# Two-leg fermionic Hubbard ladder system in the presence of state-dependent hopping

Shun Uchino and Thierry Giamarchi
*DQMP, University of Geneva, 24 Quai Ernest-Ansermet, 1211 Geneva, Switzerland*

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We study a two-leg fermionic Hubbard ladder model with a state-dependent hopping. We find that, contrary to the case without a state-dependent hopping, for which the system has a superfluid nature regardless of the sign of the interaction at incommensurate filling, in the presence of such a hopping a spin-triplet superfluid, spin-density wave, and charge-density wave phases emerge. We examine our results in the light of recent experiments on periodically driven optical lattices in cold atoms. We give protocols allowing us to realize the spin-triplet superfluid elusive in the cold atoms.

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## I. INTRODUCTION

Strongly correlated one-dimensional systems have attracted strong attention over the past decades. In general, in such systems the excitations differ strongly from their higher-dimensional counterparts, and for fermions are very different from the usual Landau quasiparticles occurring in a Fermi-liquid state [1]. Instead, many of the one-dimensional systems belong to the universality class known to be the Tomonaga-Luttinger liquid [2].

In particular, the system made of two coupled fermionic chains, namely, the two-leg ladder system, has been intensively studied in the past [2–13]. This system has been shown to exhibit superconductivity, not only for attractive interactions ($s$-wave superconductivity), but also quite remarkable for purely repulsive ones. In the latter case the superconductivity is of $d$-wave symmetry. The $d$-wave superconductivity emerges by doping of a Mott insulating phase at half filling.

While the one-dimensional system has been intensively studied as a first step towards other materials in higher dimensions, such as the high-$T_c$ superconductors, nowadays it is a major subject in itself due to the relevance for some experiments, particularly in the field of cold atomic gases [14].

Indeed, due to rapid advances in technology, cold atoms are a promising way to investigate one-dimensional systems with an unprecedented level of control on the interchain hopping and interactions. Most of the atoms utilized in experiments have internal degrees of freedom, which correspond to hyperfine states when we focus on alkali-metal species, already allowing reproduction of models such as the Hubbard model [15,16]. More recently, ladder systems have also been produced, both for bosonic and fermionic states [17–21].

In addition to simulating systems directly existing in condensed matter physics, by using the unique manipulations available in experiments, cold atoms also allow us to realize new quantum states of matter.

One such extension, which is the focus of this paper, is the time modulation of optical lattices [22–26]. By applying such a modulation with sufficiently high frequencies, it is possible to tune the hopping matrix. This technique allows one to control the hopping not just in strength but also in sign, since the renormalized hopping is essentially proportional to a Bessel function. In addition, by using the state-dependent optical lattice [27,28] or applying a magnetic field one can also control the hopping matrix element in a state-dependent manner. In fact, such a setup has motivated several theoretical studies on existence or nonexistence of exotic paired states in the two-dimensional Hubbard model [29–32], and on the presence of incommensurate density waves and segregation in the one-dimensional Hubbard model [33,34].

One may also expect the realization of an unconventional superfluid in cold atoms by means of such a unique technique. To realize a superfluid in cold atoms, so far, it is necessary to use a Feshbach resonance, since the typical temperature in the experiments is of the order of a tenth of the Fermi temperature [14]. A weak-coupling BCS transition temperature is extremely low compared to this temperature. A Feshbach resonance allows one to boost the interactions enough so that $s$-wave superfluidity can be routinely realized for attractive interactions. However, other symmetries are not so easily attainable. A $p$-wave Feshbach resonance is unstable due to the atom-molecule and molecule-molecule inelastic collisions [35]. Therefore the realization of an unconventional superfluid with cold atoms is a highly challenging issue.

In this paper, we show how one can realize a spin-triplet superfluid in a two-leg Hubbard ladder system. In the presence of a state-dependent hopping, the $d$-wave pairing state in the normal ladder is replaced by a spin-triplet superfluid and a spin-density-wave (SDW) state. We also discuss the case of an attractive interaction which would lead in the absence of state-dependent hopping to $s$-wave superconductivity and which gives an incommensurate charge-density wave (CDW) in the presence of state-dependent hopping.

With a ladder system we thus show that we can obtain a spin-triplet state with purely local ($s$-wave) repulsive interactions, which is an attainable situation in experiments. In a single chain such a state would have demanded an extended Hubbard model with on-site repulsion and nearest-neighbor attraction of the same order of magnitude [36], something which is at the moment out of reach in cold atomic systems.

This paper is organized as follows. Section II discusses the Hamiltonian we propose and its low-energy description by means of the bosonization technique. In Sec. III, the possible phases are determined by using a renormalization group analysis. In Sec. IV, we discuss the properties of the strong-coupling limit in the system and experimental protocols toward its realization. Section V is the Conclusion. Technical details can be found in the Appendix.

## II. HAMILTONIAN

We study two-component fermions confined in the two-leg ladder geometry. Our starting point is the following two-leg Hubbard ladder model:

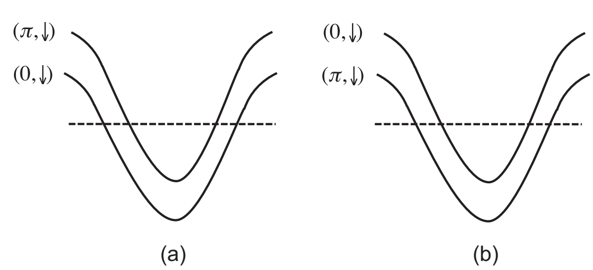
$$H = -t_{\parallel} \sum_{j=1}^{N} \sum_{s=\uparrow,\downarrow} \sum_{p=\pm 1} (c^{\dagger}_{j,s,p} c_{j+1,s,p} + \text{H.c.}) - \sum_{j,s} t_{\perp s}(c^{\dagger}_{j,s,1} c_{j,s,-1} + \text{H.c.}) + U \sum_{j,p} n_{j,\uparrow,p} n_{j,\downarrow,p}, \quad (1)$$

where $t_{\parallel}$ and $t_{\perp s}$ are, respectively, the hopping matrices along the chain and rung directions, and $j$ and $p$ indicate the chain and ladder indices. Here, the on-site Hubbard $U$ can correspond to both repulsive and attractive interactions, which indeed can be realized experimentally. We focus on a system at incommensurate filling since we are interested in the stability of the superfluids in the presence of the state-dependent hopping, in particular, in the presence of such a hopping along the rung direction. The effect of the state-dependent chain hopping has been partially discussed in Refs. [30,31,33]. In this section and Sec. III, we discuss the weak-coupling limit to analyze the possible phases using a field theory analysis. In our model, this condition implies $t_{\parallel} \gg |U|, t_{\perp s}$.

To deal with the system in the weak-coupling limit correctly, we first move to the bonding and antibonding representation for the fermion operators:

$$c_{j,s,0(\pi)} = [c_{j,s,1} + (-)c_{j,s,-1}]/\sqrt{2}, \quad (2)$$

which allows us to diagonalize the hopping terms. While in the absence of the rung hopping, the bonding and antibonding bands are energetically degenerate, these are split in the presence of the the rung hopping. In the absence of the state-dependent rung hopping, the splitting is independent of the states (or spins), and therefore, there are four different points at the Fermi level, as can be seen from Fig. 1. In the presence of the state-dependent rung hopping, however, the splitting starts to depend on the states and leads to eight different points at the Fermi level. At the same time, at $t_{\perp\uparrow} = -t_{\perp\downarrow}$, the four-point structure at the Fermi level is recovered, even though in this case the degeneracies occur between $(\pi, \uparrow)$ and $(0, \downarrow)$ and between $(0, \uparrow)$ and $(\pi, \downarrow)$ (see Fig. 1). Then, the interaction term plays the role of hybridization between the bonding and antibonding bands, which is essential to lead to nontrivial states of matter in the system.



(a) (b)

FIG. 1. Band structure of the two-leg fermionic Hubbard ladder of atoms with spin down (a) without the state-dependent hopping and (b) with state-dependent hopping as $t_{\perp\uparrow} = -t_{\perp\downarrow}$. In each case, there are four different points at the Fermi level. If the repulsive interaction is added, the latter leads to a spin-triplet superfluid while the former leads to a $d$-wave superfluid at incommensurate filling. The band structure of atoms with spin up does not change in the presence of the state-dependent hopping.

We now consider the continuum limit to use the bosonization. The fermion in the continuum limit $\psi$ can be expressed with conjugate phase fields $\phi$ and $\theta$ as [2]

$$\psi_{sqr}(x) = \frac{1}{\sqrt{2\pi\alpha}} \eta_{sq} e^{irk_F x} e^{-i[r\phi_{sq}(x) - \theta_{sq}(x)]}, \quad (3)$$

with the Fermi momentum $k_F$, index $q = 0$ or $\pi$ for the bonding and antibonding bands, index $r = -1$ or 1 for the left or right mover, cutoff parameter $\alpha$, and the phase fields $\phi_{sq}$ and $\theta_{sq}$ to be conjugate. Here, we explicitly introduce the Klein factor $\eta$, which guarantees the correct anticommutation relation of the fermions and is also important to obtain correct expressions for the bosonized Hamiltonian and correlation functions. By substituting (3) into (1), one may obtain the following low-energy effective Hamiltonian:

$$\begin{aligned} H = &\sum_{\mu=\rho,\sigma} \sum_{\nu=\pm} \int \frac{dx}{2\pi} \left[ u_{\mu\nu} K_{\mu\nu} (\nabla\theta_{\mu\nu})^2 + \frac{u_{\mu\nu}}{K_{\mu\nu}} (\nabla\phi_{\mu\nu})^2 \right] \\ &+ \int \frac{dx}{2(\pi\alpha)^2} [\cos 2\phi_{\sigma+} \{ g_1 \cos(2\phi_{\sigma-} - \delta_{\sigma-} x) \\ &+ g_2 \cos(2\phi_{\rho-} - \delta_{\rho-} x)\} \\ &+ \cos 2\theta_{\rho-} \{ g_3 \cos(2\phi_{\sigma-} - \delta_{\sigma-} x) + g_4 \cos 2\phi_{\sigma+} \} \\ &- \cos 2\theta_{\sigma-} \{ g_5 \cos(2\phi_{\rho-} - \delta_{\rho-} x) + g_6 \cos 2\phi_{\sigma+} \}], \end{aligned} \quad (4)$$

where we introduced for $\phi$ fields,

$$\phi_{\rho+} = \tfrac{1}{2}(\phi_{\uparrow 0} + \phi_{\downarrow 0} + \phi_{\uparrow\pi} + \phi_{\downarrow\pi}), \quad (5)$$

$$\phi_{\rho-} = \tfrac{1}{2}(\phi_{\uparrow 0} + \phi_{\downarrow 0} - \phi_{\uparrow\pi} - \phi_{\downarrow\pi}), \quad (6)$$

$$\phi_{\sigma+} = \tfrac{1}{2}(\phi_{\uparrow 0} - \phi_{\downarrow 0} + \phi_{\uparrow\pi} - \phi_{\downarrow\pi}), \quad (7)$$

$$\phi_{\sigma-} = \tfrac{1}{2}(\phi_{\uparrow 0} - \phi_{\downarrow 0} - \phi_{\uparrow\pi} + \phi_{\downarrow\pi}), \quad (8)$$

and similar relations for $\theta$ fields. To obtain the above, we neglect the umklapp scatterings, since the system at incommensurate filling is concerned. For our original Hamiltonian, we find $\delta_{\rho-} = 2K_{\rho-}(t_{\perp\uparrow} + t_{\perp\downarrow})/u_{\rho-}$, $\delta_{\sigma-} = 2K_{\sigma-}(t_{\perp\uparrow} - t_{\perp\downarrow})/u_{\sigma-}$, $g_i = U$ $(i = 1,2,\ldots,6)$. In addition, $u_{\mu\nu}$ and $K_{\mu\nu}$ are the velocity and the Tomonaga-Luttinger parameter, respectively. We also note that to obtain the above bosonized Hamiltonian (4), we adopt the following convention on the ordering of the Klein factors:

$$\eta_{\uparrow 0}\eta_{\downarrow 0}\eta_{\downarrow\pi}\eta_{\uparrow\pi} = 1. \quad (9)$$

## III. RENORMALIZATION GROUP ANALYSIS

Based on the bosonized Hamiltonian (4), we now determine the possible phases in this model. To this end, we employ the renormalization group (RG) approach in the bosonized Hamiltonian [2]. By performing the scaling of the cutoff $(\alpha \to \alpha' = \alpha e^{dl})$, one may obtain the set of the RG equations at the one-loop level (quadratic with respect to the coupling constants), which is given by (see Appendix)

$$\frac{dK_{\sigma-}}{dl} = -\frac{K_{\sigma-}^2 J_0(\delta_{\sigma-}\alpha)\left[y_1^2 + y_3^2\right]}{2} + \frac{J_0(\delta_{\rho-}\alpha)y_5^2 + y_6^2}{2}, \tag{10}$$

$$\frac{dK_{\sigma+}}{dl} = -\frac{K_{\sigma+}^2\left[J_0(\delta_{\sigma-}\alpha)y_1^2 + J_0(\delta_{\rho-}\alpha)y_2^2 + y_4^2 + y_6^2\right]}{2}, \tag{11}$$

$$\frac{dK_{\rho-}}{dl} = -\frac{K_{\rho-}^2 J_0(\delta_{\rho-}\alpha)\left[y_2^2 + y_5^2\right]}{2} + \frac{J_0(\delta_{\sigma-}\alpha)y_3^2 + y_4^2}{2}, \tag{12}$$

$$\frac{dy_1}{dl} = (2 - K_{\sigma-} - K_{\sigma+})y_1 - y_3 y_4, \tag{13}$$

$$\frac{dy_2}{dl} = (2 - K_{\rho-} - K_{\sigma+})y_2 - y_5 y_6, \tag{14}$$

$$\frac{dy_3}{dl} = (2 - K_{\sigma-} - 1/K_{\rho-})y_3 - y_1 y_4, \tag{15}$$

$$\frac{dy_4}{dl} = (2 - K_{\sigma+} - 1/K_{\rho-})y_4 - y_1 y_3 J_0(\delta_{\sigma-}\alpha), \tag{16}$$

$$\frac{dy_5}{dl} = (2 - K_{\rho-} - 1/K_{\sigma-})y_5 - y_2 y_6, \tag{17}$$

$$\frac{dy_6}{dl} = (2 - K_{\sigma+} - 1/K_{\sigma-})y_6 - y_2 y_5 J_0(\delta_{\rho-}\alpha), \tag{18}$$

$$\frac{d\delta_{\sigma-}}{dl} = \delta_{\sigma-} - \frac{K_{\sigma-} J_1(\delta_{\sigma-}\alpha)\left[y_1^2 + y_3^2\right]}{\alpha}, \tag{19}$$

$$\frac{d\delta_{\rho-}}{dl} = \delta_{\rho-} - \frac{K_{\rho-} J_1(\delta_{\rho-}\alpha)\left[y_2^2 + y_5^2\right]}{\alpha}, \tag{20}$$

where the initial values are given as $y_i(0) = U/(2\pi v_F)$ $(i = 1,2,\ldots,6)$, $K_{\rho-}(0) = K_{\sigma-}(0) = 1$, $K_{\rho+} = 1/\sqrt{1 + U/(2\pi v_F)}$, $K_{\sigma+}(0) = 1/\sqrt{1 - U/(2\pi v_F)}$ with the Fermi velocity $v_F$. We note that since there is no cosine term with respect to $\phi_{\rho+}$ and $\theta_{\rho+}$, which are decoupled from the other phase fields, $K_{\rho+}$ does not flow up to this order of approximation. In addition, $J_n$ $(n = 0,1)$ is the $n$th-order Bessel function, which plays a role in controlling the relevance of the corresponding cosine terms. Thus one may classify the fixed points into the following three cases:

(a) $\delta_{\rho-} \to \infty, \quad \delta_{\sigma-} \to 0,$

(b) $\delta_{\rho-} \to \infty, \quad \delta_{\sigma-} \to \infty,$

(c) $\delta_{\rho-} \to 0, \quad \delta_{\sigma-} \to \infty.$

First, let us consider the case (a), which corresponds to the limit $t_{\perp\uparrow} \approx t_{\perp\downarrow}$. In this case, the terms proportional to $g_2$, $g_5$ can be dropped due to the rapid oscillation of the cosines. Thus the RG equations reduce to ones without the state-dependent hopping [2], since this limit also allows us to do the substitutions, $J_0(\delta_{\sigma-}\alpha) = 1$ and $J_0(\delta_{\rho-}\alpha) = 0$. The RG analysis shows the fixed point is given by $g_1 \to -\infty$, $g_3 \to \infty$, $g_4 \to \infty$, $g_6 \to 0$ for $U > 0$ and $g_1 \to -\infty$, $g_3 \to -\infty$, $g_4 \to 0$, $g_6 \to \infty$ for $U < 0$. While regardless of the sign of the interaction, $\phi_{\rho-}$, $\phi_{\sigma+}$, and $\phi_{\sigma-}$ are gapped, these minimums are different for opposite signs of the interaction. It turns out that the minimum can be determined by the fixed point. Then, the dominant correlations are the $d$-wave superfluid for $U > 0$, whose pairing occurs between the different chains, and the $s$-wave superfluid for $U < 0$, whose pairing essentially occurs on site. The corresponding operators are

$$\begin{aligned} O_{\mathrm{DSF}}(j) &= \sum_p (c_{j,\uparrow,p}c_{j,\downarrow,-p} - c_{j,\downarrow,p}c_{j,\uparrow,-p}) \\ &\sim e^{-i\theta_{\rho+}}(\cos\phi_{\rho-}\sin\phi_{\sigma+}\sin\phi_{\sigma-} \\ &\quad - i\sin\phi_{\rho}\cos_{\sigma+}\cos\phi_{\sigma-}), \end{aligned} \tag{21}$$

$$\begin{aligned} O_{\mathrm{SSF}^0}(j) &= \sum_p (c_{j,\uparrow,p}c_{j,\downarrow,p} - c_{j,\downarrow,p}c_{j,\uparrow,p}) \\ &\sim e^{-i\theta_{\rho+}}(\cos\phi_{\rho-}\cos\phi_{\sigma+}\cos\phi_{\sigma-} \\ &\quad + i\sin\phi_{\rho-}\sin\phi_{\sigma+}\sin\phi_{\sigma-}), \end{aligned} \tag{22}$$

respectively [2]. In contrast to the single-chain Hubbard model, we have for the ladder a superfluid regardless of sign of the interaction.

Let us next consider the case (b), where both of the rung hoppings $t_{\perp\rho} \equiv t_{\perp\uparrow} + t_{\perp\downarrow}$ and $t_{\perp\sigma} \equiv t_{\perp\uparrow} - t_{\perp\downarrow}$ are relevant and the substitutions $J_0(\delta_{\rho-}\alpha) = J_0(\delta_{\sigma-}\alpha) = 0$ are allowed. In this case, the effects of $g_1$, $g_2$, $g_3$, $g_5$ can be dropped due to the large oscillations. By solving the RG equations under these conditions, the fixed points are shown to be $g_4 \to \infty$, $g_6 \to \infty$ for $U > 0$ and $g_4 \to -\infty$, $g_6 \to -\infty$ for $U < 0$. Thus we see $\theta_{\rho-}$, $\phi_{\sigma+}$, $\theta_{\sigma-}$ are going to be gapped. From the fixed point analysis, we find that the following SDW and CDW operators are relevant for $U > 0$ and $U < 0$, respectively:

$$\begin{aligned} O_{\mathrm{SDW}^\pi}(j) &= \sum_p p(c_{j,\uparrow,p}^\dagger c_{j,\uparrow,p} - c_{j,\downarrow,p}^\dagger c_{j,\downarrow,p}) \\ &\sim e^{-i\phi_{\rho+}}(\sin\theta_{\rho-}\cos\phi_{\sigma+}\cos\theta_{\sigma-} \\ &\quad - \cos\theta_{\rho-}\sin\phi_{\sigma+}\sin\theta_{\sigma-}), \end{aligned} \tag{23}$$

$$\begin{aligned} O_{\mathrm{CDW}^\pi}(j) &= \sum_p p(c_{j,\uparrow,p}^\dagger c_{j,\uparrow,p} + c_{j,\downarrow,p}^\dagger c_{j,\downarrow,p}) \\ &\sim e^{-i\phi_{\rho+}}(\cos\theta_{\rho-}\cos\phi_{\sigma+}\sin\theta_{\sigma-} \\ &\quad - \sin\theta_{\rho-}\sin\phi_{\sigma+}\cos\theta_{\sigma-}), \end{aligned} \tag{24}$$

where $\pi$ indicates the difference of the densities on the two legs. The presence of the state-dependent rung hopping as $|t_{\perp\uparrow}/t_{\perp\downarrow}| \neq 1$ tries to destroy the the superfluidity for the two-leg Hubbard ladder, and causes fluctuations toward crystalline orders such as the SDW or CDW. Here, one may notice the analogy with the single-chain Hubbard system in the presence of a state-dependent hopping [33], where in the wide range of parameters SDW and CDW are shown to be the dominant fluctuations for $U > 0$ and $U < 0$, respectively. Such emergences of the density wave states in the single-chain system are natural, since one of the spin components is reluctant to hop between different sites, However, now we impose the spin dependence only for the rung direction. Thus the emergence of the SDW or CDW in our model is less trivial.

Let us finally consider the case (c), which can be realized when $t_{\perp\uparrow} \approx -t_{\perp\downarrow}$ and therefore the substitutions $J_0(\delta_{\rho-}\alpha) = 1$ and $J_0(\delta_{\sigma-}\alpha) = 0$ are justified. In this case, $g_1$, $g_3$ can be dropped in a manner similar to the other cases. By solving the

RG equations, we find the fixed points to be $g_2 \to -\infty$, $g_4 \to 0$, $g_5 \to \infty$, $g_6 \to \infty$ for $U > 0$ and $g_2 \to -\infty$, $g_4 \to 0$, $g_5 \to -\infty$, $g_6 \to -\infty$ for $U < 0$, and therefore $\phi_{\rho-}, \phi_{\sigma+}, \theta_{\sigma-}$ are gapped. In accordance with the fixed points, the dominant correlations are shown to be the spin-triplet superfluid along the $z$ direction for $U > 0$ and $s$-wave superfluid for $U < 0$, where the corresponding operators are

$$O_{\mathrm{TSF}^z}(j) = \sum_p (c_{j,\uparrow,p} c_{j,\downarrow,-p} + c_{j,\downarrow,p} c_{j,\uparrow,-p}) \sim e^{-i\theta_{\rho+}} (\cos\phi_{\rho-} \cos\phi_{\sigma+} \cos\theta_{\sigma-} - i \sin\phi_{\rho-} \sin\phi_{\sigma+} \sin\theta_{\sigma-}), \tag{25}$$

$$O_{\mathrm{SSF}^\pi}(j) = \sum_p p(c_{j,\uparrow,p} c_{j,\downarrow,p} - c_{j,\downarrow,p} c_{j,\uparrow,p}) \sim e^{-i\theta_{\rho+}} (\sin\phi_{\rho-} \sin\phi_{\sigma+} \cos\theta_{\sigma-} + i \cos\phi_{\rho-} \cos\phi_{\sigma+} \sin\theta_{\sigma-}), \tag{26}$$

respectively. We first focus on the emergence of the dominant fluctuation of the spin-triplet superfluid for $U > 0$. Namely, the sign inversion in the rung hopping regarding only one of the spin components allows the change of nature of the pairings from the interchain spin singlet to the interchain spin triplet. In the bonding and antibonding representation, while the $d$-wave superfluid operator has the form $c_{j,\uparrow,0} c_{j,\downarrow,0} - c_{j,\uparrow,\pi} c_{j,\downarrow,\pi}$, the spin-triplet superfluid occurring is given as $c_{j,\uparrow,0} c_{j,\downarrow,\pi} + c_{j,\downarrow,0} c_{j,\uparrow,\pi}$. To understand the mechanism, we first point out that such a sign inversion in the rung hopping can be achieved by introducing the Peierls phases both in charge and spin sectors by $\pi/2$. Then, what is important for the

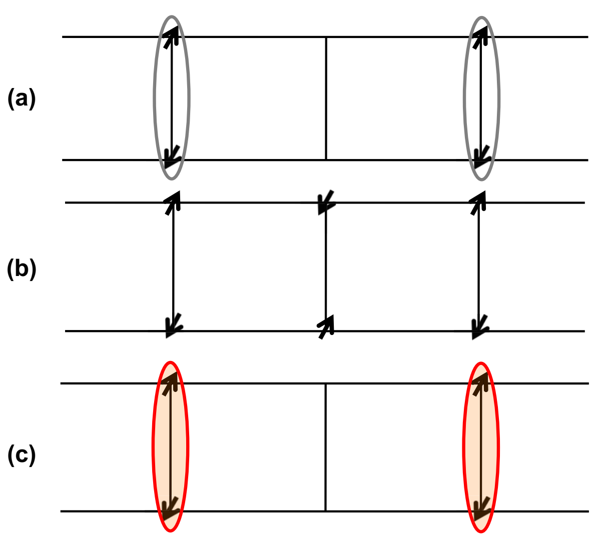



FIG. 2. (Color online) Possible phases for the repulsive Hubbard interaction, $U > 0$: $d$-wave superfluid (a), spin-density wave (b), spin-triplet superfluid along the $z$ direction (c). The arrows and ellipses (shaded ellipses) indicate the spins and spin-singlet pairing (spin-triplet pairing, especially, $|S^z = 0\rangle = |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle$), respectively. Since $U > 0$, the on-site pairing is discouraged and the interchain pairing is selected by the many-body effect for $t_{\perp\uparrow} \approx \pm t_{\perp\downarrow}$. The SDW state realized has the alternate occupation in spin on the two legs.

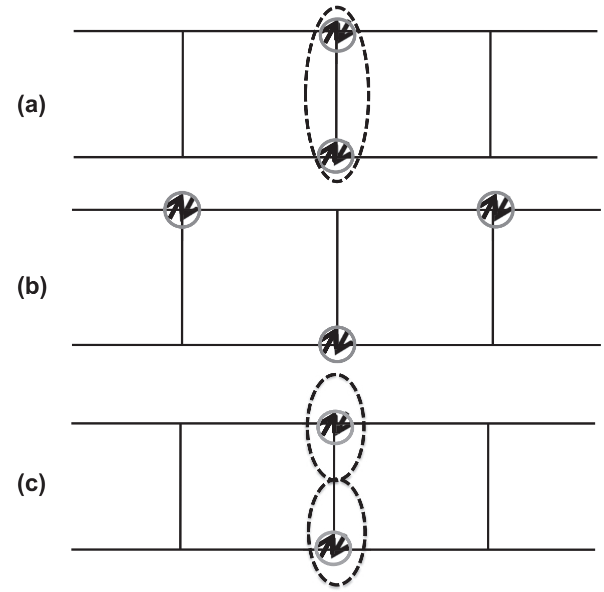



FIG. 3. Possible phases for the attractive Hubbard interaction, $U < 0$: bonding $s$-wave superfluid (a), charge-density wave (b), antibonding $s$-wave superfluid (c). The difference of the dashed curves is that the $s$-wave superfluid (a) occurs for the bonding band of the Cooper pairs while the superfluid (c) occurs for the antibonding band of the Cooper pairs. The CDW (b) has the alternate occupation on the two legs.

pairing is the Peierls phase in the spin sector. In fact, it has been shown in Ref. [37] that such a Peierls phase causes the spin rotation of the fermions for one of the chains and transforms a spin-singlet into a spin-triplet pairing. For $U < 0$, on the other hand, the difference between the $s$-wave superfluids in Eq. (26) and in Eq. (22) is that if we treat the Cooper pairs occurring in each chain as the bosons, the superfluid in the absence of the state-dependent hopping occurs for the bonding band of the bosons, while the superfluid in the presence of it occurs for the antibonding band of the bosons. Compared with the situation from the spin-singlet to spin-triplet pairings for $U > 0$, the important ingredient for this change of the $s$-wave superfluids for $U < 0$ is the Peierls phase in charge sector. One may also accept this situation recalling that in a Bose-Einstein condensate on a double-well potential, a BEC on the bonding band is normally the ground state while a BEC on the antibonding band becomes the ground state in the presence of the sign inversion hopping [38].

The possible phases are summarized in Figs. 2 and 3.

## IV. DISCUSSION

### A. Strong-coupling limit

So far, we have discussed the weak-coupling limit by means of the bosonization and RG analysis; it is also interesting to see what happens in the strong-coupling limit in which naively a similar phase diagram may be expected. For the $U > 0$ case, in fact, it may be difficult to depict a general phase diagram analytically since a faithful effective Hamiltonian has yet to be known, except for commensurate filling such as half filling. In addition, the rung hopping is a relevant perturbation, which prevents one from starting at the single-chain Hubbard model where the Bethe ansatz approach is available. At the same time, the previous numerical analyses in the absence of the state-dependent hopping show that the $d$-wave superfluid state emerges even in the strong-coupling limit [2,10–13]. In addition, since the hybridization among the four different Fermi points by the on-site repulsive interaction shown in Fig. 1(a) is an essential ingredient of the $d$-wave state, we obtain the $d$-wave superfluid not only for $t_\parallel \gg t_\perp$ but also for $t_\parallel \approx t_\perp$, in which the numerical calculation has been performed [10–13]. Therefore the presence of the spin-triplet superfluid in the strong-coupling limit can also be shown with the argument in Sec. III. Namely, by using the canonical transformations $c_{j,s,1} \to a_{j,s,1} c_{j,s,-1} \to \pm a_{j,s,-1}$ where the sign is $+$ for $s=\uparrow$ and $-$ for $s=\downarrow$, the Hamiltonian with $t_{\perp\uparrow} = -t_{\perp\downarrow}$ is mapped onto one with $t_{\perp\uparrow} = t_{\perp\downarrow}$, that is, a normal two-leg fermionic Hubbard ladder can be obtained. Accordingly, the operator of the spin-triplet superfluid is transformed into that of the $d$-wave superfluid. Therefore, once we confirm the emergence of the $d$-wave superfluid in the normal two-leg fermionic Hubbard ladder system, we see that the spin-triplet superfluid occurring in $t_{\perp\uparrow} \approx -t_{\perp\downarrow}$ is robust. We also note that the essence of the spin-triplet superfluid is the manipulation on the rung hopping, and thus, nothing happens and the $d$-wave superfluid remains even if such a manipulation on the hopping is performed for the chain direction. Thus to see the spin-triplet superfluid, the manipulation on the hopping along the rung direction is required. Another interesting but remaining issue may be the possibility of segregation in the limit $t_\uparrow = 0$ or $t_\downarrow = 0$ [39,40].

On the other hand, for the $U < 0$ case, we can discuss the possible phases in the strong-coupling limit by means of an effective Hamiltonian approach. To see this, we first perform the so-called particle-hole transformation [41] in this model. Then the original model is mapped onto the system with $U > 0$ and spin imbalance at half filling, and therefore the effective Hamiltonian is shown to be

$$H = J_\parallel \sum_j (\vec{S}_{j,1}\cdot\vec{S}_{j+1,1} + \vec{S}_{j,-1}\cdot\vec{S}_{j+1,-1}) - h\sum_j \left(S^z_{j,1} + S^z_{j,-1}\right) + J^{xy}_\perp \sum_j \left(S^x_{j,1}S^x_{j,-1} + S^y_{j,1}S^y_{j,-1}\right) + J^z_\perp \sum_j S^z_{j,1}S^z_{j,-1}, \tag{27}$$

where $J_\parallel = 4t_\parallel^2/|U|$, $J^{xy}_\perp = 4t_{\perp\uparrow}t_{\perp\downarrow}/|U|$, $J^z_\perp = 2(t^2_{\perp\uparrow} + t^2_{\perp\downarrow})/|U|$, and $h$ is a magnetic field corresponding to filling in the original attractive model. By performing bosonization for the above Hamiltonian [2], one may obtain

$$H^{\rm eff} = \sum_{\mu=s,a} \int \frac{dx}{2\pi}\left(u_\mu K_\mu (\nabla\theta_\mu)^2 + \frac{u_\mu}{K_\mu}(\nabla\phi_\mu)^2\right) + \frac{1}{2(\pi\alpha)^2}\int dx [J^{xy}_\perp \cos(\sqrt{2}\theta_a) + J^z_\perp \cos(2\sqrt{2}\phi_a)], \tag{28}$$

where $\phi_{s(a)} = [\phi_1 + (-)\phi_{-1}]/\sqrt{2}$ is the phase field in chain $p$, $\phi_p$ $(p=\pm 1)$, and similar relations for the $\theta$ field. The original spin fields and phase fields are related as $S^z_p(x) = -\nabla\phi_p(x)/\pi + (-1)^x \cos(2\phi_p(x))/(\pi\alpha)$ and $S^+_p(x) = e^{-i\theta_p(x)}[(-1)^x + \cos 2\phi_p(x)]/\sqrt{2\pi\alpha}$. Since $J_\parallel \gg J_\perp$ is concerned, we can determine the Tomonaga-Luttinger parameters as

$$K_{s,a} = K\left(1 \mp \frac{KJ^z_\perp}{2\pi u}\right). \tag{29}$$

Here $K$ and $u$ are the Tomonaga-Luttinger parameter and velocity in the single-chain Heisenberg model, respectively. The Tomonaga-Luttinger parameter $K$ can be determined by means of Bethe ansatz, and it is known that the possible range is $1/2 \leqslant K \leqslant 1$, where $K = 1/2$ corresponds to the no-magnetization case and $K = 1$ to the fully polarized case [42]. Since the above consists of the linear combination of the simple cosine terms, one can determine the ground state with a simple scaling argument. In fact, $\cos\sqrt{2}\theta_a$ and $\cos\sqrt{8}\phi_a$ have the scaling dimensions of $(2K_a)^{-1}$ and $2K_a$, respectively. Thus we see that $\cos\sqrt{2}\theta_a$ is ordered for $K_a > 1/2$ and the situation is reversed for $K_a < 1/2$. As can be seen from Eqs. (28) and (29), $K_a > 1/2$, and we expect that $\theta_a$ is ordered except for the limit $J^{xy}_\perp \to 0$ where $\phi_a$ is ordered. To specify the ground state in the spin language, let us introduce bonding and antibonding spin operators as $\vec{S}_0 = \vec{S}_1 + \vec{S}_{-1}$ and $\vec{S}_\pi = \vec{S}_1 - \vec{S}_{-1}$, respectively. Then, one finds that the bonding (antibonding) transverse spin-spin correlation $\langle S^+_0(r)S^-_0(0)\rangle(\langle S^+_\pi(r)S^-_\pi(0)\rangle)$ is dominant for $\theta_a$ to be gapped with $J^{xy}_\perp < 0 (J^{xy}_\perp > 0)$, while the antibonding longitudinal spin-spin correlation $\langle S^z_\pi(r)S^z_\pi(0)\rangle$ is dominant for $\phi_a$ to be gapped [2]. Now, we can determine the dominant correlation in the original model by using the particle-hole transformation again. Since by this transformation

$$S^-_0 \to \sum_p p c_{j,\uparrow,p} c_{j,\downarrow,p} = O_{{\rm SSC}^\pi}, \tag{30}$$

$$S^-_\pi \to \sum_p c_{j,\uparrow,p} c_{j,\downarrow,p} = O_{{\rm SSC}^0}, \tag{31}$$

$$S^z_\pi \to \sum_{p,s} p c^\dagger_{j,s,p} c_{j,s,p} = O_{{\rm CDW}^\pi}, \tag{32}$$

we conclude that the $s$-wave superfluid is dominant except for $t_{\perp\uparrow}t_{\perp\downarrow} \to 0$, where the CDW correlation is dominant. In particular, the bonding $s$-wave pairing state is realized for $t_{\perp\uparrow}t_{\perp\downarrow} > 0$ while the antibonding $s$-wave pairing state is realized for the opposite sign case. Thus the phase structure is compatible with the weak-coupling analysis while in the weak-coupling limit the region of the $s$-wave superfluid is rather narrow but in the strong-coupling limit the situation is reversed. This may be explained by the observation that the pairing gap becomes larger as the attractive interaction is increased and the pairing in the $s$-wave superfluid essentially occurs in a single site; therefore the introduction of the small state-dependent rung hopping may not cause the disappearance of the superfluid correlation.

### B. Experimental protocol

We now discuss the realization of our model and its ground states in cold atoms.

In order to realize the two-leg ladder geometry, we can consider an optical superlattice [17,18]. By using this technology, we can obtain a system where there are a number of two-leg ladders, each of which is weakly coupled by some hopping parameter. To ensure the one-dimensional character in the system, this hopping parameter should be much smaller than a temperature [2]. Then, the two-leg ladder system in the absence of state-dependent hopping is obtained. As another route to realize such a ladder geometry, one may also utilize internal degrees of freedom in an atom as discussed in Refs. [43,44].

On the other hand, the Hubbard interaction $U$ can be tuned by selecting atomic species and by changing lattice depth. Typically, $^{40}$K and $^{6}$Li have been utilized to realize the system for $U > 0$ and $U < 0$, respectively [14]. In addition, the Feshbach resonance is available to change the strength and sign of the interaction.

The most important ingredient in the system discussed is a state-dependent hopping. When it comes to a positive state-dependent hopping, heteronuclear mixtures such as $^{6}\mathrm{Li}-^{40}\mathrm{K}$ [45] and $^{6}\mathrm{Li}-^{173}\mathrm{Yb}$ [46] are available. However, since we are also interested in a state-dependent hopping whose sign is different between spin up and down, another scheme is necessary.

To this end, we start with the two-leg Hubbard ladder system in the absence of a state-dependent hopping. To obtain a state-dependent rung hopping, we consider adding the following time-dependent term in the Hamiltonian:

$$\sum_{s=\uparrow,\downarrow} A_s \cos(\omega t) \sum_{p=\pm 1} p n_{j,s,p}. \tag{33}$$

If $A_\uparrow = A_\downarrow$, the above time-varying linear potential can be obtained with a sinusoidal shaking of an optical lattice along the rung direction [24]. In order to exactly obtain the above time-dependent term for $A_\uparrow \neq A_\downarrow$, a sinusoidal shaking of a state-dependent optical lattice [27,28] or of a magnetic field gradient [47] can be utilized. Then, an essential point is that when $\hbar\omega \gg t_\parallel, t_\perp, |U|$, we may perform the time average of the above oscillation term, which causes the renormalization of the hopping parameter as $t_\perp \to t_\perp J_0(A_s/(\hbar\omega))$ [23,26]. Since the argument of the Bessel function is now state dependent, the Hamiltonian (1) can be obtained. We note that the Bessel function can take a negative value, which allows us to consider a negative hopping parameter. Indeed, such a negative hopping parameter by the time-dependent oscillation term has been observed in Refs. [24,25].

A particularly interesting challenge is to make the spin-triplet superfluid realized around $t_{\perp\uparrow} \approx -t_{\perp\downarrow}$ for a repulsive $U$. Here, we note that $'' \approx ''$ implies $t_{\perp\rho}/T \ll 1$, where $T$ is a temperature. In this case, the effect of $t_{\perp\rho}$ can be dropped in the Hamiltonian, and the effective Hamiltonian reduces to one of $t_{\perp\uparrow} = -t_{\perp\downarrow}$ [2]. In the two-leg fermionic ladder system at incommensurate filling, we have a one-charge gap and two spin gaps, which are exponentially small for the weak-coupling limit but are of the order of the exchange energy for the strong-coupling limit. (See Refs. [10–13] for numerical estimations for the strong-coupling limit.) Since the gaps are essential to characterize the spin-triplet superfluid state, the temperature should be smaller than them as well as the other Hamiltonian parameters $t_\parallel, U, t_{\perp,s}$. Thus the dominant spin-triplet superfluid correlation should show up at the temperature satisfying these conditions. We note that a similar argument for the realizations of the other phases is also possible.

When it comes to the spin-triplet superfluid, we can also utilize the technique of synthetic gauge fields [48]. Recently, by using a Raman laser and lattice driving [47,49–52], it is possible to introduce the Peierls phase in the hopping parameter. If such a Peierls phase has a state dependency, which is indeed possible experimentally, the hopping parameter is modified as $t \to t e^{i\Phi_s}$, where $\Phi_s$ is the Peierls phase. When $\Phi_\uparrow = -\Phi_\downarrow$, such a hopping term can also be regarded as a spin-orbit coupling. As explained in Sec. III, the essence of the spin-triplet superfluid is the emergence of the state-dependent Peierls phase along the rung direction as $t_\perp \to t_\perp e^{is\Phi}$. Thus the spin-triplet superfluid realized with this manipulation is essentially the same mechanism as one discussed in this paper.

Finally, we give a few comments on the experimental observability of the superfluids. An important feature is the presence of the gaps, which may be measured by rf spectroscopy [53]. However, the measurement of gaps alone is not enough to distinguish different superfluids. One of the possible solutions to this problem is to use the particle-hole transformation [41]. Then the Hamiltonian for $U > 0$ at incommensurate filling without a spin imbalance is transformed into one for $U < 0$ at half filling with a spin imbalance. We also note that the hopping terms in Eq. (1) are invariant under such a particle-hole transformation. On the other hand, the operators of the $d$-wave and spin-triplet superfluids are transformed into those of staggered spin-flux and bond SDW phases, respectively [37]. The properties of such phases may be captured by the local addressing of the flux [49,54] and spin correlations [20,55] or by spin-sensitive Bragg scattering of light [56].

## V. CONCLUSION

We have examined a two-leg fermionic Hubbard ladder model in the presence of a state-dependent hopping. We have focused on a case where such a hopping exists for the rung direction. This system can be treated as the minimal construction of the physics of mixed dimensions [57] if the rung hopping in one of the states (spins) is zero since another rung hopping plays a role in connecting different chains. Due to the state-dependent hopping, the original $d$-wave and $s$-wave superfluids realized in the normal two-leg fermionic Hubbard ladder model for the repulsive and attractive interactions, respectively, become unstable. We have demonstrated that instead the spin-triplet superfluid, SDW, and CDW states become stable depending on the ratio $t_{\perp\uparrow}/t_{\perp\downarrow}$. In particular, our proposal shows a spin-triplet superfluid state for purely local interactions can be realized. We have also discussed the experimental protocol and observability toward the spin-triplet superfluid.

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## APPENDIX: RENORMALIZATION GROUP EQUATIONS

In this Appendix, we wish to outline the derivation of the renormalization group equations in a similar way as Ref. [2]. We first consider the following correlation function:

$$R(r_1 - r_2) = \langle T_\tau e^{i\phi_{\sigma+}(x_1,\tau_1)} e^{-i\phi_{\sigma+}(x_2,\tau_2)} \rangle \tag{A1}$$

where $T_\tau$ denotes the time-ordered product. By expanding the above correlation function in terms of $g_i$ up to third order, we obtain

$$R(r_1 - r_2) \approx e^{-\frac{K_{\sigma+}}{2} F_1(r_1 - r_2)} + (\mathrm{S}) + (\mathrm{T}), \tag{A2}$$

where $F_1(r) = \ln(r/\alpha)$,

$$\begin{aligned}
(\mathrm{S}) = {} & \frac{1}{2}\left(\frac{g_1}{8(\pi\alpha)^2 v_F}\right)^2 \sum_{\epsilon_1,\epsilon_2=\pm 1} \int d^2r' d^2r'' [\langle e^{i\{\phi_{\sigma+}(\vec{r}_1)-\phi_{\sigma+}(\vec{r}_2)+2\epsilon_1[\phi_{\sigma+}(\vec{r}')-\phi_{\sigma+}(\vec{r}'')]+2\epsilon_2[\phi_{\sigma-}(\vec{r}')-\phi_{\sigma-}(\vec{r}'')-\delta_{\sigma-}(x'-x'')]\}}\rangle_0 \\
& -\langle e^{i[\phi_{\sigma+}(\vec{r}_1)-\phi_{\sigma+}(\vec{r}_2)]}\rangle_0 \langle^{i\{2\epsilon_1[\phi_{\sigma+}(\vec{r}')-\phi_{\sigma+}(\vec{r}'')]+2\epsilon_2[\phi_{\sigma-}(\vec{r}')-\phi_{\sigma-}(\vec{r}'')-\delta_{\sigma-}(x'-x'')]\}}\rangle_0] \\
& +\frac{1}{2}\left(\frac{g_2}{8(\pi\alpha)^2 v_F}\right)^2 \sum_{\epsilon_1,\epsilon_2=\pm 1} \int d^2r' d^2r'' [\langle e^{i\{\phi_{\sigma+}(\vec{r}_1)-\phi_{\sigma+}(\vec{r}_2)+2\epsilon_1[\phi_{\sigma+}(\vec{r}')-\phi_{\sigma+}(\vec{r}'')]+2\epsilon_2[\phi_{\rho-}(\vec{r}')-\phi_{\rho-}(\vec{r}'')-\delta_{\rho-}(x'-x'')]\}}\rangle_0 \\
& -\langle e^{i[\phi_{\sigma+}(\vec{r}_1)-\phi_{\sigma+}(\vec{r}_2)]}\rangle_0 \langle^{i\{2\epsilon_1[\phi_{\sigma+}(\vec{r}')-\phi_{\sigma+}(\vec{r}'')]+2\epsilon_2[\phi_{\rho-}(\vec{r}')-\phi_{\rho-}(\vec{r}'')-\delta_{\rho-}(x'-x'')]\}}\rangle_0] \\
& +\frac{1}{2}\left(\frac{g_4}{8(\pi\alpha)^2 v_F}\right)^2 \sum_{\epsilon_1,\epsilon_2=\pm 1} \int d^2r' d^2r'' [\langle e^{i\{\phi_{\sigma+}(\vec{r}_1)-\phi_{\sigma+}(\vec{r}_2)+2\epsilon_1[\phi_{\sigma+}(\vec{r}')-\phi_{\sigma+}(\vec{r}'')]+2\epsilon_2[\theta_{\rho-}(\vec{r}')-\theta_{\rho-}(\vec{r}'')]\}}\rangle_0 \\
& -\langle e^{i[\phi_{\sigma+}(\vec{r}_1)-\phi_{\sigma+}(\vec{r}_2)]}\rangle_0 \langle^{i\{2\epsilon_1[\phi_{\sigma+}(\vec{r}')-\phi_{\sigma+}(\vec{r}'')]+2\epsilon_2[\theta_{\rho-}(\vec{r}')-\theta_{\rho-}(\vec{r}'')]\}}\rangle_0] \\
& +\frac{1}{2}\left(\frac{g_6}{8(\pi\alpha)^2 v_F}\right)^2 \sum_{\epsilon_1,\epsilon_2=\pm 1} \int d^2r' d^2r'' [\langle e^{i\{\phi_{\sigma+}(\vec{r}_1)-\phi_{\sigma+}(\vec{r}_2)+2\epsilon_1[\phi_{\sigma+}(\vec{r}')-\phi_{\sigma+}(\vec{r}'')]+2\epsilon_2[\theta_{\sigma-}(\vec{r}')-\theta_{\sigma-}(\vec{r}'')]\}}\rangle_0 \\
& -\langle e^{i[\phi_{\sigma+}(\vec{r}_1)-\phi_{\sigma+}(\vec{r}_2)]}\rangle_0 \langle^{i\{2\epsilon_1[\phi_{\sigma+}(\vec{r}')-\phi_{\sigma+}(\vec{r}'')]+2\epsilon_2[\theta_{\sigma-}(\vec{r}')-\theta_{\sigma-}(\vec{r}'')]\}}\rangle_0],
\end{aligned} \tag{A3}$$

and

$$\begin{aligned}
(\mathrm{T}) = {} & -g_1 g_3 g_4 \left(\frac{1}{8(\pi\alpha)^2 v_F}\right)^3 \sum_{\epsilon_1,\epsilon_2,\epsilon_3=\pm 1} \int d^2r' d^2r'' d^2r''' \\
& \times [\langle e^{i\{\phi_{\sigma+}(\vec{r}_1)-\phi_{\sigma+}(\vec{r}_2)+2\epsilon_1[\phi_{\sigma+}(\vec{r}')-\phi_{\sigma+}(\vec{r}''')]+2\epsilon_2[\phi_{\sigma-}(\vec{r}')-\phi_{\sigma-}(\vec{r}'')-\delta_{\sigma-}(x'-x'')]+2\epsilon_3[\theta_{\rho-}(\vec{r}'')-\theta_{\rho-}(\vec{r}''')]\}}\rangle_0 \\
& -\langle e^{i[\phi_{\sigma+}(\vec{r}_1)-\phi_{\sigma+}(\vec{r}_2)]}\rangle_0 \langle^{i\{2\epsilon_1[\phi_{\sigma+}(\vec{r}')-\phi_{\sigma+}(\vec{r}''')]+2\epsilon_2[\phi_{\sigma-}(\vec{r}')-\phi_{\sigma-}(\vec{r}'')-\delta_{\sigma-}(x'-x'')]+2\epsilon_3[\theta_{\rho-}(\vec{r}'')-\theta_{\rho-}(\vec{r}''')]\}}\rangle_0] \\
& -g_2 g_5 g_6 \left(\frac{1}{8(\pi\alpha)^2 v_F}\right)^3 \sum_{\epsilon_1,\epsilon_2,\epsilon_3=\pm 1} \int d^2r' d^2r'' d^2r''' \\
& \times [\langle e^{i\{\phi_{\sigma+}(\vec{r}_1)-\phi_{\sigma+}(\vec{r}_2)+2\epsilon_1[\phi_{\sigma+}(\vec{r}')-\phi_{\sigma+}(\vec{r}'')]+2\epsilon_2[\phi_{\rho-}(\vec{r}')-\phi_{\rho-}(\vec{r}''')-\delta_{\rho-}(x'-x''')]+2\epsilon_3[\theta_{\rho-}(\vec{r}'')-\theta_{\rho-}(\vec{r}''')]\}}\rangle_0 \\
& -\langle e^{i[\phi_{\sigma+}(\vec{r}_1)-\phi_{\sigma+}(\vec{r}_2)]}\rangle_0 \langle^{i\{2\epsilon_1[\phi_{\sigma+}(\vec{r}')-\phi_{\sigma+}(\vec{r}'')]+2\epsilon_2[\phi_{\rho-}(\vec{r}')-\phi_{\rho-}(\vec{r}''')-\delta_{\rho-}(x'-x''')]+2\epsilon_3[\theta_{\rho-}(\vec{r}'')-\theta_{\rho-}(\vec{r}''')]\}}\rangle_0].
\end{aligned} \tag{A4}$$

In the above, $\langle \cdots \rangle_0$ denotes the average without the cosine terms, that is, one with the Tomonaga-Luttinger Hamiltonian. When we focus on (T), the dominant contributions come from $\vec{r}''' = \vec{r}'' + \vec{r}$ or $\vec{r}'' = \vec{r}' + \vec{r}$ for the term proportional to $g_1 g_3 g_4$, and from $\vec{r}''' = \vec{r}'' + \vec{r}$ or $\vec{r}''' = \vec{r}' + \vec{r}$ for one proportional to $g_2 g_5 g_6$ with a small $r$. Therefore, by expanding around $\vec{r} = 0$, after a straightforward calculation, we can obtain the following renormalization relations on the effective quantities:

$$\begin{aligned}
K_{\sigma+}^{\mathrm{eff}} = {} & K_{\sigma+} - \frac{K_{\sigma+}^2}{2} \int \frac{dr}{\alpha} \left[ y_1^2 \left(\frac{r}{\alpha}\right)^{3-2(K_{\sigma+}+K_{\sigma-})} J_0(2\delta_{\sigma-} r) + y_2^2 \left(\frac{r}{\alpha}\right)^{3-2(K_{\sigma+}+K_{\rho-})} J_0(2\delta_{\rho-} r) \right. \\
& \left. + y_4^2 \left(\frac{r}{\alpha}\right)^{3-2(K_{\sigma+}+1/K_{\rho-})} + y_6^2 \left(\frac{r}{\alpha}\right)^{3-2(K_{\sigma+}+1/K_{\sigma-})} \right],
\end{aligned} \tag{A5}$$

$$\left(y_1^{\mathrm{eff}}\right)^2 = y_1^2 - 2 y_1 y_3 y_4 \int \frac{r}{\alpha} \left(\frac{r}{\alpha}\right)^{1-2/K_{\rho-}}, \tag{A6}$$

$$\left(y_2^{\mathrm{eff}}\right)^2 = y_2^2 - 2 y_2 y_5 y_6 \int \frac{r}{\alpha} \left(\frac{r}{\alpha}\right)^{1-2/K_{\sigma-}}, \tag{A7}$$

$$\left(y_4^{\mathrm{eff}}\right)^2 = y_4^2 - 2y_1y_3y_4 \int \frac{r}{\alpha}\left(\frac{r}{\alpha}\right)^{1-2K_{\rho-}} J_0(2\delta_{\sigma-}r), \tag{A8}$$

$$\left(y_6^{\mathrm{eff}}\right)^2 = y_6^2 - 2y_2y_5y_6 \int \frac{r}{\alpha}\left(\frac{r}{\alpha}\right)^{1-2K_{\rho-}} J_0(2\delta_{\rho-}r). \tag{A9}$$

By changing the cutoff $\alpha \to e^l\alpha = \alpha + d\alpha$, we obtain Eqs. (10), (13), (14), (16), and (18). In a way similar to the above, the other RG equations can also be obtained.


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