Universal entanglement decay of photonic-orbital-angular-momentum qubit states in atmospheric turbulence

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We consider the propagation of two photonic qubits, initially maximally entangled in their orbital angular momenta (OAM), across a weakly turbulent atmosphere. By introducing the *phase correlation length* of an OAM beam, we show that the photonic entanglement exhibits a universal exponential decay.

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I. INTRODUCTION

The ability of photons carrying orbital angular momentum (OAM) to encode quantum states in a high-dimensional Hilbert space makes them potentially very useful for quantum information purposes [1–8], among which free space quantum communication is one of the most promising future applications. Proof-of-principle experiments have shown a significant increase of the classical channel capacity using OAM multiplexing [9]. However, before the realization of (quantum) OAM multiplexing in free space, reliable transport of OAM photons through atmospheric turbulence has to be accomplished [10].

The transmission of photons carrying OAM across turbulence is challenging because the intrinsic refractive index fluctuations associated with turbulence distort the photons' wave fronts [11] that encode quantum information, resulting in a deterioration thereof. The first successful quantum key distribution protocol with OAM photons was demonstrated [12] over a distance of only 210 m—much shorter than over 100-km-long free space links which were attained for quantum information transmission with polarized photons [13,14]. In recent years, experimental [15–19] and theoretical [20–25] efforts have been dedicated to clarifying and to partially mitigating [18,19,25] the impact of turbulence on single and entangled OAM photons. Despite some progress, there are still fundamental open issues concerning the behavior of OAM photons in turbulence, one of them being the description of the OAM photons' entanglement evolution.

A crucial difficulty for the quantitative description of the latter stems from the fact that optical inhomogeneities induced by turbulence induce coupling of the initially excited, finite number of OAM modes to all modes of the infinite-dimensional OAM space. Therefore, theoretical methods to treat the open system entanglement evolution of finite-dimensional quantum systems [26,27] are not directly applicable. Indeed, to deal with a necessarily finite-dimensional output state upon detection, we need to truncate the Hilbert space, which unavoidably leads to a loss of norm. Therefore, it is more appropriate to use the tools for entanglement characterization of decaying states [28].

In this contribution we report on some insights on the entanglement evolution in weak turbulence, whose impact on the propagating beam is reduced to phase aberrations. Thereby we neglect the turbulence-caused intensity scintillations as well as the beam's diffraction [29]. Note that diffraction does not influence the OAM of a beam, but leads to a change of the beam width along the propagation path. Since the single phase screen model used here assumes a constant beam width it can only be used if diffraction effects are small. In the optical domain and for $w_0 \simeq 0.1$ m diffraction can be ignored for distances L of about 1 km [29]. As it turns out, a model of weakly turbulent atmosphere is valid within the same range of distances L (see Sec. II). Specifically we consider the example of the simplest decaying OAM state—a maximally entangled OAM qubit, that is, a twin-photon state (in short, "biphoton") whose wave front represents a superposition of two spatial Laguerre-Gaussian (LG) modes. We introduce the *phase correlation length* $\xi(l)$—an inherent property of an LG beam with angular momentum l, reflecting its complex spatial structure—and show that the entanglement exhibits a universal exponential decay as a function of $\xi(l)/r_0$, vanishing at $\xi(l)/r_0 \approx 1$, where r_0 is the turbulence's correlation length defining the characteristic scale of the turbulent "granularity" over a given propagation distance L, also called the Fried parameter. If $\xi(l) \ll r_0$, the turbulent atmosphere appears as a homogeneous medium to the OAM biphoton, and its spatial entanglement remains high. As r_0 approaches $\xi(l)$, the phase errors become sufficiently large to destroy the wave front structure, and the entanglement vanishes.

We now proceed as follows. The next section introduces our model and defines the turbulence map which acts on the initial photonic OAM state. Using the properties of this transformation, in Sec. III we derive an analytical expression for the evolution of the concurrence of the initially maximally entangled OAM qubit state under weak turbulence. Section IV concludes the article.

II. MODEL

The setup we have in mind is illustrated in Fig. 1. The source produces pairs of photons that are maximally entangled in their OAM, there encoded by LG modes with the opposite azimuthal quantum number l [8]. We assume that the input LG modes have a waist w_0 (which coincides with the waist of the Gaussian TEM_{00} mode [30]), a radial quantum number $p_0 = 0$, and azimuthal quantum numbers l_0 and $-l_0$. The generated Bell state thus reads

$$|\Psi_0\rangle = \frac{1}{\sqrt{2}}(|l_0,-l_0\rangle + e^{i\gamma}|-l_0,l_0\rangle), \qquad (1)$$

where γ is a relative phase.

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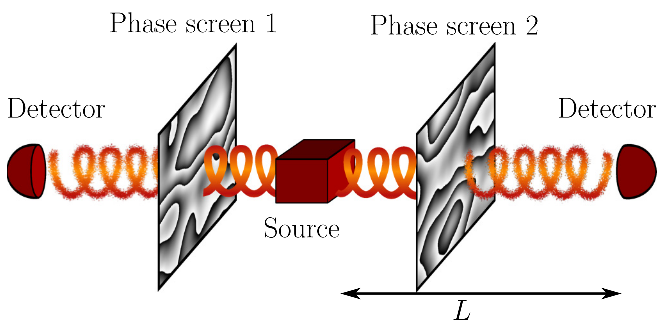

FIG. 1. (Color online) Sketch of the setup. A source produces pairs of OAM-entangled qubits whose wave fronts get deteriorated as they propagate along the (horizontal) z axis, through independent layers of a weakly turbulent atmosphere, modeled as random phase screens. The characteristic scale of the screens' phase patterns is defined by the Fried parameter r_0, which in turn depends on the distance L from the source to the detector [see Eq. (4)].

The entangled OAM photons are then sent along the (horizontal) z axis through independent weak turbulences modeled as phase screens [11,20,21]. We note that the validity of the phase screen model to describe entanglement transport through turbulence was experimentally tested in Ref. [15], where a turbulence cell was used as a realistic emulation of an about 2-km-thick atmospheric layer seen by one of the entangled photons. For free space links exceeding a few kilometers, a single phase screen model becomes invalid because it ignores important wave propagation and turbulence-induced effects, such as diffraction and intensity scintillations [31]. The assumption of independent turbulence experienced by both photons [20–22] is a corollary of the Markov approximation with respect to the refractive index spatial correlation function along the z direction, which can be rigorously justified for the optical wave propagation in the atmosphere [31].

Each phase screen introduces random phase errors into the beam's transverse profile which lead to the entanglement decay of the output quantum state. Our fundamental quantity of interest is the output density operator ρ of the biphoton state upon transmission through the weakly turbulent media of thickness $z = |L|$ (for each photon).

It is useful to recall the properties of the linear map Λ that represents the action of an ensemble-averaged phase screen on a single photon density operator [25]. Λ relates the input and the output density matrices of a single photon in the OAM basis, $\sigma^{(0)}$ and σ, respectively, through the equation

$$\sigma_{pl,p'l'} = \sum_{p_0l_0,p_0'l_0'} \Lambda_{pl,p'l'}^{p_0l_0,p_0'l_0'} \sigma_{p_0l_0,p_0'l_0'}^{(0)} \qquad (2)$$

and includes the propagation distance implicitly, through its dependence on the Fried parameter [see Eqs. (4) and (5) below]. The matrix elements $\Lambda_{pl,p'l'}^{p_0l_0,p_0'l_0'}$ have the following meaning: The ones with coinciding indices $p = p_0$, $l = l_0$, $p' = p_0'$, and $l' = l_0'$ describe the mapping of the initially populated OAM modes onto themselves (the "survival amplitude"); all other matrix elements control the crosstalk to distinct modes (at least one of the indices changes its value).

By generalizing the above description to biphotons, it is easy to show that the two-photon output state ρ is related to the input state $\rho^{(0)} = |\Psi_0\rangle\langle\Psi_0|$ by the formula

$$\rho = (\Lambda_1 \otimes \Lambda_2)\rho^{(0)}, \qquad (3)$$

where Λ_i $(i = 1,2)$ is a linear transformation representing the phase screens seen by either photon, respectively. In the following, we assume that the phase screens are characterized by the same statistical properties, which allows us to write $\Lambda_1 = \Lambda_2 = \Lambda$. This corresponds to the setup where two photons traverse horizontally equal distances across a uniform turbulence, which leads to the same transverse correlation length of turbulence (Fried parameter) [30]

$$r_0 = \left(0.423\, C_n^2 k^2 L\right)^{-3/5}, \qquad (4)$$

where C_n^2 is the index-of-refraction structure constant, L is the propagation distance, and k is the optical wave number.

It should be mentioned that our model can easily be adapted to an alternative communication protocol—the so-called one-sided noisy channel—wherein only one of the twin photons is sent through the atmosphere to a distant party, whereas another photon is detected locally. Then the output state is obtained from the input state via the map $(\mathbb{1} \otimes \Lambda)$, where $\mathbb{1}$ is the identity operator acting in the Hilbert space of the locally detected photon. In this scenario entanglement evolution of an arbitrary bipartite qubit state is determined by that of the maximally entangled state [32]. Recently, this law has been used to verify a numerical simulation method of the entanglement decay in strong turbulence [33].

We now proceed to describe the entanglement evolution under the influence of turbulence, keeping track of the azimuthal quantum number which encodes quantum information. The radial quantum number of the output state remains thereby unobserved and is traced over. The elements of the resulting transformation, $\Lambda_{l,l'}^{l_0,l_0'} \equiv \sum_p \Lambda_{pl,pl'}^{0l_0,0l_0'}$, read

$$\Lambda_{l,\pm l}^{l_0,l_0'} = \frac{\delta_{l_0-l_0',l\mp l}}{2\pi} \int_0^\infty dr\, R_{0l_0}(r) R_{0l_0'}^*(r) r \int_0^{2\pi} d\vartheta \times e^{-i\vartheta[l\pm l-(l_0+l_0')]/2} e^{-0.5 D_\phi(2r|\sin(\vartheta/2)|)}, \qquad (5)$$

where $R_{0l_0}(r)$ is the radial part of the input LG beam at $z = 0$, with radial and azimuthal quantum numbers 0 and l_0, respectively [34]. $D_\phi(r) = 6.88(r/r_0)^{5/3}$ is the phase structure function of the Kolmogorov model of turbulence [30].

III. ENTANGLEMENT EVOLUTION IN TURBULENCE

Due to the crosstalk as described by Eq. (2), the matrix elements of the output state spread over the entire—infinite-dimensional—OAM basis. To deal with a finite-dimensional Hilbert space, the transmitted state is post-selected [17,20] in the basis of the injected qubit state, $\{|-l_0,-l_0\rangle, |-l_0,l_0\rangle, |l_0,-l_0\rangle, |l_0,l_0\rangle\}$. Since such postselection entails the decay of the output state, it needs to be renormalized by its trace [28] before we can quantify the output entanglement—here in terms of concurrence [35].

To evaluate the output entanglement in the truncated Hilbert space, we make use of the inversion symmetry,

$$\Lambda_{l,l'}^{l_0,l_0'} = \Lambda_{-l,-l'}^{-l_0,-l_0'}, \qquad (6)$$

of the linear map (5), which stems from the isotropy of turbulence along horizontal paths [30,36]. Consequently, there are only two distinct nonzero matrix elements which fully determine the map Λ:

$$a = \Lambda^{l_0,l_0}_{l_0,l_0} = \Lambda^{-l_0,-l_0}_{-l_0,-l_0} = \Lambda^{-l_0,l_0}_{-l_0,l_0} = \Lambda^{l_0,-l_0}_{l_0,-l_0}, \quad (7)$$

$$b = \Lambda^{-l_0,-l_0}_{l_0,l_0} = \Lambda^{l_0,l_0}_{-l_0,-l_0}, \quad (8)$$

and which we recognize as the survival (a) and crosstalk (b) amplitudes, respectively [see our discussion following Eq. (2)]. By virtue of Eqs. (3)–(8), the biphoton output state can now be analytically parametrized in terms of a and b, and evaluated numerically, for arbitrary l_0.

Wootters' formula for the concurrence of a mixed bipartite qubit state [35] then immediately yields an analytical expression for the output entanglement:

$$C(\rho) = \max\left[0, \frac{(1-2\tilde{a})}{(1+\tilde{a})^2}\right], \quad (9)$$

where $\tilde{a} \equiv b/a$. It follows that $C(\rho) = 1$ for $\tilde{a} = 0$, and $C(\rho) = 0$ for $\tilde{a} \geqslant 1/2$. Moreover, from the definitions (7) and (8), $\tilde{a} = 0$ implies $b = 0$ and $a = 1$, which corresponds to $r_0 \to \infty$, that is, to the absence of turbulence. For finite values of the Fried parameter r_0, there emerges a nonzero crosstalk ($b > 0$), which is accompanied by a decrease of the survival amplitude ($a < 1$) and results in a monotonic growth of $\tilde{a}$ until $\tilde{a} = 1/2$ (i.e., a steady decrease of $C(\rho)$ to zero).

We now want to gain some insight into the physical mechanism governing the entanglement evolution of OAM biphotons in turbulence. Entangled qubits become more robust with increasing l_0 [20] because their spatial phase structure gets finer—as the OAM beam widens, its phase front oscillates more rapidly with increasing l_0 [see Fig. 2 (bottom row of the left panel)]. As a result, for fixed turbulence correlation length r_0, OAM-entangled qubits whose wave fronts have a shorter characteristic length than r_0 "see" turbulence as a homogeneous medium which does not affect their quantum entanglement. When we plot $C(\rho)$ against the ratio w_0/r_0, this property of the wave fronts of OAM beams does implicitly come into play as the increased longevity of the concurrence for larger l_0, since w_0 is l_0 independent [20–22]. However, this effect can be made strikingly obvious by a proper rescaling.

To this end, we introduce [38] the *phase correlation length* $\xi(l_0)$ which we define as the average distance between the points in the LG beam cross-section that have a phase difference of $\pi/2$. The idea comes from the basic fact that two monochromatic waves having a phase difference $\lesssim\pi/2$ are "in phase" and interfere constructively. From the azimuthal phase dependence of OAM beams, proportional to $e^{il_0\vartheta}$, it is easy to see that the angle between two such points is equal to $\alpha = \pi/2|l_0|$ [see Fig. 2(right panel)]. Now, choose a point at a distance r from the origin. Then the distance $\Delta s(r)$ from this point to the line along which the phase differs by $\pi/2$ from the phase of the departure point coincides with the leg of the right triangle shown in Fig. 2 (right panel) and is given by $\Delta s(r) = r\sin\alpha = r\sin(\pi/2|l_0|)$. Also note that such a triangle cannot be constructed for $|l_0| = 1$, which will manifest in a specific concurrence evolution for this case (see below).

Finally, $\xi(l_0)$ is defined as the average of Δs, or $\xi(l_0) = \langle r\rangle \sin(\pi/2|l_0|)$, with the intensity distribution of the initial beam as the weighting function:

$$\xi(l_0) = \int_0^\infty |R_{0l_0}(r)|^2 \Delta s(r) r dr. \quad (10)$$

The integral in Eq. (10) can be evaluated exactly [39], yielding our final expression for the phase correlation length:

$$\xi(l_0) = \sin\left(\frac{\pi}{2|l_0|}\right) \frac{w_0}{\sqrt{2}} \frac{\Gamma(|l_0|+3/2)}{\Gamma(|l_0|+1)}, \quad (11)$$

where $\Gamma(x)$ is the Gamma function. For $|l_0| > 2$, the function $\xi(l_0)$ is monotonically decreasing with $|l_0|$, which is consistent with the faster phase oscillations of LG beams for larger $|l_0|$.

In Fig. 3 we plot our numerical results for the concurrence $C(\rho)$ [Eq. (9)] as a function of the ratio $x \equiv \xi(l_0)/r_0$ and for different values of l_0: Apart from a finite-size effect for $|l_0| = 1$, the output state entanglement for all variable-l_0 initial states collapses onto one universal curve, $C(\rho) \approx \exp(-4.16x^{3.24})$, where the exponential fit is obtained from the x dependence of $C(\rho)$ for $l_0 \gtrsim 50$, when all curves become indistinguishable

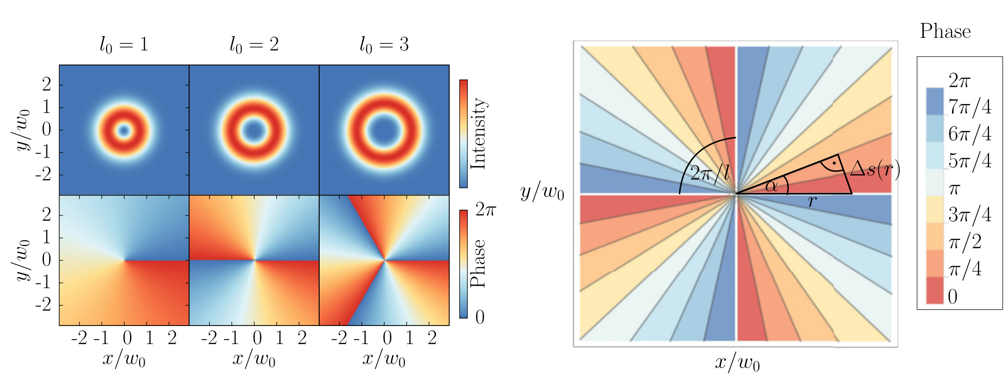

FIG. 2. (Color online) Illustration of the complex spatial structure of LG (or vortex) beams carrying nonvanishing OAM. Top left panel (from left to right): Inhomogeneous intensity profile of LG beams for $l_0 = 1,2,3$. Note the widening of the beam with increasing l_0. Bottom left panel (from left to right): Augmented phase oscillations with increasing l_0, for $l_0 = 1,2,3$, respectively. Right panel: Sketch for the calculation of the *phase correlation length* $\xi(l)$. The phase variation of a LG beam with $l_0 = 4$ is encoded in color with a step width of $\pi/4$. The length $\Delta s(r)$ is defined as the shortest distance from a point (here, chosen on the x axis), located at a distance r from the origin, to the line of points with a phase difference of $\pi/2$.

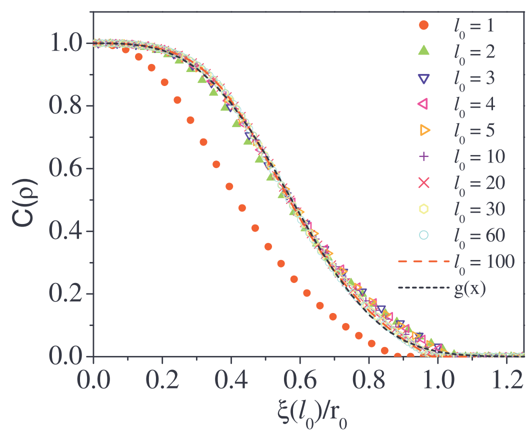

FIG. 3. (Color online) Output state concurrence (3) as a function of the ratio $\xi(l_0)/r_0$ for different l_0. Concurrences for $l_0 > 1$ essentially collapse onto a universal, exponential decay law which is best fitted by $g(x) = \exp(-4.16x^{3.24})$.

for different l_0 values. It can furthermore be shown that in the case of the one-sided channel the concurrence is given by the formula $\max[0,(1-\tilde{a})/(1+\tilde{a})]$ and also exhibits a universal decay (fitted by an exponential function which differs from the one in Fig. 3). The thus demonstrated universality of OAM entanglement decay in turbulence is the key result of our present contribution. It shows that the entanglement evolution of OAM qubit states in turbulence is governed by the sole parameter $\xi(l_0)/r_0$. It should be mentioned that a different rescaling was done in Ref. [11], where only the broadening, but not the phase oscillations, of the LG beam with increasing l_0 was taken into account. Therefore, such rescaling cannot unveil the here uncovered universality of the concurrence decay. Using Fig. 3 together with the definition (11), and recalling the properties of the Gamma function [40], we obtain an asymptotic ($l_0 \to \infty$) scaling law, $L \sim |l_0|^{5/6}$, for the dependence of the propagation distance L (over which the concurrence remains finite) on l_0. This provides a rigorous justification of an earlier, phenomenological result of Ref. [17].

IV. CONCLUSION

To conclude, we introduced the phase correlation length $\xi(l)$ of OAM beams, which fully determines the entanglement evolution of OAM qubit states in weak turbulence. Since $\xi(l)$ reflects the complex spatial structure of OAM beams and is independent of the turbulence model, it is suggestive to apply this quantity to characterize entanglement evolution in strong turbulence. A recent successful communication of OAM superpositions across 3 km of strong turbulence [41] will potentially render long-distance OAM entanglement distribution possible. Another direction of future work will be to see whether a generalization of the phase correlation length to high dimensions—as a weighted sum of partial phase correlation lengths of individual OAM components—can be useful for an improved understanding of the entanglement evolution of OAM qudit states in turbulence and/or for the identification of high-dimensional and *robust* entangled OAM states [24].

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