}^\dagger\beta^*)^\dagger = \beta\hat{b}\alpha^*\hat{a}^\dagger. \tag{3.4}$$

At this point an important note should be made. For bosons, the integrations are carried out over commuting variables, while for fermions the integrations are taken over anticommuting variables. So, one should be typically concerned with integration over such pairs of anticommuting numbers, for, e.g., α and α^* and for such pairs we will confine ourselves to a typical notation $\int d^2\boldsymbol{\alpha} = \int d\alpha^* d\alpha$, in which the differential of the conjugated variable $d\alpha^*$ comes first and we should keep in mind that $d\alpha^* d\alpha = -d\alpha d\alpha^*$. It is worth pointing out therefore that the minus sign in Eq. (3.1) results from the chosen convention $d^2\boldsymbol{\alpha}$ as $d\alpha^* d\alpha$. If the differential had been chosen as $d^2\boldsymbol{\alpha} = d\alpha d\alpha^*$ instead of $d\alpha^* d\alpha$, the sign would have been positive.

Finally, a system described by an arbitrary density operator $\hat{\rho}$ of a single fermionic mode may be expressed in terms of its characteristic function $\chi(\eta,\eta^*)$

$$\chi(\eta,\eta^*) = Tr[\hat{\rho}\exp(\eta\hat{a}^\dagger - \hat{a}\eta^*)] \tag{3.5}$$

by means of an expansion due to Cahill and Glauber

$$\hat{\rho} = \int d^2\boldsymbol{\eta}\,\chi(\eta,\eta^*)\exp(\hat{a}\eta^* - \eta\hat{a}^\dagger). \tag{3.6}$$

Equation (3.6) represents an operator analog of the Fourier transform over Grassmann arguments η and η^* while the Fourier transformation of Eq. (3.5) gives the fermionic P distribution.

B. Properties of two-mode characteristic function

In the Schrödinger picture, the density operator $\hat{\rho}(t)$ behaves like a state vector and becomes a time-dependent quantity. The density operator $\hat{\rho}(t)$ is however related to the time-independent Heisenberg density operator $\hat{\rho}$ for fermionic fields [Eq. (3.1)] by the relation

$$\hat{\rho}(t) = \hat{U}(t)\hat{\rho}\hat{U}^{-1}(t), \tag{3.7}$$

where, $\hat{U}(t)$ is the unitary time evolution operator as defined in Eqs. (2.11)–(2.12). Now, our target is to find out a solution of the two-mode density operator at an arbitrary time t, when it is specified at $t = 0$. The density operator that describes the two-mode system at time t, may be expressed by means of its characteristic function by generalizing Eq. (3.5).

The characteristic function is then defined for two complex Grassmann pairs (η,ξ) and (η^*,ξ^*) by the relation

$$\chi(\eta,\xi;\eta^*,\xi^*;t) = \mathrm{Tr}\{\hat{\rho}(t)\exp(\eta\hat{a}^\dagger + \xi\hat{b}^\dagger - \hat{a}\eta^* - \hat{b}\xi^*)\}. \tag{3.8}$$

Since both the operators and the Grassmann numbers anticommute with each other, it is worth emphasizing that special care must be taken to the ordering of all the fermionic quantities, i.e., both operators and anticommuting numbers. Apart from these considerations, Eq. (3.8) takes a form analogous to the one that defines the bosonic two-mode characteristic function. Using Eq. (3.7) we may further express $\chi(\eta,\xi;\eta^*,\xi^*;t)$ in terms of the initial density operator $\hat{\rho}$ and the Heisenberg operators $(\hat{a}(t),\ \hat{b}^\dagger(t))$ and their adjoints. The above equation may then be rewritten as follows:

$$\chi(\eta,\xi;\eta^*,\xi^*;t) = \mathrm{Tr}\{\hat{\rho}\exp(\eta\hat{a}^\dagger(t) + \xi\hat{b}^\dagger(t) - \hat{a}(t)\eta^* - \hat{b}(t)\xi^*)\}. \tag{3.9}$$

As the characteristic function $\chi(\eta,\xi;\eta^*,\xi^*;t)$ determines the density operator $\hat{\rho}(t)$ uniquely, a solution can be determined to the initial value density operator problem by calculating the characteristic function $\chi(\eta,\xi;\eta^*,\xi^*;t)$ in terms of its initial form.

To calculate the solution in this way we substitute Eqs. (2.11)–(2.12) for $\hat{a}(t)$ and $\hat{b}^\dagger(t)$ and their Hermitian adjoints into Eq. (3.9) to obtain

$$\chi(\eta,\xi;\eta^*,\xi^*;t) = \mathrm{Tr}\{\hat{\rho}\exp[(\eta\hat{C}_a^*(t) + \xi^*\hat{S}_b(t))\hat{a}^\dagger + (\xi\hat{C}_b^*(t) + \eta^*\hat{S}_a(t))\hat{b}^\dagger - \mathrm{H.c.}]\} \tag{3.10}$$

Next, we define the Grassmann functions

$$\overline{\eta}(\eta,\xi;\eta^*,\xi^*;t) = \eta C_a^*(t) + \xi^* S_b(t) \tag{3.11}$$

$$\overline{\xi}(\eta,\xi;\eta^*,\xi^*;t) = \xi C_b^*(t) + \eta^* S_a(t), \tag{3.12}$$

where the c-number functions are given by

$$C_a^*(t) \equiv e^{i\omega_a t}\cos(\Delta\phi) \quad S_b(t) \equiv -ie^{-i\omega_b t}\sin(\Delta\phi) \tag{3.13}$$

$$C_b^*(t) \equiv e^{i\omega_b t}\cos(\Delta\phi) \quad S_a(t) \equiv -ie^{-i\omega_a t}\sin(\Delta\phi) \tag{3.14}$$

and $\Delta\phi$ corresponds to the relative phase difference between the modes. Using Eqs. (3.11)–(3.12) in Eq. (3.10) we immediately find out that the function $\chi(\eta,\xi;\eta^*,\xi^*;t)$ obeys the following functional identity:

$$\chi(\eta,\xi;\eta^*,\xi^*;t) = \chi(\overline{\eta}(\eta,\xi;\eta^*,\xi^*;t),\overline{\xi}(\eta,\xi;\eta^*,\xi^*;t);\overline{\eta}^*(\eta,\xi;\eta^*,\xi^*;t),\overline{\xi}^*(\eta,\xi;\eta^*,\xi^*;t);0), \tag{3.15}$$

where $\overline{\eta}^*(\eta,\xi;\eta^*,\xi^*;t)$ and $\overline{\xi}^*(\eta,\xi;\eta^*,\xi^*;t)$ are the complex conjugates of $\overline{\eta}(\eta,\xi;\eta^*,\xi^*;t)$ and $\overline{\xi}(\eta,\xi;\eta^*,\xi^*;t)$. The characteristic function $\chi(\eta,\xi;\eta^*,\xi^*;t)$ is thus specified in terms of its initial form at $t = 0$ and by the Grassmann functions. This property of the characteristic function will be further used in Sec. III to analyze fermionic Wigner function.

C. General expression for the two-mode density operator:

Now, it is straightforward to obtain a formal solution of the time-dependent density operator. A generalized form of the two-mode density operator $\hat{\rho}(t)$ may be written down from Eq. (3.6) as

$$\hat{\rho}(t) = \int d^2\boldsymbol{\eta} d^2\boldsymbol{\xi}\,\chi(\eta,\xi;\eta^*,\xi^*;t) \times \exp(\hat{a}\eta^* + \hat{b}\xi^* - \eta\hat{a}^\dagger - \xi\hat{b}^\dagger). \tag{3.16}$$

Substituting Eq. (3.15) for $\chi(\eta,\xi;\eta^*,\xi^*;t)$ into this equation, we have

$$\hat{\rho}(t) = \int d^2\boldsymbol{\eta} d^2\boldsymbol{\xi}\,\chi(\overline{\eta}(\eta,\xi;\eta^*,\xi^*;t),\overline{\xi}(\eta,\xi;\eta^*,\xi^*;t);\overline{\eta}^*(\eta,\xi;\eta^*,\xi^*;t),\overline{\xi}^*(\eta,\xi;\eta^*,\xi^*;t);0) \times \exp(\hat{a}\eta^* + \hat{b}\xi^* - \eta\hat{a}^\dagger - \xi\hat{b}^\dagger). \tag{3.17}$$

The integration may now be carried out most conveniently by change the variables from $(\eta;\xi)$ to $[\overline{\eta} = \overline{\eta}(\eta,\xi;\eta^*,\xi^*;t);\overline{\xi} = \overline{\xi}(\eta,\xi;\eta^*,\xi^*;t)]$ and $(\eta^*;\xi^*)$ to $[\overline{\eta}^* = \overline{\eta}^*(\eta,\xi;\eta^*,\xi^*;t);\overline{\xi}^* = \overline{\xi}^*(\eta,\xi;\eta^*,\xi^*;t)]$ with the provision $d^2\boldsymbol{\eta} d^2\boldsymbol{\xi} = d^2\overline{\boldsymbol{\eta}} d^2\overline{\boldsymbol{\xi}}$, which ensures that the anticommutation properties of the Grassmann numbers are preserved under the transformation.

Then, carrying out the change of variables in Eq. (3.17) we find out the density operator takes the form of

$$\hat{\rho}(t) = \int d^2\overline{\boldsymbol{\eta}} d^2\overline{\boldsymbol{\xi}}\,\chi(\overline{\eta},\overline{\xi};\overline{\eta}^*,\overline{\xi}^*;0) \times \exp(\hat{a}\eta^*(\overline{\eta},\overline{\xi};\overline{\eta}^*,\overline{\xi}^*;t) + \hat{b}\xi^*(\overline{\eta},\overline{\xi};\overline{\eta}^*,\overline{\xi}^*;t) - \eta(\overline{\eta},\overline{\xi};\overline{\eta}^*,\overline{\xi}^*;t)\hat{a}^\dagger - \xi(\overline{\eta},\overline{\xi};\overline{\eta}^*,\overline{\xi}^*;t)\hat{b}^\dagger), \tag{3.18}$$

where $(\eta(\overline{\eta},\overline{\xi};\overline{\eta}^*,\overline{\xi}^*;t);\xi(\overline{\eta},\overline{\xi};\overline{\eta}^*,\overline{\xi}^*;t))$ are the solutions of the inverted Eqs. (3.11)–(3.12), which are given by

$$\eta(\overline{\eta},\overline{\xi};\overline{\eta}^*,\overline{\xi}^*;t) = \overline{\eta}C_a(t) + \overline{\xi}^* S_a(t) \tag{3.19}$$

$$\xi(\overline{\eta},\overline{\xi};\overline{\eta}^*,\overline{\xi}^*;t) = \overline{\xi}C_b(t) + \overline{\eta}^* S_b(t) \tag{3.20}$$

and $(\eta^*(\overline{\eta},\overline{\xi};\overline{\eta}^*,\overline{\xi}^*;t);\xi^*(\overline{\eta},\overline{\xi};\overline{\eta}^*,\overline{\xi}^*;t))$ are their complex conjugates. The time-dependent density operator is thus expressed in terms of the initial form of the characteristic function and its time dependence is completely contained within the linear functions $(\overline{\eta},\overline{\xi})$ and $(\overline{\eta}^*,\overline{\xi}^*)$ of Fourier arguments of the *fermionic mode amplitudes*.

This section closes with a few remarks on *fermionic field amplitudes* or *Grassmann amplitudes* from a physical point of view [38,39]. Keeping in mind the essential difference between the ordinary variables and the Grassmann variables, one may conclude that Grassmann variables do not bear any classical analogy, since they obey anticommutation relations and anticommutations do not have any classical analog. This may lead to a misinterpretation and needs further clarification. For a field to be classically measurable, field amplitude has to be very strong, which is only possible when a large number of particles are accommodated in one state so that the fields get summed up coherently. In other words, particles have to obey Bose-Einstein statistics for, e.g., light quanta are bosons because strong electromagnetic fields can be produced and measured classically. On the contrary, for fermionic fields obeying Fermi-Dirac statistics, only quantities such as charge, energy, current density, and particle number variance, which are only bilinear in field operators, say for, $\hat{a}$ and $\hat{a}^\dagger$, can be measured classically. To illustrate the point a bit further, the expectation values and the variance of the number of quanta are considered, when the initial state of the system is taken as the pure coherent state $|\alpha_0,\beta_0\rangle$, where α_0 and β_0 are respectively the Grassmann amplitudes of the A and B modes. The mean values of the operator $\hat{a}(t)$ and $\hat{b}(t)$ are then given by [32]

$$\overline{\alpha}(t) = \alpha_0 C_a(t) + \beta_0^* S_a(t) \tag{3.21}$$

$$\overline{\beta}(t) = \beta_0 C_b(t) + \alpha_0^* S_b(t), \tag{3.22}$$

while the variance of the number of quanta for the A mode may be calculated as [32]

$$\begin{aligned} \text{var} &= \langle \alpha_0,\beta_0 | [\hat{a}^\dagger(t) - \overline{\alpha}^*(t)][\hat{a}(t) - \overline{\alpha}(t)] | \alpha_0,\beta_0 \rangle \\ &= |S_a^2(t)| = \sin^2(\Delta\phi). \end{aligned} \tag{3.23}$$

It may be noted that Eq. (3.21) for $\overline{\alpha}(t)$ with Grassmann field amplitudes α_0 and β_0^* or Eq. (3.22) for $\overline{\beta}(t)$ with Grassmann field amplitudes β_0 and α_0^* has the same form as those of bosonic field amplitudes with c numbers. The mean number of quanta in Eq. (3.21) is linear in $\hat{a}$ and $\hat{b}^\dagger$ and hence linear in Grassmann variables, represents the *amplitude* of the fermionic field mode and is not an experimentally measurable quantity while the variance, which is bilinear in Grassmann amplitudes makes it experimentally relevant. These remarks may be corroborated by another observation. The number operator $\hat{N} = \sum_n \hat{N}_n$ and the energy operator $\hat{H} = \sum_n \epsilon_n \hat{N}_n$ have classical limits because they are bilinear in $\hat{a}$ and $\hat{a}^\dagger$ and hence commute with each other. Anticommutation relations in quantum mechanics are something special because they incorporate the Pauli exclusion principle, which does not make sense at the classical level. Extrapolating this idea, we emphasize that Grassmann fields themselves and fermionic field operators are, by construction, fermionic while the c numbers and bosonic field operators are bosonic. A product of an even number of Grassmann variables or fermionic quantities is bosonic, which makes it experimentally relevant [38,39]. This discussion will be resumed in the next section when fermionic mode amplitudes of the Wigner function are considered.

IV. TIME EVOLUTION OF WIGNER AND *P* FUNCTION

A. Fermionic Liouville's equation: Properties of two-mode Wigner function

The Wigner function, which was originally introduced as a quantum analog of the classical phase-space distribution function, has various applications in quantum optics, quantum kinetic theory, radiation transport, and others [42,43]. In this section the Wigner function is constructed for the fermionic parametric amplifier in terms of its initial form and it is shown that for arbitrary initial states of the quantum system, the Wigner function satisfies fermionic analog of Liouville's equation.

We may define the Wigner function $W(\alpha,\beta;\alpha^*,\beta^*;t)$ for the coupled two-mode systems as the Fourier transform of the characteristic function as

$$\begin{aligned} W(\alpha,\beta;\alpha^*,\beta^*;t) = \int d^2\boldsymbol{\eta} d^2\boldsymbol{\xi} \exp[\alpha\eta^* + \beta\xi^* - \eta\alpha^* - \xi\beta^*] \\ \times \chi(\eta,\eta^*;\xi,\xi^*;t). \end{aligned} \tag{4.1}$$

The identity Eq. (3.15), which permits us to express the characteristic function in terms of its initial form, and allows us to do the same with the Wigner function. Substituting Eq. (3.15) for $\chi(\eta,\xi;\eta^*,\xi^*;t)$ into Eq. (4.1) we find

$$\begin{aligned} W(\alpha,\beta;\alpha^*,\beta^*;t) = \int d^2\boldsymbol{\eta} d^2\boldsymbol{\xi} \exp[\alpha\eta^* + \beta\xi^* - \eta\alpha^* - \xi\beta^*] \\ \times \chi(\overline{\eta}(\eta,\xi;\eta^*,\xi^*;t),\overline{\xi}(\eta,\xi;\eta^*,\xi^*;t); \\ \overline{\eta}^*(\eta,\xi;\eta^*,\xi^*;t),\overline{\xi}^*(\eta,\xi;\eta^*,\xi^*;t);0). \end{aligned} \tag{4.2}$$

As usual, by changing the variables of integration as carried out previously from $(\eta;\xi)$ to $[\overline{\eta} = \overline{\eta}(\eta,\xi;\eta^*,\xi^*;t);\overline{\xi} = \overline{\xi}(\eta,\xi;\eta^*,\xi^*;t)]$ and $(\eta^*;\xi^*)$ to $[\overline{\eta}^* = \overline{\eta}^*(\eta,\xi;\eta^*,\xi^*;t);\overline{\xi}^* = \overline{\xi}^*(\eta,\xi;\eta^*,\xi^*;t)]$ and making use of Eqs. (3.19)–(3.20) we can write down the above integral as

$$\begin{aligned} W(\alpha,\beta;\alpha^*,\beta^*;t) = \int d^2\overline{\boldsymbol{\eta}} d^2\overline{\boldsymbol{\xi}} \exp\{[\alpha C_a^*(t) - \beta^* S_b(t)]\overline{\eta}^* \\ + [\beta C_b^*(t) - \alpha^* S_a(t)]\overline{\xi}^* - \text{H.c.}\} \\ \times \chi(\overline{\eta},\overline{\xi};\overline{\eta}^*,\overline{\xi}^*;0). \end{aligned} \tag{4.3}$$

Now the following functions are introduced, which are defined by

$$\alpha_{0c}(\alpha,\beta;\alpha^*,\beta^*;t) = \alpha C_a^*(t) - \beta^* S_b(t) \tag{4.4}$$

$$\beta_{0c}(\alpha,\beta;\alpha^*,\beta^*;t) = \beta C_b^*(t) - \alpha^* S_a(t). \tag{4.5}$$

Equations (4.4)–(4.5) when substituted in Eq. (4.3), we obtain the identity relation satisfied the Wigner function as follows

$$\begin{aligned} W(\alpha,\beta;\alpha^*,\beta^*;t) &= \int d^2\overline{\boldsymbol{\eta}} d^2\overline{\boldsymbol{\xi}} \exp[\alpha_{0c}(\alpha,\beta;\alpha^*,\beta^*;t)\overline{\eta}^* \\ &\quad + \beta_{0c}(\alpha,\beta;\alpha^*,\beta^*;t)\overline{\xi}^* - \text{H.c.}] \\ &\quad \times \chi(\overline{\eta},\overline{\xi};\overline{\eta}^*,\overline{\xi}^*;0) \\ &= W(\alpha_{0c},\beta_{0c};\alpha_{0c}^*,\beta_{0c}^*;0). \end{aligned} \tag{4.6}$$

Equation (4.6) expresses the Wigner function at time t in terms of its initial form. To obtain a better insight about the above identity, we denote $\alpha_{0c}(\alpha,\beta;\alpha^*,\beta^*;t) = \alpha_0$; $\beta_{0c}(\alpha,\beta;\alpha^*,\beta^*;t) = \beta_0$ and define functions $\alpha_c(\alpha_0,\beta_0;\alpha_0^*,\beta_0^*;t)$ and $\beta_c(\alpha_0,\beta_0;\alpha_0^*,\beta_0^*;t)$ as the solutions of Eqs. (4.4)–(4.5) for α and β as follows:

$$\alpha_c(\alpha_0,\beta_0;\alpha_0^*,\beta_0^*;t) = \alpha_0 C_a(t) + \beta_0^* S_a(t) \tag{4.7}$$

$$\beta_c(\alpha_0,\beta_0;\alpha_0^*,\beta_0^*;t) = \beta_0 C_b(t) + \alpha_0^* S_b(t). \tag{4.8}$$

In terms of these functions the identity Eq. (4.6) for Wigner function takes the form of

$$\begin{aligned} W(\alpha_c(\alpha_0,\beta_0;\alpha_0^*,\beta_0^*;t),\beta_c(\alpha_0,\beta_0;\alpha_0^*,\beta_0^*;t); \\ \alpha_c^*(\alpha_0,\beta_0;\alpha_0^*,\beta_0^*;t),\beta_c^*(\alpha_0,\beta_0;\alpha_0^*,\beta_0^*;t);t) \\ = W(\alpha_0,\beta_0;\alpha_0^*,\beta_0^*;0), \end{aligned} \tag{4.9}$$

which is valid for arbitrary Grassmann amplitude (α_0,β_0) and its complex pairs.

It is important to note that the structure of Eqs. (4.7)–(4.8) is identical to the earlier Eqs. (3.21)–(3.22) for the mean values of the operators $\hat{a}(t)$ and $\hat{b}(t)$. This immediately suggests that the mean value of the quantum mechanical operators $\hat{a}(t)$ and $\hat{b}(t)$ are governed by the same equations of motion that are obeyed by the complex Grassmann amplitudes.

The functions $\alpha_c(\alpha_0,\beta_0;\alpha_0^*,\beta_0^*;t)$ and $\beta_c(\alpha_0,\beta_0;\alpha_0^*,\beta_0^*;t)$ are just the complex Grassmann amplitudes evaluated at time t for the coupled modes of parametric fermionic amplifier with initial amplitudes α_0 and β_0. On the contrary, α_0 and β_0 in Eqs. (3.21)–(3.22) represent the initial values of the average amplitudes of the fermionic modes. In other words, the functions $\alpha_{0c}(\alpha,\beta;\alpha^*,\beta^*;t)$ and $\beta_{0c}(\alpha,\beta;\alpha^*,\beta^*;t)$ are the initial amplitudes, which correspond *classically* to the amplitudes α and β at time t. Now, the sense of *classicality* for fermionic fields require further investigation. To illustrate the point, first the differential form of Eq. (4.9) is constructed.

To express Eq. (4.9) or Eq. (4.6) in the differential form, the line of work on Wigner functional for fermionic fields [44] is followed and the total time derivative of the Wigner function is written in the following form:

$$\left[\frac{\partial}{\partial t}+\frac{i}{\hbar}\sum_{\{k\}}\left(\frac{\partial H}{\partial \Pi_k}\frac{\partial}{\partial \Psi_k}+\frac{\partial H}{\partial \Psi_k}\frac{\partial}{\partial \Pi_k}\right)\right]W(\Psi,\Pi,t)=0, \tag{4.10}$$

where H represents the *classical* form of the starting Hamiltonian operator [Eq. (2.1)] that we are looking for and Ψ and Π are the canonical conjugate pairs for the fermionic fields. For this model the index $\{k\}$ takes the values a and b, corresponding to the Grassmann amplitudes $\Psi_a=\alpha$; $\Psi_b=\beta$ and $\Pi_a=\alpha^*$; $\Pi_b=\beta^*$. The classical Hamiltonian for the fermionic counterpart of parametric amplifier is then obtained by replacing the operators $\hat{a}(t)$, $\hat{b}(t)$ in Eq. (2.1) by α and β and similarly for their Hermitian adjoints. Therefore, the form of the classical Hamiltonian is given by

$$H=\hbar\omega_a\alpha^*\alpha+\hbar\omega_b\beta^*\beta-\hbar\kappa[\alpha^*\beta^*e^{-i\omega t}+\alpha\beta e^{i\omega t}]. \tag{4.11}$$

It is noted that the classical Hamiltonian H is different from the Hamiltonian operator $\hat{H}$ in Eq. (2.1) since the classical field variables in Eq. (4.11) belong to Grassmann numbers. Equation (4.10) forms the structure of classical-like Liouville's equation for fermionic field. It is important to note that the positive sign in Eq. (4.10) carries the signature of anticommutation relations that are obeyed by the fermionic operators or by the Grassmann variables. The sign is significantly negative for the more familiar bosonic fields for which the corresponding operators obey commutation relations instead of anticommutation relations or in turn the c-number variables commutes with each other. Now by substituting the form of the classical Hamiltonian in Eq. (4.10) we identify the form of the classical like Liouville operator as

$$\mathbf{L}=\left[(\omega_a\alpha-\kappa\beta^*e^{-i\omega t})\frac{\partial}{\partial\alpha}+(\omega_b\beta-\kappa\alpha^*e^{-i\omega t})\frac{\partial}{\partial\beta}\right.$$
$$\left.-\frac{\partial}{\partial\alpha^*}(\omega_a\alpha^*-\kappa\beta e^{i\omega t})-\frac{\partial}{\partial\beta^*}(\omega_b\beta^*-\kappa\alpha e^{i\omega t})\right] \tag{4.12}$$

and Eq. (4.10) can be written in the following form

$$\frac{dW}{dt}=\frac{\partial W}{\partial t}+i\mathbf{L}W=0. \tag{4.13}$$

Equation (4.13) thus asserts that the Wigner function $W(\alpha,\beta;\alpha^*,\beta^*;t)$ for the fermionic parametric amplifier satisfies Liouville's equation and is therefore constant along a classical trajectory. This is the property that the Wigner function for fermionic fields always shares with the classical phase-space distribution functions and satisfied by bosonic Wigner function in their classical limit. It should be emphasized that the result is valid for both kind of fields and for arbitrary density operators. The fact that the Wigner function has this property is a consequence of the form taken by the initial Hamiltonian Eq. (2.1). It may be shown that whenever the Hamiltonian of a system of oscillators (both bosonic or fermionic) is given by a bilinear form in the creation and annihilation operators, the Wigner function is constant [45] along classical trajectories. This property does not extend to systems of arbitrary Hamiltonians of bosonic fields, as happens in case of classical phase-space distributions. Here lies an important difference between the fermionic Wigner function when compared with Wigner functional of bosonic fields where the Liouville's equation is obtained only when the fields are either free or the quantum correction to the interaction are neglected [29,44]. In this sense, the fermionic analogs of the Wigner function, which is linear in field variables, is always classical except for the effect that they incorporate Pauli exclusion principle. This is not surprising because of the simple reason that the Grassmann algebra does not allow for any derivative higher than second order. This is the same reason that guides the Dirac equation for a fermionic field to take a simple linear form [44].

B. *P*-distribution function

The Wigner function obviously is not the only quasiprobability distribution, which allows for the description of the phase space of fermionic systems. The Wigner function for fermionic fields may be related to other quasiprobability functions by the convolution of Gaussian function in Grassmann variables [31]

$$W(\alpha,s)=\int\prod_j\left[\frac{(r-s)}{2}d^2\boldsymbol{\beta}_j\right]$$
$$\times\exp\left[\frac{2}{(r-s)}\sum_i(\alpha_i-\beta_i)(\alpha_i^*-\beta_i^*)\right]W(\beta,r), \tag{4.14}$$

where an ordering of operators are specified by the real parameters r and s, which can take values from -1 for antinormal to $+1$ for normal ordering. For the special case of $r=1$ and $s=0$, Eq. (4.14) relates the Wigner function with the P-distribution function as

$$W(\alpha,\beta;\alpha^*,\beta^*;t)=\frac{1}{4}\int d^2\boldsymbol{\mu}d^2\boldsymbol{\nu}\exp[2(\alpha-\mu)(\alpha^*-\mu^*)$$
$$+2(\beta-\nu)(\beta^*-\nu^*)]P(\mu,\nu;\mu^*,\nu^*;t). \tag{4.15}$$

Substituting Eq. (4.15) into Eq. (4.10) for the Wigner function with a little bit of algebra we obtain

$$\frac{dW}{dt}=\frac{1}{4}\int d^2\boldsymbol{\mu}d^2\boldsymbol{\nu}\exp[2(\alpha-\mu)(\alpha^*-\mu^*)$$
$$+2(\beta-\nu)(\beta^*-\nu^*)]$$
$$\times\left\{\left[\frac{d}{dt}+I+II+I_{hc}+II_{hc}\right]\right.$$
$$\left.\times P(\mu,\nu;\mu^*,\nu^*;t)\right\}\equiv 0, \tag{4.16}$$

where the I and I_{hc} have the following forms:

$$I = i(\omega_a\alpha - \kappa\beta^* e^{-i\omega t})\frac{\partial}{\partial\alpha}\exp[2(\alpha-\mu)(\alpha^*-\mu^*) + 2(\beta-\nu)(\beta^*-\nu^*)] \quad (4.17)$$

$$I_{hc} = -i\frac{\partial}{\partial\alpha^*}\exp[2(\alpha-\mu)(\alpha^*-\mu^*) + 2(\beta-\nu)(\beta^*-\nu^*)] \times(\omega_a\alpha^* - \kappa\beta e^{i\omega t}) \quad (4.18)$$

and similarly for II and its Hermitian conjugate II_{hc}. By calculating all these differentials in Grassmann variables and using the anticommuting properties of the respective variables one may simplify Eq. (4.16) in the following form

$$\frac{1}{4}\int d^2\boldsymbol{\mu} d^2\boldsymbol{\nu}\exp[2(\alpha-\mu)(\alpha^*-\mu^*) + 2(\beta-\nu)(\beta^*-\nu^*)] \times\left\{\left[\frac{d}{dt} + i(\omega_a\alpha\alpha^* + \omega_b\beta\beta^*)\right]P(\mu,\nu;\mu^*,\nu^*;t)\right\} \equiv 0. \quad (4.19)$$

Now, from Eqs. (4.17)–(4.18) and its Hermitian conjugations, it is possible to recognize the differential forms of the Grassmann variables in the present case as follows:

$$\alpha \equiv \frac{1}{4}\left(-\frac{\partial}{\partial\alpha^*} + \frac{\partial}{\partial\mu^*}\right); \quad \alpha^* \equiv \frac{1}{4}\left(\frac{\partial}{\partial\alpha} - \frac{\partial}{\partial\mu}\right) \quad (4.20)$$

$$\beta \equiv \frac{1}{4}\left(-\frac{\partial}{\partial\beta^*} + \frac{\partial}{\partial\nu^*}\right); \quad \beta^* \equiv \frac{1}{4}\left(\frac{\partial}{\partial\beta} - \frac{\partial}{\partial\nu}\right) \quad (4.21)$$

Substituting Eqs. (4.20)–(4.21) in Eq. (4.19) we have

$$\int d^2\boldsymbol{\mu} d^2\boldsymbol{\nu}\exp[2(\alpha-\mu)(\alpha^*-\mu^*) + 2(\beta-\nu)(\beta^*-\nu^*)] \times\left[\frac{d}{dt} + \frac{i}{4}\left(\omega_a\frac{\partial^2}{\partial\mu\partial\mu^*} + \omega_b\frac{\partial^2}{\partial\nu\partial\nu^*}\right)\right] \times P(\mu,\nu;\mu^*,\nu^*;t) = 0. \quad (4.22)$$

Then changing the dummy variables from $\mu=\alpha$ and $\nu=\beta$ and similarly for the complex conjugates, we find that $P(\alpha,\beta;\alpha^*,\beta^*;t)$ fulfills the following differential equation

$$\left[\frac{d}{dt} + \frac{i}{4}\left(\omega_a\frac{\partial^2}{\partial\alpha\partial\alpha^*} + \omega_b\frac{\partial^2}{\partial\beta\partial\beta^*}\right)\right] \times P(\alpha,\beta;\alpha^*,\beta^*;t) = 0, \quad (4.23)$$

where d/dt is the total time derivative as defined by Eq. (4.13). Equation (4.23) is the Liouville analog of the P function, which is satisfied by the fermionic Wigner function. In this sense $P(\alpha,\beta;\alpha^*,\beta^*;t)$ is a fermionic analog of the bosonic joint-probability distribution for finding the A and B modes of the system. It is evident that the P function, unlike the Wigner function does not remain constant along the classical trajectory. The positive sign in Eq. (4.23) as usual bears the true fermionic nature of the underlying dynamics which appears negative for the traditional bosonic fields.

V. CALCULATION OF CORRELATION FUNCTION: CONNECTION TO EXPERIMENT

To investigate the time-dependent correlation between the modes, let us define the following fluctuation operators:

$$\Delta\hat{a}(t) = \hat{a}(t) - \overline{\alpha}(t) \quad (5.1)$$

$$\Delta\hat{b}(t) = \hat{b}(t) - \overline{\beta}(t), \quad (5.2)$$

which describes the quantum fluctuations of the mode amplitudes about their mean values $\overline{\alpha}(t)$ and $\overline{\beta}(t)$ respectively. One may infer from the quantum regression theorem that the fluctuation operators $\Delta\hat{a}(t)$, $\Delta\hat{b}(t)$ obey the same set of linear equations of motion as obeyed by the operators $\hat{a}(t)$ and $\hat{b}(t)$. The time dependence of the correlation functions that characterize the density operator $\hat{\rho}(t)$ may now be expressed in simple terms by carrying out a transformation of variables that decouples the basic equations of motion of the system. Therefore two such operators $\hat{\sigma}_+(t)$ and $\hat{\sigma}_-(t)$ are introduced that are defined in terms of the Heisenberg operators $\hat{a}(t)$ and $\hat{b}^\dagger(t)$ by the relations

$$\hat{\sigma}_+(t) = \frac{1}{\sqrt{2}}[\hat{a}(t)e^{i\omega_a t} + i\hat{b}^\dagger(t)e^{-i\omega_b t}] \quad (5.3)$$

$$\hat{\sigma}_-(t) = \frac{1}{\sqrt{2}}[\hat{a}(t)e^{i\omega_a t} - i\hat{b}^\dagger(t)e^{-i\omega_b t}]. \quad (5.4)$$

With the help of Eqs. (2.5)–(2.6), the newly defined operators take a simple form

$$\hat{\sigma}_\pm(t) = \hat{\sigma}_\pm[\cos(\Delta\hat{\phi}) \mp \sin(\Delta\hat{\phi})], \quad (5.5)$$

where the operators $\hat{\sigma}_\pm$ are the initial values of $\hat{\sigma}_\pm(t)$ and are given by $\hat{\sigma}_\pm = 2^{-1/2}(\hat{a} \pm i\hat{b}^\dagger)$. It is clear from Eq. (5.5) that the operators $\hat{\sigma}_\pm(t)$ are decoupled from one another. Thus the transformation Eqs. (5.3)–(5.4) actually define a species of normal coordinates for the system. The operators $\hat{\sigma}_\pm(t)$ and their Hermitian adjoints satisfy the following algebraic properties:

$$\{\hat{\sigma}_+(t),\hat{\sigma}_+^\dagger(t)\} = \{\hat{\sigma}_-(t),\hat{\sigma}_-^\dagger(t)\} = 1 \quad (5.6)$$

$$\{\hat{\sigma}_+(t),\hat{\sigma}_-^\dagger(t)\} = \{\hat{\sigma}_-(t),\hat{\sigma}_+^\dagger(t)\} = 0. \quad (5.7)$$

Now with the Heisenberg density operator $\hat{\rho}$, one can similarly define operators such as $\Delta\hat{\sigma}_\pm(t)$ as the deviation of the operators $\hat{\sigma}_\pm(t)$ from their mean values as follows:

$$\Delta\hat{\sigma}_\pm(t) \equiv \hat{\sigma}_\pm(t) - \mathrm{Tr}\{\hat{\rho}\hat{\sigma}_\pm(t)\}. \quad (5.8)$$

Furthermore, using the argument of quantum regression theorem we can state that the solutions of the equations of motion for the fluctuation operators $\Delta\hat{\sigma}_\pm(t)$ would be identical with the operators $\hat{\sigma}_\pm(t)$.

To this end we define the correlation function of the fluctuations of the mode amplitudes as $t\to\infty$ by

$$\mathscr{G}_-(t) = \mathrm{Tr}\{\hat{\rho}[\Delta\hat{a}^\dagger(t)e^{-i\omega_a t} + i\Delta\hat{b}(t)e^{i\omega_b t}] \times[\Delta\hat{a}(t)e^{i\omega_a t} - i\Delta\hat{b}^\dagger(t)e^{-i\omega_b t}]\} = 2\mathrm{Tr}\{\hat{\rho}\Delta\hat{\sigma}_-^\dagger(t)\Delta\hat{\sigma}_-(t)\}. \quad (5.9)$$

Similarly, to discuss the correlation of fluctuations as $t \to -\infty$ we define function

$$\mathscr{G}_{+}(t) = 2\mathrm{Tr}\{\hat{\rho}\Delta\hat{\sigma}_{+}^{\dagger}(t)\Delta\hat{\sigma}_{+}(t)\}. \tag{5.10}$$

Since $\hat{\rho}$ is Hermitian and positive definite, the correlation functions must be real positive quantities, i.e., $\mathscr{G}_{\pm}(t) \geqslant 0$. Equation (5.5) and its adjoints imply that time dependence of the correlation functions $\mathscr{G}_{\pm}(t)$ may be expressed as

$$\mathscr{G}_{\pm}(t) = 2\mathscr{G}_{\pm}(0)[1 \pm \sin(2\Delta\phi)]. \tag{5.11}$$

In order to make a better connection with the JILA experiment [27], we now consider the form of the correlation functions $\mathscr{G}_{\pm}(t)$ in the Schrödinger picture. Making use of Eq. (3.7) and Eqs. (2.11)–(2.12) in Eq. (5.9) we obtain

$$\mathscr{G}_{\pm}(t) = (\mathrm{Tr}\hat{\rho}(t)\{\hat{a}^{\dagger} - \overline{\alpha}^{*}(t)]e^{-i\omega_a t} \mp i[\hat{b} - \overline{\beta}(t)]e^{i\omega_b t}\} \times \{[\hat{a} - \overline{\alpha}(t)]e^{i\omega_a t} \pm i[\hat{b}^{\dagger} - \overline{\beta}^{*}(t)]e^{-i\omega_b t}\}). \tag{5.12}$$

Equation (5.12) allows us to calculate the maximum value that the correlation functions can attain. It is evident from Eq. (5.12) that correlations of the mode amplitudes attain their maximum value when the state of the joint system is specified by the pure coherent state $|\alpha,\beta\rangle$. Then, at any time t, if the density operator $\hat{\rho}(t) = |\alpha,\beta\rangle\langle\alpha,\beta|$, we obtain by using Eq. (5.12) and the anticommutation relation for $\hat{b}$ and $\hat{b}^{\dagger}$ that

$$\mathscr{G}_{\pm}^{max}(t) = 1. \tag{5.13}$$

Equation (5.13) gives the upper bound for the correlation functions, which directly correspond to experimental measurement of correlation functions that are carried out after time-of-flight expansion of the dissociation of $^{40}K_2$ molecules near magnetic Feshbach resonance [27]. To correlate the spatial correlation measurement performed at JILA, previous efforts have been made based on a fermionic analog of the standard quantum optical squeezing Hamiltonian [15]. Their observations correspond to the results here, i.e., correlation functions for atom optics counterparts of parametric down-conversion possess an upper bound of value 1, which is in complete contrast to the bosonic case for which the correlation functions possess only lower bound, i.e., $\mathscr{G}_{\pm}(t) \geqslant 0$. Equation (5.13), therefore suggests a good agreement between the present theory and the experimental observations. The result is a hallmark of the fermionic character of the constituent atoms, which makes the properties of the correlation functions completely distinct from the traditional parametric amplifier of bosonic case.

Finally, it is asked in more general terms what constrains the correlation functions impose on the joint P-distribution function. For that, let us assume that $\hat{\rho}(t)$ has a two-mode P representation as

$$\hat{\rho}(t) = \int d^2\boldsymbol{\alpha} d^2\boldsymbol{\beta}\, P(\alpha,\beta;\alpha^*,\beta^*,t)|\alpha,\beta\rangle\langle\alpha,\beta|. \tag{5.14}$$

Then by making use of Eq. (5.12) and the anticommutation of $\hat{b}$ and $\hat{b}^{\dagger}$ we find

$$0 \leqslant \mathscr{G}_{\pm}(t) = 1 - \int |\{[\alpha - \overline{\alpha}(t))]e^{i\omega_a t} \pm i[\beta^* - \overline{\beta}^*(t)]e^{-i\omega_b t}\}|^2 \times P(\alpha,\beta;\alpha^*,\beta^*,t)d^2\boldsymbol{\alpha} d^2\boldsymbol{\beta} \leqslant 1, \tag{5.15}$$

which implies that fermionic two-mode $P(\alpha,\beta;\alpha^*,\beta^*,t)$ function can not take negative values, whereas a single mode fermionic P distribution is exclusively negative. This result is also remarkable when compared with bosonic two-mode P distribution, which may take negative values and even sometimes fails to exist.

VI. CONCLUSION

In this paper the statistical formulation of the fermionic counterpart of a conventional parametric amplifier of bosonic case is explored. The key elements of the analysis are based on two-mode generalization of the time-dependent density operator for fermionic fields to describe the joint quantum state of the full interacting system, which was originally developed by Cahill and Glauber in the equilibrium domain. Since fermionic operators anticommute, it is necessary to work with anticommuting numbers or Grassmann variables for expressing the solution of the joint density operator in terms of fermionic quasiprobability functions, such as Wigner or P distribution. The main conclusions are summarized as follows.

(i) It has been shown that the time evolution of the density operator, correlation functions, and the weight factors $P(\alpha,\beta;\alpha^*,\beta^*;t)$ and $W(\alpha,\beta;\alpha^*,\beta^*;t)$ possess identical structures as their corresponding bosonic counterparts. For arbitrary initial states of the density operator, the Wigner function has been shown to satisfy the fermionic counterpart of Liouville's equation. Although the anticommuting nature of Grassmann variables precludes the possibility of interpreting the fermionic field amplitude in physical terms, one can identify classical Liouville operator for matter wave counterpart of parametric amplifier.

(ii) It is interesting to note that fermionic correlation functions are marked by a characteristic upper bound due to Fermi statistics obeyed by the underlying dynamical variables, which make it distinct from bosonic correlation functions. Properties of such correlation functions can be verified in atom optics counterpart of parametric down-conversion that can be realized through dissociation of BEC molecular dimers consisting of fermionic atoms.

(iii) The relation between the correlation functions and the fermionic P distribution reveals a remarkable property of the two-mode $P(\alpha,\beta;\alpha^*,\beta^*;t)$ function, that it is always positive definite while it is predominantly negative for the single-mode case. This is in sharp contrast to bosonic two-mode P distribution, which may take negative values and can be even singular.

(iv) Due to the peculiar properties of the Grassmann algebra, fermionic field amplitudes become linear in Grassmann variables, which ensures that, unlike the bosonic Wigner functions (and irrespective of any interaction in the Hamiltonian), Wigner functions for fermionic fields are bound to satisfy fermionic analog of classical Liouville's equation and therefore most closely resemble to classical phase-space distribution functions.

ACKNOWLEDGMENTS

Thanks are due to the Council of Scientific and Industrial Research, Government of India for partial financial support. The author is grateful to Professor Deb Shankar Ray for valuable suggestions and encouragement.

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