Generalized rotating-wave approximation for the two-qubit quantum Rabi model

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(Received 21 October 2014; revised manuscript received 4 December 2014; published 8 January 2015)

The generalized rotating-wave approximation (GRWA) is presented for the two-qubit quantum Rabi model. The analytical expressions in the zeroth-order approximation recover the previous adiabatic ones. The counter-rotating-wave terms can be eliminated by performing the first-order corrections. An effective solvable Hamiltonian with the same form as the ordinary RWA one is then obtained, giving very accurate eigenvalues and eigenstates. Energy levels in the present GRWA are in accordance with the numerical exact diagonalization ones in a wide range of coupling strengths. The population dynamics in the GRWA are in quantitative agreement with the numerical results and exhibit the absence of collapses clearly, revealing the effects of the counter-rotating wave.

DOI: 10.1103/PhysRevA.91.013814 PACS number(s): 42.50.Pq, 42.50.Lc, 03.65.Ge

I. INTRODUCTION

Recent experimental progress related to qubit-oscillator systems using superconducting qubit circuits has made it possible to achieve the so-called ultrastrong-coupling regime, where the coupling strength between a single qubit and a single oscillator is comparable to the bare frequencies of the two constituents [1–6]. In this regime, the conventional rotating-wave approximation (RWA) [7] is expected to break down, leading to a mass of unexplored physics and giving rise to fascinating quantum phenomena, such as the asymmetry of vacuum Rabi splitting [8,9], collapse and revival dynamics [10–13], a Bloch-Siegert shift [2], super-radiance transition [14–16], and radiation processes based on virtual photons [17–19]. It is highly desirable to understand the behavior of the qubit oscillator in the whole coupling regime.

Since the Hamiltonian of a qubit-oscillator system contains counter-rotating-wave terms, the total excitation number is not conserved. It is very challenging to obtain the analytical solutions in the ultrastrong-coupling regime. There is much ongoing interest in this field. The single-qubit quantum Rabi model [20] describing a single qubit interacting with a quantum harmonic oscillator has been studied extensively beyond RWA with various analytical methods [21–23] in the recent years. Two or more qubits coupled to a common harmonic oscillator in the ultrastrong-coupling regime have more potential applications in quantum information processing than the single-qubit Rabi model, such as the implementation of quantum-information protocols with the oscillator transferring information coherently between qubits [24], quantum entanglement of multiple qubits [25], and super-radiance phase transition in the Dicke model describing a two-level atom ensemble in a cavity [15,16]. We investigate the two-qubit quantum Rabi model beyond the RWA, in which a quantum harmonic oscillator interacts with two identical qubits symmetrically. One of our motivations is the absence of extensive study of two and more qubits in the ultrastrong-coupling regime. Recently, an adiabatic approximation functions well when the qubit frequency is much smaller than the oscillator frequency [26], a Bargmann space technique [27] and an extended coherent state method [28] have been adopted to solve the two-qubit quantum Rabi model [27], and the variational treatment [29] reasonably captures the properties of the ground state.

We focus here on the analytic energy spectrum and eigenstates of the two-qubit quantum Rabi model with two identical qubits beyond the RWA in the ultrastrong-coupling regime by the generalized rotating-wave approximation (GRWA). By mapping the two-qubit quantum Rabi model with counter-rotating-wave interactions into a solvable Hamiltonian with the same form as the ordinary RWA term, we show that all eigenvalues and eigenstates can be approximately determined by the analytical expression based on our method, which agree well with the exactly numerical simulation in the ultrastrong-coupling regime under different detunings. We recover the same results with the zeroth-order approximation as that in Ref. [26] and make great improvement of energy spectrum by the first-order corrections. The two-qubit population dynamics is calculated to discuss the collapse and revivals of the quantum Rabi's oscillation, justifying the validity of the GRWA within a wide range of parameters.

II. HAMILTONIAN AND THE ZEROTH-ORDER APPROXIMATION

The Hamiltonian of the two-qubit quantum Rabi model, where two identical qubits couple to a harmonic oscillator with the counter-rotating-wave interactions, is ($\hbar = 1$)

$$H = \omega_0 J_x + \omega a^\dagger a + g(a^\dagger + a)J_z, \tag{1}$$

where a and $a^\dagger$ are, respectively, the annihilation and creation operators of the harmonic oscillator with frequency ω. The collective spin-1 angular momentum operators $J_z = \frac{1}{2}(\sigma_z^1 + \sigma_z^2)$ and $J_x = \frac{1}{2}(\sigma_x^1 + \sigma_x^2)$. Physically, the spin-1 system can be formed by the two identical qubits in their triplet space. ω_0 is the atomic transition frequency, and g denotes the collective qubit-oscillator coupling strength.

To begin, a brief review of the standard RWA is given in order to establish the arguments used in deriving the generalized approximation. The first step is to rewrite Eq. (1)

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in the form

$$H = -\omega_0 J_z + \omega a^\dagger a + \frac{g}{2}(a^\dagger + a)(J_+ + J_-), \tag{2}$$

where $J_\pm$ are the collective atomic raising and lowering operators of a spin-1 system. In the bases $|j_z = 1,n-1\rangle$, $|j_z = 0,n\rangle$, and $|j_z = -1,n+1\rangle$, which are the eigenstates of the noninteracting Hamiltonian $-\omega_0 J_z + \omega a^\dagger a$, the interaction term $a^\dagger J_- + a J_+$ couples the states $|j_z = 1,n-1\rangle$ with $|j_z = 0,n\rangle$ as well as the states $|j_z = 0,n\rangle$ with $|j_z = -1,n+1\rangle$, which have nearly equal energies for near resonance $\omega_0 \approx \omega$ in the absence of the interaction. On the other hand, the terms $a^\dagger J_+ + a J_-$ couple the off-resonant states, such as $|j_z = 0,n\rangle$ with $|j_z = 1,n+1\rangle$ and $|j_z = -1,n-1\rangle$. To eliminate the counter-rotating-wave terms, the Hamiltonian under the RWA is $H_{\mathrm{RWA}} = -\omega_0 J_z + \omega a^\dagger a + \frac{g}{2}(a^\dagger J_- + a J_+)$ and can be written in a matrix form:

$$H_{\mathrm{RWA}} = \begin{pmatrix} \omega(n-1)-\omega_0 & \frac{\sqrt{2}}{2}g\sqrt{n} & 0 \\ \frac{\sqrt{2}}{2}g\sqrt{n} & \omega n & \frac{\sqrt{2}}{2}g\sqrt{n+1} \\ 0 & \frac{\sqrt{2}}{2}g\sqrt{n+1} & \omega(n+1)+\omega_0 \end{pmatrix}. \tag{3}$$

In the RWA, one can diagonalize the above Hamiltonian easily.

Including the counter-rotating-wave terms, the total excitation number is not conserved and the above subspace related to n is not closed, rendering the complication of the solution. Here, we present a treatment of the Hamiltonian (1) based on the unitary transformation [30–33]: $H' = U^\dagger H U$ with the following displaced operator:

$$U = \exp\left[\frac{g}{\omega} J_z (a^\dagger - a)\right]. \tag{4}$$

The transformed Hamiltonian is

$$H' = H_0 + H_1 + H_2, \tag{5}$$

$$H_0 = \omega a^\dagger a - g^2/\omega J_z^2, \tag{6}$$

$$H_1 = \omega_0 J_x G_0(a^\dagger a) + i J_y \omega_0 F_1(a^\dagger a)(a^\dagger - a), \tag{7}$$

$$H_2 = \omega_0 J_x \left\{ \cosh\left[\frac{g}{\omega}(a^\dagger - a)\right] - G_0(a^\dagger a)\right\} + i J_y \omega_0 \left\{ \sinh\left[\frac{g}{\omega}(a^\dagger - a)\right] - F_1(a^\dagger a)(a^\dagger - a)\right\}, \tag{8}$$

where $G_0(a^\dagger a)$ and $F_1(a^\dagger a)$ are the coefficients that depend on the oscillator number operator $a^\dagger a = n$ and the dimensionless parameter g/ω. When $\cosh[\frac{g}{\omega}(a^\dagger - a)]$ is expanded as $1 + \frac{1}{2!}[\frac{g}{\omega}(a^\dagger - a)]^2 + \frac{1}{4!}[\frac{g}{\omega}(a^\dagger - a)]^4 + \cdots$, it is performed by keeping the terms containing the number operator n with the coefficient $G_0(n)$

$$G_0(n) = \langle n| \cosh\left[\frac{g}{\omega}(a^\dagger - a)\right]|n\rangle = e^{-\frac{g^2}{2\omega^2}} L_n\left(\frac{g^2}{\omega^2}\right), \tag{9}$$

with the Laguerre polynomials $L_n(g^2/\omega^2)$. Higher-order excitation terms such as $a^{\dagger 2}, a^2, \ldots$, which are accounted for in the multiphoton process, are neglected within this approximation. Similarly, by expanding $\sinh[\frac{g}{\omega}(a^\dagger - a)] = \frac{g}{\omega}(a^\dagger - a) + \frac{1}{3!}[\frac{g}{\omega}(a^\dagger - a)]^3 + \frac{1}{5!}[\frac{g}{\omega}(a^\dagger - a)]^5 + \cdots$, the one-excitation terms are kept as $F_1(n)a^\dagger - a F_1(n)$ with the coefficient $F_1(n)$ to be determined. Since the terms $a F_1(n)$ and $F_1(n)a^\dagger$ involve creating and eliminating a single photon of the oscillator, it can be evaluated as

$$\begin{aligned} F_1(n) &= \frac{1}{\sqrt{n+1}}\langle n+1| \sinh\left[\frac{g}{\omega}(a^\dagger - a)\right]|n\rangle \\ &= \frac{1}{n+1}\frac{g}{\omega} e^{-\frac{g^2}{2\omega^2}} L_n^1\left(\frac{g^2}{\omega^2}\right), \end{aligned} \tag{10}$$

with the Laguerre polynomials $L_n^1(\frac{g^2}{\omega^2}) = \sum_{i=0}^{n+1}(-1)^{n-i}\frac{(n+1)!(g^2/\omega^2)^{n-i}}{(n+1-i)!(n-i)!i!}$. The simplicity of the GRWA is based on its close connection to the standard RWA. Consequently, the terms retained in H_1 correspond to the energy-conserving one-excitation terms, just as in the standard RWA.

In the zeroth-order approximation, we neglect the terms $F_1(a^\dagger a)(a^\dagger - a)$ involving creating and eliminating a single photon, and the Hamiltonian is then approximated as

$$H' = \omega a^\dagger a - g^2/\omega J_z^2 + \omega_0 J_x G_0(a^\dagger a). \tag{11}$$

Note that only the oscillator excitation operator $n = a^\dagger a$ emerges, so the Hilbert space can be decomposed into different n manifolds spanned by the spin and oscillator basis of $|1,n\rangle$, $|0,n\rangle$, and $|-1,n\rangle$. For nth manifold, the Hamiltonian takes the form

$$H_n' = \begin{pmatrix} \omega n - \frac{g^2}{\omega} & \frac{\omega_0}{\sqrt{2}}G_0(n) & 0 \\ \frac{\omega_0}{\sqrt{2}}G_0(n) & \omega n & \frac{\omega_0}{\sqrt{2}}G_0(n) \\ 0 & \frac{\omega_0}{\sqrt{2}}G_0(n) & \omega n - \frac{g^2}{\omega} \end{pmatrix}. \tag{12}$$

The corresponding eigenvalues and eigenfunctions are straightforwardly given by

$$\begin{aligned} \varepsilon_{\pm,n} &= \omega n + \frac{\omega_0 G_0(n)}{2\sqrt{2}}\left(-\chi_n \pm \sqrt{\chi_n^2 + 8}\right), \\ \varepsilon_{0,n} &= \omega n - \frac{g^2}{\omega}, \end{aligned} \tag{13}$$

and

$$|\varepsilon_{0,n}\rangle = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad |\varepsilon_{\pm,n}\rangle = \begin{pmatrix} 1 \\ \left(\chi_n \pm \sqrt{8+\chi_n^2}\right)/2 \\ 1 \end{pmatrix}, \tag{14}$$

where $\chi_n = \frac{\sqrt{2}g^2}{\omega\omega_0 G_0(n)}$. Interestingly, the eigenvalues and eigenstates obtained in this way are exactly the same as those obtained by the adiabatic approximation [26], which are also obtained in the nth manifold.

The zeroth-order energy spectrum is plotted in Fig. 1 with blue (gray) dashed lines. For comparison, the energies obtained from numerically exact diagonalization and in the RWA are also given with black solid lines and green (gray) dashed lines. The ground-state energy and low excited energies agree well with the numerical results for $\omega_0/\omega = 0.5$. It is obvious that the RWA results become worse as the coupling strength increases. The adiabatic approximate results also deviate from the numerical ones in the ultrastrong coupling regime, and this deviation gets worse with increasing atomic transition frequency. By neglecting the term $i J_y F_1(a^\dagger a)(a^\dagger - a)$ in

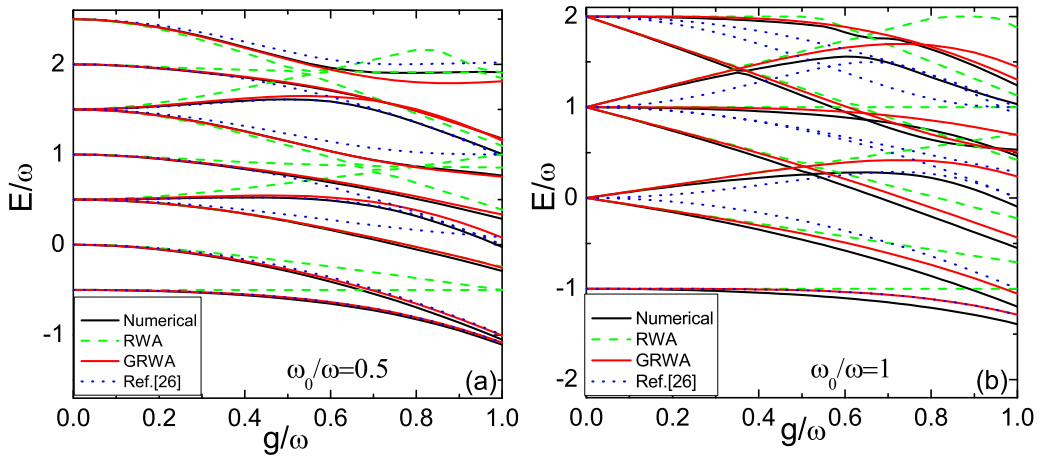

FIG. 1. (Color online) Ground-state energy E_0/ω in Eq. (25), the first excited state energy $E_{1\pm}/\omega$ in Eq. (24), and the energies obtained by solving Eq. (21) for $n > 0$ by the GRWA method (red [gray] solid lines) for different $\omega_0/\omega = 0.5$ (a) and $\omega_0/\omega = 1$ (b). The energies by the numerically exact diagonalization (black solid lines), results of RWA in Eq. (3) (green [gray] dashed line) and results in Ref. [26] expressed in Eq. (13) (green [gray] dotted lines) obtained by the zeroth-order approximation are plotted for comparison.

the zeroth-order approximation, there exists transition only between states in the same manifold spanned by $|0,n\rangle$ and $|\pm1,n\rangle$. Hence, the validity of the adiabatic approximation is restricted to the large detuning regime $\omega_0 \ll \omega$. The transitions between states belonging to the different manifolds for large value of ω_0 will be considered in the next section.

III. GENERALIZED ROTATING-WAVE APPROXIMATION

As the first-order correction, the term $iJ_yF_1(a^\dagger a)(a^\dagger - a)$ in $H' = H_0 + H_1$ will be included. The Hamiltonian now consists of two parts:

$$H_0' = \omega a^\dagger a - \frac{g^2}{\omega}J_z^2 + \omega_0\beta J_x, \tag{15}$$

$$H_1' = \omega_0 J_x[G_0(a^\dagger a) - \beta] + iJ_y\omega_0 F_1(a^\dagger a)(a^\dagger - a), \tag{16}$$

where $\beta = G_0(0) = e^{-\frac{g^2}{2\omega^2}}$.

Obviously, the spin and oscillator in H_0' are decoupled and its spin part can be diagonalized in the spin basis of $|-1\rangle,|0\rangle$, and $|1\rangle$ by a unitary matrix S as

$$S = \begin{pmatrix} 1/\lambda_- & 1/\sqrt{2} & 1/\lambda_+ \\ \mu_-/\lambda_- & 0 & \mu_+/\lambda_+ \\ 1/\lambda_- & -1/\sqrt{2} & 1/\lambda_+ \end{pmatrix}, \tag{17}$$

where $\mu_\pm = \frac{\chi_0}{2} \pm \frac{\sqrt{\chi_0^2+8}}{2}, \chi_0 = \frac{\sqrt{2}g^2}{\omega\omega_0\beta}, \lambda_\pm = \sqrt{2+\mu_\pm^2}$. The corresponding eigenvalues are $\varepsilon_\pm = \frac{\omega_0\beta}{2\sqrt{2}}(-\chi_0 \pm \sqrt{\chi_0^2+8})$ and $\varepsilon_0 = -g^2/\omega$. Therefore the diagonal H_0' takes the form

$$\tilde{H}_0 = \begin{pmatrix} \omega n + \varepsilon_- & 0 & 0 \\ 0 & \omega n + \varepsilon_0 & 0 \\ 0 & 0 & \omega n + \varepsilon_+ \end{pmatrix}, \tag{18}$$

The first-order term H_1' is transformed by the unitary matrix

$$\begin{aligned}\tilde{H}_1 &= S^+ H_1' S \\ &= \begin{pmatrix} \frac{2\sqrt{2}\mu_-}{\lambda_-^2} & 0 & \frac{\sqrt{2}(\mu_+ + \mu_-)}{\lambda_+\lambda_-} \\ 0 & 0 & 0 \\ \frac{\sqrt{2}(\mu_+ + \mu_-)}{\lambda_+\lambda_-} & 0 & \frac{2\sqrt{2}\mu_+}{\lambda_+^2} \end{pmatrix} \omega_0[G_0(a^\dagger a) - \beta] \\ &\quad + \begin{pmatrix} 0 & -\frac{\mu_-}{\lambda_-} & 0 \\ \frac{\mu_-}{\lambda_-} & 0 & \frac{\mu_+}{\lambda_+} \\ 0 & -\frac{\mu_+}{\lambda_+} & 0 \end{pmatrix} \omega_0 F_1(a^\dagger a)(a^\dagger - a).\end{aligned} \tag{19}$$

Neglecting the counter-rotating-wave terms $a^\dagger J_+ + aJ_-$ and the remote matrix elements $\frac{\sqrt{2}(\mu_+ + \mu_-)}{\lambda_+\lambda_-}$, we give the total Hamiltonian as

$$\begin{aligned}H_{\mathrm{GRWA}} &= \omega a^\dagger a + \left\{\varepsilon_+ + \frac{2\sqrt{2}\mu_+\omega_0}{\lambda_+^2}[G_0(a^\dagger a) - \beta]\right\}|1\rangle\langle 1| \\ &\quad + \left\{\varepsilon_- + \frac{2\sqrt{2}\mu_-\omega_0}{\lambda_-^2}[G_0(a^\dagger a) - \beta]\right\}|-1\rangle\langle -1| \\ &\quad + \varepsilon_0|0\rangle\langle 0| + \frac{\mu_+\omega_0}{\lambda_+}F_1(a^\dagger a)(a|1\rangle\langle 0| + a^\dagger|0\rangle\langle 1|) \\ &\quad - \frac{\mu_-\omega_0}{\lambda_-}F_1(a^\dagger a)(a|0\rangle\langle -1| + a^\dagger|-1\rangle\langle 0|),\end{aligned} \tag{20}$$

where there are only the energy-conserving terms $a|1\rangle\langle 0| + h.c$ and $a|0\rangle\langle -1| + h.c$ with renormalized coefficients $\frac{\mu_+\omega_0}{\lambda_+}F_1(a^\dagger a)$ and $-\frac{\mu_-\omega_0}{\lambda_-}F_1(a^\dagger a)$ respectively, originating from the counter-rotating-wave terms $iJ_yF_1(a^\dagger a)(a^\dagger - a)$. So it is exactly the two-qubit quantum Rabi model with renormalized parameters in the RWA form. In this sense, we can also call the first-order corrections as the GRWA. Due to the presence of the energy-conserving terms, there exist state transitions with different oscillator excitations n and $n \pm 1$. It exhibits an improvement on the adiabatic approximation and would display the effect of the counter-rotating wave, especially in the ultrastrong coupling regime.

Note that the individual bosonic creation (annihilation) operator $a^\dagger(a)$ also appears in the GRWA, so the transitions between states belonging to different manifolds should be involved. In the basis of $|-1,n+1\rangle$, $|0,n\rangle$, and $|1,n-1\rangle, (n=1,2,\ldots)$, H_{GRWA} takes the following matrix form:

$$H_{\mathrm{GRWA}}=\begin{pmatrix}\omega(n+1)+\xi_{-,n+1} & -\frac{\mu_-}{\lambda_-}R_{n,n+1}\sqrt{n+1} & 0\\ -\frac{\mu_-}{\lambda_-}R_{n,n+1}\sqrt{n+1} & \omega n+\varepsilon_0 & \frac{\mu_+}{\lambda_+}R_{n-1,n}\sqrt{n}\\ 0 & \frac{\mu_+}{\lambda_+}R_{n-1,n}\sqrt{n} & \omega(n-1)+\xi_{+,n-1}\end{pmatrix}, \tag{21}$$

where $\xi_{+,n-1}=\varepsilon_+ + \frac{2\sqrt{2}\mu_+\omega_0[G_0(n-1)-\beta]}{\lambda_+^2}$, $\xi_{-,n+1}=\varepsilon_- + \frac{2\sqrt{2}\mu_-\omega_0[G_0(n+1)-\beta]}{\lambda_-^2}$, and $R_{n,n+1}=\omega_0F_1(n)\sqrt{n+1}$, $R_{n-1,n}=\omega_0F_1(n-1)\sqrt{n}$.

Similar to the usual RWA Hamiltonian (3), the eigenvalues E_n and eigenstates $|\phi_n\rangle_{\mathrm{GRWA}}$ can be easily obtained in the closed form

$$|\phi_n\rangle_{\mathrm{GRWA}}=\alpha_{-1,n}|-1,n+1\rangle+\alpha_{0,n}|0,n\rangle+\alpha_{1,n}|1,n-1\rangle, \tag{22}$$

where the coefficients $\{\alpha_n\}$ are given in Appendix A in detail. Under the GRWA, the eigenfuntions consist of the states $|\mp 1,n\pm 1\rangle$ and $|0,n\rangle$, which are different from the states in the same manifold in the adiabatic approximation. The GRWA combines the states with different value of oscillator excitation, providing an excellent approximation to the actual energies and eigenfunctions of the system in the ultrastrong-coupling regime.

There is a special case for $n=0$. In the basis $|-1,1\rangle$ and $|0,0\rangle$, we have

$$H_{\mathrm{GRWA}}=\begin{pmatrix}\varepsilon_0 & -\frac{\mu_-R_{0,1}}{\lambda_-}\\ -\frac{\mu_-R_{0,1}}{\lambda_-} & \omega+\xi_{-,1}\end{pmatrix}, \tag{23}$$

which results in the first and second excited eigenvalues

$$E_{0,\pm}=\frac{\varepsilon_0+\omega+\xi_{-,1}}{2}\pm\frac{1}{2}\sqrt{(\varepsilon_0-\omega-\xi_{-,1})^2+4\left(\frac{\mu_-\omega_0R_{0,1}}{\lambda_-}\right)^2} \tag{24}$$

and eigenstates $|\phi\rangle_{0,\pm}=\alpha_{0,\pm}|-1,1\rangle+|0,0\rangle$ with the coefficients $\alpha_{0,\pm}=\{\frac{\lambda_-}{2\mu_-\omega_0R_{0,1}}[(\varepsilon_0-\omega-\xi_{-,1})\pm\sqrt{(\varepsilon_0-\omega-\xi_{-,1})^2+4(\frac{\mu_-\omega_0R_{0,1}}{\lambda_-})^2}]\}$.

The ground-state energy for the state $|-1,0\rangle$ is

$$E_0=\frac{\omega_0\beta}{2\sqrt{2}}\left(-\chi_0-\sqrt{\chi_0^2+8}\right). \tag{25}$$

The GRWA results for $\omega_0/\omega=0.5$ and 1 are presented in Fig. 1 using red (gray) lines. It is obvious that the GRWA results for the energy spectrum are much better than the adiabatic approximated ones [26], compared with those in the numerically exact diagonalization. Remarkably, the GRWA works reasonably well even at resonance with $\omega_0/\omega=1$. As illustrated in Fig. 1(a), the ground-state energy E_0 in Eq. (25) agrees well with the numerical one in the whole coupling regime and there is qualitative agreement for high energy levels. The level crossing is present in both the GRWA results and the exact ones, as shown in Fig. 1(b). The RWA reproduces the correct limiting behavior as $g/\omega\to 0$, but breaks down in the ultrastrong-coupling regime $g/\omega\geqslant 0.3$. The RWA requires weak coupling due to the complete neglect of counter-rotating-wave terms. The adiabatic approximation in Ref. [26] is derived under the assumption that $\omega_0\ll\omega$, so it is valid only for small detuning. Our approach is basically a perturbative expansions in terms of ω_0/ω and includes the dominant contribution of the counter-rotating-wave terms. In the framework of the present approach, the adiabatic approximated is actually the zeroth-order one and the GRWA is the first-order correction. As the increase of ω_0/ω and g/ω, the present GRWA should be better than both the RWA and the adiabatic approximation.

IV. POPULATION DYNAMICS

The quantum dynamical effects are one fundamental issue in quantum optics. We will explore the qubit population dynamics in the two-qubit and cavity coupling system in the ultrastrong-coupling regime.

In the zeroth-order approximation, the eigenstates of the Hamiltonian (1) with the counter-rotating-wave terms can be given by using the unitary transformation U

$$\begin{aligned}|\varphi_{0,n}^0\rangle&=U^\dagger|\varepsilon_{0,n}\rangle=\begin{pmatrix}-|n\rangle_1\\ 0\\ |n\rangle_{-1}\end{pmatrix},\\ |\varphi_{\pm,n}^0\rangle&=U^\dagger|\varepsilon_{\pm,n}\rangle=\begin{pmatrix}|n\rangle_1\\ (\chi\pm\sqrt{8+\chi^2})/2|n\rangle_0\\ |n\rangle_{-1}\end{pmatrix},\end{aligned} \tag{26}$$

with the oscillator extended coherent states $|n\rangle_j=\exp[\frac{jg}{\omega}(a^\dagger-a)]|n\rangle, (j=0,\pm 1)$. Similarly, the GRWA eigenstates $|\varphi_n\rangle_{\mathrm{GRWA}}$ of Eq. (1) are evaluated in detail in Appendix B.

The initial state is set $|\varphi(0)\rangle=|-1\rangle|\alpha_{-1}\rangle$ with $|\alpha_{-1}\rangle=e^{g/\omega(a^\dagger-a)}|\alpha\rangle$. The wave function evolves as $|\varphi(t)\rangle=e^{-iHt}|\varphi(0)\rangle$, which can be expanded by the eigenvalues and eigenstates under the adiabatic approximation and the GRWA method.

The population for the qubits remaining in the initial state $|-1\rangle$ is

$$P_{-1}(t)=|\langle-1|\mathrm{Tr}_{\mathrm{ph}}|\varphi(t)\rangle\langle\varphi(t)|-1\rangle, \tag{27}$$

which is evaluated in detail in Appendix B. From the population formula (B4), one can get function $S_n(t)$ in Eq. (B5), which exhibits the collapse and revivals of the Rabi's oscillation. It is interesting to note that the first line in Eq. (B5) basically give the results in the adiabatic approximation where only the transitions between states in the same manifold are considered.

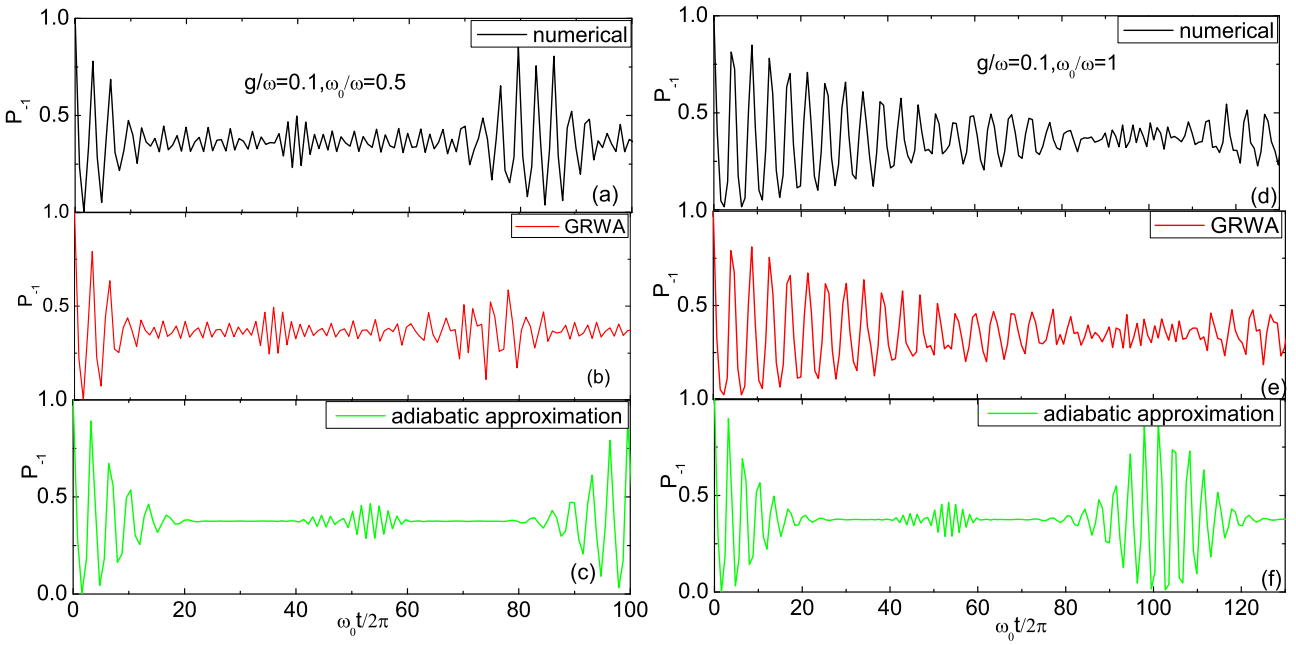

FIG. 2. (Color online) Collapse and revivals of the population $P_{1,-1}(t)$ for the two-qubit quantum Rabi model using GRWA, numerical exact diagonalization, and the adiabatic approximation are given for $\omega_0/\omega = 0.5$, $g/\omega = 0.1$ [(a)–(c)] and $\omega_0/\omega = 1$, $g/\omega = 0.1$ [(d)–(f)].

The second line of this equation will cover the effects of the transitions between states from the different manifolds, which are the crucial to the quantum dynamics and destroy the collapse given in the adiabatic approximation.

Population dynamics within adiabatic approximation and the GRWA method are plotted in Fig. 2 for $g/\omega = 0.1$ and two typical detunings $\omega_0/\omega = 0.5$ and 1. The numerically exact ones are also presented for comparison. Obviously, the GRWA results agree well with the numerical ones. The results by the adiabatic approximation obviously deviate from the exact ones and becomes worse with ω_0/ω. More seriously, collapse found in the adiabatic approximation is absent in the numerically exact solutions, following that the results in adiabatic approximation are qualitatively different from the exact ones. The reason is that the transitions between states belonging to different manifolds in the true physical process are neglected in the adiabatic approximation. Interestingly, in the GRWA one can also show the absence of the collapse as exhibited in the exact study, indicating that the dominant mechanisms have been considered. The transitions between states from different manifolds beyond the GRWA will only quantitatively modify the GRWA results slightly, as shown in Figs. 2(a), 2(b), 2(d), and 2(e). Because the agreements between the GRWA and the exact ones are quite good, so the further corrections beyond the GRWA are generally not necessary.

Population dynamics for a single-qubit case has been studied [11,33,34]. For comparison, the single-qubit population from the initial coherent state $|\alpha\rangle = e^{g/\omega(a^\dagger - a)}|0\rangle$ in the lower spin level for $\omega_0/\omega = 0.5, g = 0.1$ by the numerical diagonalization, GRWA, and the adiabatic approximation [34] are collected in Fig. 3. As we can see, there are still quantum oscillations in the GRWA in certain regions where there are collapses in the adiabatic solution, which deviates from the numerical ones. It is observed that the dynamical properties of collapses and revivals for both the single- and two-qubit cases are qualitatively similar.

V. CONCLUSION

In summary, the effective solvable Hamiltonian for the two-qubit quantum Rabi model beyond RWA is derived by a unitary transformation, which can in turn gives accurate eigenvalues and eigenstates. The zeroth-order approximation produces the analytical eigenvalues and eigenstates of the

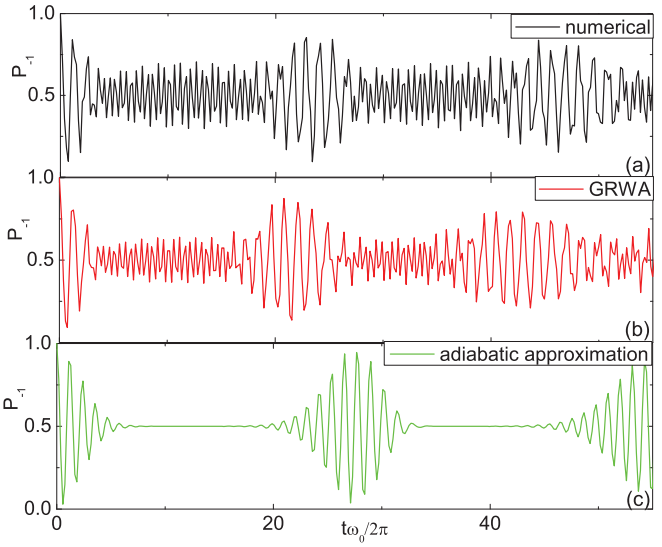

FIG. 3. (Color online) For the single-qubit quantum Rabi model, population dynamics $P_{1,-1}(t)$ using the numerically exact diagonalization (a), GRWA [33] (b), and the adiabatic approximation (c), given $\omega_0/\omega = 0.5$, $g/\omega = 0.1$.

adiabatic approximation completely. The first-order approximation, called GRWA, is mainly performed where the rotating-wave interacting coupling strength is renormalized and the counter-rotating-wave interactions include the renormalized coefficients. In the GRWA, the mathematical simplicity of the ordinary RWA is retained, which facilitates further study. The obtained energy spectra are in good agreement with the numerically exact diagonalization ones in a wide range of coupling strength and are much better than the previous adiabatic approximation. The population dynamics obtained using the GRWA is also in quantitative agreement with the numerical ones, indicating the validity of the eigenstates and eigenvalues in the ultrastrong coupling regime. Moreover, the GRWA one can show the absence of collapses clearly, indicating that the dominate mechanism of the counter-rotating wave have been considered. By the analytical eigensolutions, all properties for this two-qubit quantum Rabi model can be easily explored. Our approach can be extended to the multiple-qubit case, such as the Dicke model.

ACKNOWLEDGMENTS

This work was supported by National Natural Science Foundation of China (Grants No. 11174254, No. 11104363, and No. 11274403) and Research Fund for the Central Universities (No. CQDXWL-2013-Z014 and No. CQDXWL-2012-Z005.

APPENDIX A: SOLUTIONS TO UNIVARIATE CUBIC EQUATIONS

From the GRWA Hamiltonian matrix H_{GRWA} (21), the analytic expression of eigenenergies E_n and eigenstates $|\phi_n\rangle$ can be clearly derived with the following procedure. The determinant can be reduced to

$$\begin{vmatrix} v_- - E & z & 0 \\ z & v_0 - E & y \\ 0 & y & v_+ - E \end{vmatrix} = 0, \tag{A1}$$

where

$$\begin{aligned} v_- &= \omega(n+1) + \xi_{-,n+1}, v_0 = \omega n + \varepsilon_0, \\ v_+ &= \omega(n-1) + \xi_{+,n-1}, \\ z &= -\frac{\sqrt{n+1}\mu_-}{\lambda_-} R_{n,n+1}, \\ y &= \frac{\sqrt{n}\mu_+}{\lambda_+} R_{n-1,n}. \end{aligned}$$

It gives the following cubic equation $E^3 + bE^2 + cE + d = 0$, where

$$\begin{aligned} b &= -v_- - v_0 - v_+, \\ c &= v_- v_0 + v_+(v_- + v_0) - z^2 - y^2, \\ d &= -v_- v_0 v_+ + z^2 v_+ + y^2 v_-. \end{aligned}$$

Then we can easily obtain three real eigenvalues $E_{1,n}$, $E_{2,n}$, and $E_{3,n}$ for each $n > 0$ in any mathematics manually:

$$E_{n,1} = -\frac{-b - 2\sqrt{b^2 - 3c}\cos\theta}{3}, \tag{A2}$$

$$E_{n,2} = -\frac{-b - 2\sqrt{b^2 - 3c}\cos\left(\theta - \frac{2\pi}{3}\right)}{3}, \tag{A3}$$

$$E_{n,3} = -\frac{-b - 2\sqrt{b^2 - 3c}\cos\left(\theta + \frac{2\pi}{3}\right)}{3}, \tag{A4}$$

where $\theta = \frac{1}{3} arc\cos[\frac{2b(b^2-3c)-3(bc-9d)}{2\sqrt{(b^2-3c)^3}}]$ when $(bc-9d)^2 - 4(b^2-3c)(c^2-3bd) < 0$.

The eigenstates $|\phi_n\rangle$ can be expressed as

$$|\phi_n\rangle_{\mathrm{GRWA}} = \alpha_{-1,n}|-1,n+1\rangle + \alpha_{0,n}|0,n\rangle + \alpha_{1,n}|1,n-1\rangle, \tag{A5}$$

where the coefficients are evaluated as

$$\alpha_{-1,n} = \frac{z(E_n - v_-)}{\eta}, \quad \alpha_{0,n} = \frac{(E_n - v_+)(E_n - v_-)}{\eta}, \quad \alpha_{1,n} = \frac{y(E_n - v_+)}{\eta},$$

where the normalized parameter $\eta^2 = z^2(E_n - v_-)^2 + (E_n - v_+)^2(E_n - v_-)^2 + y^2(E_n - v_+)^2$.

APPENDIX B: ANALYTIC FORMULA FOR THE POPULATION DYNAMICS

In the adiabatic approximation, the coefficient $C_{-1,\mathrm{adia}}$ of the qubit state $|-1\rangle$ is

$$C_{-1,\mathrm{adia}} = \sum_{n=0} f_n(e^{-i\varepsilon_{\pm,n}t} + e^{-i\varepsilon_{0,n}t})|n\rangle_{-1}, \tag{B1}$$

where $f_n = {}_{-1}\langle n|\alpha\rangle$. The transition frequencies of the population is determined by $\varepsilon_{+,n} - \varepsilon_{-,n}$ and $\varepsilon_{\pm,n} - \varepsilon_{0,n}$. In weak-coupling regime and with the assumption $\omega_0 \ll \omega$, the eigenenergies with adiabatic approximation are simplified as $\varepsilon_{\pm,n} = \omega n \pm \omega_0 G_0(n)$ and $\varepsilon_{0,n} = \omega n$. Hence, the oscillation of the population is determined by two frequencies $\omega_0 G_0(n)$ and $2\omega_0 G_0(n)$, which is the same as that in Ref. [26].

Because the GRWA eigenstates $\{|\varphi_n\rangle_{\mathrm{GRWA}}\}$ in the original Hamiltonian(1) are

$$\begin{aligned} &|\varphi_n\rangle_{\mathrm{GRWA}} \\ &= U^\dagger S^\dagger |\phi_n\rangle_{\mathrm{GRWA}} \\ &= \begin{pmatrix} \frac{1}{\lambda_-}(\alpha_{-1,n}|n+1\rangle_{-1} + \alpha_{1,n}|n-1\rangle_{-1} + \mu_-\alpha_{0,n}|n\rangle_{-1}) \\ \frac{1}{\sqrt{2}}(\alpha_{-1,n}|n+1\rangle_0 - \alpha_{1,n}|n-1\rangle_0) \\ \frac{1}{\lambda_+}(\alpha_{-1,n}|n+1\rangle_1 + \alpha_{1,n}|n-1\rangle_1 + \mu_+\alpha_{0,n}|n\rangle_1) \end{pmatrix}, \end{aligned} \tag{B2}$$

the dynamical wave function can be expanded as by the eigenstates $\{E_n\}$ and eigenstates $\{|\varphi_n\rangle_{\mathrm{GRWA}}\}$ as

$$\begin{aligned} |\varphi(t)\rangle &= e^{-iHt}|\varphi(0)\rangle \\ &= f_0 e^{-iE_0 t}|0\rangle_{-1}|-1\rangle + f_{0,\pm}e^{-iE_{0,\pm}t}|\varphi_{0,\pm}\rangle \\ &\quad + \sum_{n=1} f_n e^{-iE_n t}|\varphi_n\rangle_{\mathrm{GRWA}}, \end{aligned}$$

where $f_n =_{\mathrm{GRWA}} \langle\varphi_n|\varphi(0)\rangle$. By substituting $\{|\varphi_n\rangle_{\mathrm{GRWA}}\}$ into the above function, the coefficient C_{-1}^{GRWA} of the qubit state $|-1\rangle$ becomes

$$\begin{aligned} C_{-1}^{\mathrm{GRWA}} = & \Big(\beta_0 e^{-iE_0 t} + \beta_{0,0}^2 e^{-iE_{0,+}t} \\ & + \beta_{0,0}^1 e^{-iE_{0,-}t} + \sum_{j=1}^{3} \beta_{1,1}^j e^{-iE_{1,j}t}\Big)|0\rangle_{-1} \\ & + \sum_{n>0}\sum_{j=1}^{3}\big(e^{-iE_{n-1,j}t}\beta_{-1,n-1}^j + e^{-iE_{n,j}t}\beta_{0,n}^j \\ & + e^{-iE_{n+1,j}t}\beta_{1,n+1}^j\big)|n\rangle_{-1}, \end{aligned} \tag{B3}$$

with the coefficients $\beta_0 = f_0/\lambda_-$, $\beta_{\pm1,n}^j = \alpha_{\pm1,n}^j f_n^j/\lambda_-$, and $\beta_{0,n}^j = u_-\alpha_{0,n}^j f_n^j/\lambda_-$. It indicates that there exist energy transitions among levels $E_{n,1}$, $E_{n,2}$, and $E_{n,3}$ belonging to the same nth manifold. Besides, the level transitions between $E_{n,j}$ and $E_{n\pm1,j}$ corresponding to states in different manifolds also exist.

The population $P_{-1}(t) = |C_{-1}^* C_{-1}|$ can be evaluated as

$$\begin{aligned} P_{-1}(t) = & \beta_0\beta_{0,0}^2\cos(E_0 - E_{0,+}) + \beta_0\beta_{0,0}^1\cos(E_0 - E_{0,-}) \\ & + \sum_{j=1}^{3}\beta_0\beta_{1,1}^j\cos(E_0 - E_{1,j}) + \sum_{n>0} S_n(t), \end{aligned} \tag{B4}$$

where

$$\begin{aligned} S_n(t) = & \sum_{j,k=1}^{3}\frac{e^{-|\alpha|^2}|\alpha|^{2n}}{n!}\big(\beta_{0,n}^j\beta_{0,n}^k\cos\Omega_{n,n}^{j,k} \\ & + \beta_{1,n}^j\beta_{1,n}^k\cos\Omega_{n+1,n+1}^{j,k} + \beta_{-1,n}^j\beta_{-1,n}^k\cos\Omega_{n-1,n-1}^{j,k}\big) \\ & + \sum_{j,k=1}^{3}\frac{e^{-|\alpha|^2}|\alpha|^{2n}}{n!}\big(\beta_{0,n}^j\beta_{1,n+1}^k\cos\Omega_{n,n+1}^{j,k} \\ & + \beta_{0,n}^j\beta_{-1,n-1}^k\cos\Omega_{n,n-1}^{j,k} \\ & + \beta_{-1,n-1}^j\beta_{1,n+1}^k\cos\Omega_{n-1,n+1}^{j,k}\big), \end{aligned} \tag{B5}$$

with

$$\Omega_{n,m}^{j,k} = E_{n,j} - E_{m,k}(n > 0), \quad (m = n, n \pm 1). \tag{B6}$$

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