Decay of metastable excited states of two qubits in a waveguide

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For a system of two spatially separated qubits (two-level atoms) coupled to a one-dimensional waveguide we have described the time evolution of singly or doubly excited states of the atomic subsystem. When the interatomic distance l takes special ("resonant" or "antiresonant") values, the singly excited system of resonant atoms can form metastable (dark) states. If l slightly deviates from one of the special values or the atomic frequencies do not coincide, the dark states slowly decay and we have calculated the decay rate. Also, we have found that the *doubly* excited state of two resonant atoms located at the special positions does not completely decay but, with a *finite* probability, can evolve (with the emission of a single photon) to one of the metastable singly excited states. Metastable states of pairs of qubits may find applications (e.g., as memory elements) in information processing or as detectors sensitive to external perturbations.

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I. INTRODUCTION AND RESULTS

Qubits talking to each other through a quantum field are a central object of quantum optics and information processing [1–3]. Possible physical realizations of qubit systems include two-level atoms (TLAs) interacting through a resonant electromagnetic field and quantum dots coupled to plasmon modes of a semiconductor or to Josephson circuits [4–8]. A study of collective effects in TLA systems was initiated by Dicke [9], who described enhanced spontaneous emission (a superradiance effect) of some of the excited states of a multiatomic system located in a domain of a small size (compared with the resonant wavelength λ), as well as a reduced emission rate of other excited states (subradiance effect). Since that time superradiance effects have been intensively studied in various geometries (see, e.g., the review in Ref. [10]). Dark (subradiant) collective excited states of several TLAs (qubits) are also of considerable interest. For instance, they can be used for information storage. In the present paper we consider dark states in a system of TLAs and study their stability.

The simplest system in which dark states may exist consists of only two TLAs. There have been many investigations of the dynamics, entanglement, and emission of two TLAs coupled with an electromagnetic field in free space, in a cavity, and, in particular, in a one-dimensional (1D) waveguide (see the review in Refs. [11] and recent papers [12–14]). The latter 1D case is of special interest: on one hand, in contrast to the case of a higher dimension, there is no spatial spreading of the emitted radiation; on the other hand, the electromagnetic field in a thin 1D waveguide comprises an infinite number of modes for right- and left-moving photons of arbitrary frequencies. Therefore, one deals with a system where an excited atomic state decays due to the coupling with an infinite photon thermostat, but at the same time the field, emitted by one atom in the direction of another one, propagates without attenuation. Interference of different decay channels results in interesting effects. In particular, as shown in Ref. [15], some of the collective states

$$\left|\Psi_{\rm in}^{(1)}\right\rangle = C_1|e_1\rangle \otimes |g_2\rangle + C_2|g_1\rangle \otimes |e_2\rangle \tag{1}$$

of a *singly* excited system of two spatially separated *identical* TLAs (e and g denote the excited and the ground states of the corresponding atoms, respectively) in a 1D waveguide are metastable with respect to the single-photon emission if the distance l between the two atoms equals an integer number of $\lambda/2$. In the general case, the dynamics of state $|\Psi^{(1)}(t)\rangle$, (1), is characterized by the "fast" decay rate $\sim\Gamma$, where Γ is the decay rate of a single TLA in the considered 1D waveguide. However, as mentioned above, the situation changes when the distance l between the two resonant TLAs takes one of the special values $l_{\rm res} = 2\pi Nc/\omega_0$ with an integer N [the "antiresonant" case $l_{\rm res} = \pi(2N+1)c/\omega_0$ is treated similarly]; here $\omega_0 = ck_0$ is the TLA transition frequency, k_0 is the corresponding wave vector, and c is the speed of light in the waveguide. In this case only a specially chosen superposition ($C_1 = C_2$) of the two excited atomic states decays rapidly, while the orthogonal state (with $C_1 = -C_2$) demonstrates a more complicated behavior: a short stage of a fast partial decay transforms into a stationary regime where the excited atomic state does not decay any more. Such a metastable state has been interpreted [15] as the superposition of the excited atomic state in the absence of photons and of the ground atomic state in the presence of a photon bouncing between the TLAs; each atom works as a perfect mirror for the resonant 1D photon [16].

It is instructive to compare the considered physical situation with the case of two TLAs located very close to each other ($l \to 0$). In this elementary case the survival probability of the symmetric singly excited state ($C_1 = C_2$) decays at the rate 2Γ, giving the simplest example of the collective ("superradiant") emission [9]. On the contrary, the antisymmetric superposition ($C_1 = -C_2$) does not decay at all (due to the zero transition dipole moment between the excited and the ground state) and is "absolutely dark" in the sense that the probability of finding the atomic system in the initial state remains equal to unity (respectively, no photon is emitted).

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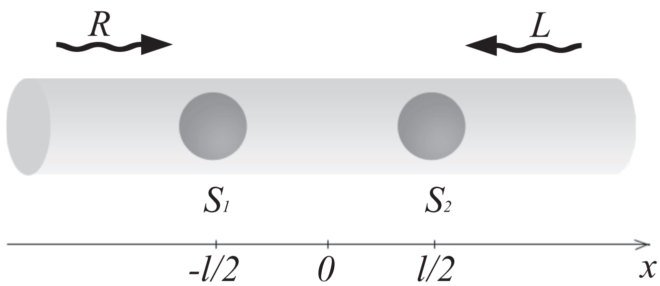

FIG. 1. Two qubits (two-level "atoms") coupled with a one-dimensional waveguide

On the contrary, the antisymmetric singly excited state of two spatially separated TLAs ($l \neq 0$) located at the resonant distance $l_{\rm res}$ is not absolutely dark: the probability of finding the atomic system in the initial state is less than unity. At small values of the parameter $\Gamma l/c$ this probability is of the order of unity, while at large values of $\Gamma l/c$, it is small $[\sim(c/\Gamma l)^2]$. But what is important is that this probability remains finite when the time goes to infinity, so the state can be referred to as a "mestastable."

However, these metastable ("dark") states are fragile with respect to deviations of the TLA's positions from the special ones [15]. Similarly, one can expect a decay of the metastable states if there is a detuning of the atomic transition frequencies (hence the "atomic mirrors" are not perfect anymore). The aim of this paper is to study the decay rates of the metastable (dark) states in a system of two TLAs coupled to a 1D waveguide (Fig. 1). We consider two related problems:

(i) the long-time dynamics of the spontaneous decay of a metastable excited state $|\Psi_{\rm in}^{(1)}\rangle$, (1), of two spatially separated TLAs in a 1D waveguide when the interatomic distance l deviates from the special values $l_{\rm res}$ (so that $k_0 l = k_0 l_{\rm res} + \delta$) and in the case of an inhomogeneous detuning (i.e., when there is a difference Ω between the two atomic transition frequencies) and

(ii) the long-time dynamics of the upper (doubly) excited state of two TLAs,

$$\left|\Psi_{\rm in}^{(2)}\right\rangle = |e_1\rangle \otimes |e_2\rangle. \tag{2}$$

For problem (i) we have obtained the following results.

(a) For small deviations ($\delta \ll 1$) of the interatomic distance from the "resonant" (or "antiresonant") value, we have obtained the expression for the decay rate of a metastable antisymmetric (or symmetric) state, $\Gamma\delta^2/\{2[1+\Gamma l/(2c)]^3\}$, which is valid for an arbitrary value of the parameter Γl [17].

(b) In the presence of a small frequency detuning Ω ($|\Omega| \ll \Gamma$), the decay rate acquires an additional contribution, $\Omega^2/\{2\Gamma[1+\Gamma l/(2c)]\}$. These expressions determine the requirements for the accuracy of locating and tuning TLA systems in order to have long-lived metastable states.

For problem (ii) we ask and answer the question whether the doubly excited state $|\Psi_{\rm in}^{(2)}\rangle$ of two identical TLAs can evolve to one of the metastable configurations. We have found that for an arbitrary distance l between the atoms the system will decay rapidly (with a decay rate on the scale of Γ) to the ground state with the emission of two photons. However, when l is close to one of the special positions $l_{\rm res}$, the system will rapidly decay (with the emission of a single photon) to the superposition of a bright and a dark (metastable) state of type $|\Psi_{\rm in}^{(1)}\rangle$ with certain coefficients C_1 and C_2. The bright component of this superposition will decay rapidly to the ground atomic state with the second photon emitted, while the dark state will remain forever (in the case of ideal resonance conditions) or decay slowly in accordance with the results for item (i). For the case $\Gamma l/c \ll 1$, we have calculated the probability of the formation of a metastable state. This probability is low $[\Gamma l/(2c)]^2$ and increases quadratically with an increase in l (as long as $l \lesssim c/\Gamma$). This study shows the possibility of creating metastable states with a strong external pulse.

As in the case of a singly excited state, it is instructive to compare the described behavior of a doubly excited state of spatially separated TLAs with that in the case of closely located TLAs ($l \to 0$). The doubly excited state, (2), is symmetric; it decays (with the emission of the first photon) to the symmetric singly excited state, and the latter decays to the ground state (with the second photon emitted). The antisymmetric singly excited state (which would be the dark state in the considered case $l = 0$) is not involved in the decay process.

For a nonzero interatomic distance ($l \neq 0$) the optical transition between the doubly excited and the singly excited antisymmetric states is not strictly forbidden. If the TLAs are located at a "resonant" distance $l_{\rm res}$, the antisymmetric singly excited state becomes a dark metastable one. This is the physical mechanism of the possibility of an incomplete decay of the doubly excited state of two spatially separated TLAs.

This paper has the following structure. In Sec. II, we describe the model and write the Hamiltonian of our system. In Sec. III we consider the evolution of the singly excited state $|\Psi_{\rm in}^{(1)}\rangle$ of two TLAs and obtain results for the metastable decay rate briefly described above [item (i)]. To study the more complicated case of the doubly excited state $|\Psi_{\rm in}^{(2)}\rangle$ we introduce the Weyl basis and write the basic equations (Sec. IV). These equations are solved in Sec. V, where we describe the time evolution of the state. Finally, the probability of the formation of a metastable state is calculated in Sec. VI. In the last section (Sec. VII) we summarize the obtained results and discuss possible applications and extensions.

II. DESCRIPTION OF TWO QUBITS IN A WAVEGUIDE MODEL

To study the dynamics of a system of TLAs coupled to 1D photons [18] we use the Hamiltonian formalism, which allows one to describe simultaneously the atomic and the photon systems, while the alternative density matrix approach excludes the photon degrees of freedom. The Hamiltonian approach has been successfully explored [19,20] to show the integrability of a model with an arbitrary number of TLAs coupled with a *chiral* waveguide, where photons propagate in only one direction [21]. In the usual nonchiral waveguides with right- and left-propagating photons, the dynamics is more complicated due to multiple reflections of photons, so that a general description of the system dynamics is considerably more difficult. Nevertheless, some particular problems can still be described explicitly. Among them is the problem of photon scattering by a system of few TLAs where the (inelastic) scattering matrix of several photons can be explicitly found [12,14,22–25]. However, the problem of the spontaneous decay of an initially excited system of TLAs is more difficult because it requires describing the system in time rather than in the spectral domain, and the transition from one to another is quite sophisticated. To describe the 1D propagation of an electromagnetic field in a waveguide we use the coordinate representation for "smooth" (envelope) field operators rather than the momentum representation. This widely implemented approach (see, for instance, [3,14,18,20,22,23,25]) is well suited for the nonperturbative treatment of multiple absorption-emission-reflection events in a system with spatially separated TLAs. The more traditional momentum representation is usually applied for configurations where there is no back-and-forth propagation of photons between the TLAs (e.g., for collocated TLAs [12], chiral waveguides [24], etc.).

The system we study is shown schematically in Fig. 1. It consists of a waveguide of infinite length where two kinds of photons (left and right) can propagate, coupled with two TLAs located at the points $x = \pm l/2$ along the waveguide axis (x axis). Assuming that l is greater than the resonant wavelength λ one may neglect the short-range dipole-dipole interaction. Restricting the consideration to the lowest transverse mode of the thin waveguide we consider only 1D propagation of the field and represent the field operator in the following form:

$$A(x) \to a_R(x)e^{ik_0x} + a_L(x)e^{-ik_0x}. \tag{3}$$

The envelope functions $a_R(x)$ and $a_L(x)$ are the operators of right- and left-moving photons (R- and L-photons), respectively. They obey the usual commutation relations:

$$[a_i(x),a_j^\dagger(x')] = \delta_{ij}\delta(x-x'), \quad i,j = (R,L). \tag{4}$$

TLAs at the points $x = -l/2$ and $x = l/2$ can be represented by the spin-$\frac{1}{2}$ operators $\vec{S}_1$ and $\vec{S}_2$, with the commutation relations

$$[S_a^+,S_b^-] = 2\delta_{ab}S_a^z, \quad [S_a^z,S_b^\pm] = \pm\delta_{ab}S_a^\pm, \tag{5}$$

where $S_a^\pm = S_a^x \pm iS_a^y$, $a,b = (1,2)$.

The Hamiltonian of the system has the form

$$\begin{aligned}\hat{H} = &-ic\int dx a_R^\dagger(x)\partial_x a_R(x) + ic\int dx a_L^\dagger(x)\partial_x a_L(x)\\ &+ \sqrt{\gamma c}\{S_1^+[a_R(-l/2)e^{-ik_0l/2} + a_L(-l/2)e^{ik_0l/2}]\\ &+ S_2^+[a_R(l/2)e^{ik_0l/2} + a_L(l/2)e^{-ik_0l/2}] + \text{H.c.}\}\\ &+ \frac{\Omega}{2}\left(S_1^z - S_2^z\right).\end{aligned} \tag{6}$$

Here the first line corresponds to free photons, the second and third lines describe the interaction of photons with the TLAs, and the last line introduces the detuning Ω between the transition frequencies of the two TLAs. The interaction constant γ is the amplitude decay rate connected with the decay rate $\Gamma = 2\gamma$ of the survival probability

$$P(t) = \exp(-\Gamma t) \tag{7}$$

of the excited state of a single TLA coupled with the waveguide modes.

The ground state $|0\rangle = |g\rangle \otimes |0\rangle_{\text{ph}}$ of the system is the product of the photon vacuum state (no photons) $|0\rangle_{\text{ph}}$ and of the ground state $|g\rangle \equiv |g_1\rangle \otimes |g_2\rangle$ of the two-atom system. The considered spin representation of TLAs is chosen so that the state $|g_i\rangle$ ($i = 1,2$) corresponds to the ith "spin" state $|\downarrow\rangle$, so the ground state $|g\rangle$ of the two TLAs system in "spin language" is represented as $|\downarrow,\downarrow\rangle$. We are interested in the time evolution of initial states with singly or doubly excited atomic systems, $|\Psi_{\text{in}}^{(1)}\rangle$, (1), or $|\Psi_{\text{in}}^{(2)}\rangle$, (2), respectively. In the next section we consider the initial state $|\Psi_{\text{in}}^{(1)}\rangle$, (1).

III. DECAY OF METASTABLE STATES IN A SYSTEM WITH ONE EXCITATION

We search the time-dependent state $|\Psi(t)\rangle$ which corresponds to the initial state $|\Psi_{\text{in}}^{(1)}\rangle$, (1), with the normalization condition $|C_1|^2 + |C_2|^2 = 1$. The state $|\Psi(t)\rangle$ can be represented in the form

$$\begin{aligned}|\Psi(t)\rangle = &A_1(t)S_1^+|0\rangle + A_2(t)S_2^+|0\rangle\\ &+ \int dx B_1(x,t)a_R^\dagger(x)|0\rangle + \int dx B_2(x,t)a_L^\dagger(x)|0\rangle.\end{aligned} \tag{8}$$

It is convenient to use the Laplace transform in Eq. (8),

$$|\Psi(t)\rangle \to |\tilde{\Psi}[s]\rangle = \int_0^\infty dt\, e^{-st}|\Psi(t)\rangle, \tag{9}$$

so that the time-dependent Schrödinger equation transforms as

$$\begin{aligned}i\frac{\partial}{\partial t}|\Psi(t)\rangle = \hat{H}|\Psi(t)\rangle \to s|\tilde{\Psi}[s]\rangle\\ - |\Psi(t=0)\rangle = -iH|\tilde{\Psi}[s]\rangle,\end{aligned} \tag{10}$$

where $|\Psi(t=0)\rangle$ is given by (1). Equation (10) is equivalent to the system of equations for the Laplace transforms of the amplitudes:

$$\begin{aligned}s\tilde{A}_1[s] - C_1 = &-i\sqrt{\gamma c}\tilde{B}_1[-l/2,s]e^{-ik_0l/2}\\ &- i\sqrt{\gamma c}\tilde{B}_2[-l/2,s]e^{ik_0l/2} - i\Omega/2\tilde{A}_1[s];\end{aligned} \tag{11}$$

$$\begin{aligned}s\tilde{A}_2[s] - C_2 = &-i\sqrt{\gamma c}\tilde{B}_1[l/2,s]e^{ik_0l/2}\\ &- i\sqrt{\gamma c}\tilde{B}_2[l/2,s]e^{-ik_0l/2} + i\Omega/2\tilde{A}_2[s];\end{aligned} \tag{12}$$

$$\begin{aligned}s\tilde{B}_1[x,s] = &-c\partial_x\tilde{B}_1[x,s] - i\sqrt{\gamma c}\tilde{A}_1[s]\delta(x+l/2)e^{ik_0l/2}\\ &- i\sqrt{\gamma c}\tilde{A}_2[s]\delta(x-l/2)e^{-ik_0l/2};\end{aligned} \tag{13}$$

$$\begin{aligned}s\tilde{B}_2[x,s] = &c\partial_x\tilde{B}_2[x,s] - i\sqrt{\gamma c}\tilde{A}_1[s]\delta(x+l/2)e^{-ik_0l/2}\\ &- i\sqrt{\gamma c}\tilde{A}_2[s]\delta(x-l/2)e^{ik_0l/2}.\end{aligned} \tag{14}$$

As there are no photons in the initial state, (1), no right-moving photons can appear in the domain $x < -l/2$ and no left-moving photons can appear in the domain $x > l/2$. This means that $B_1(x < -l/2,t) = 0 = B_2(x > l/2,t)$ for any t, and therefore, $\tilde{B}_1[x < -l/2,s] = 0 = \tilde{B}_2[x > l/2,s]$. Accounting for these conditions, the solutions of Eqs. (13) and (14) for $\tilde{B}_1[x,s]$ and $\tilde{B}_2[x,s]$, for all points except $\pm l/2$, are determined by linear homogeneous equations and have the form

$$\tilde{B}_1[x,s] = \begin{cases} 0 & \text{if} \quad x < -l/2, \\ D_{1l}e^{-sx/c} & \text{if} \quad -l/2 < x < l/2, \\ D_R e^{-sx/c} & \text{if} \quad x > l/2 \end{cases} \quad (15)$$

and

$$\tilde{B}_2[x,s] = \begin{cases} D_L e^{sx/c} & \text{if} \quad x < -l/2, \\ D_{2l}e^{sx/c} & \text{if} \quad -l/2 < x < l/2\,, \\ 0 & \text{if} \quad x > l/2, \end{cases} \quad (16)$$

where D_{1l}, D_{2l}, D_R, and D_L are some undefined coefficients. Because of the δ-function singularities in Eqs. (13) and (14), the functions $\tilde{B}_{1,2}$ are discontinuous at points $x = \pm l/2$. The jumps of $\tilde{B}_{1,2}$ at these points are obtained by integration of Eqs. (13) and (14) over an infinitesimal interval around the points $x = \pm l/2$,

$$\tilde{B}_1[\pm l/2+0,s] - \tilde{B}_1[\pm l/2-0,s] = -i\sqrt{\frac{\gamma}{c}}\tilde{A}_{2(1)}[s]e^{\mp ik_0l/2}, \quad (17)$$

$$\tilde{B}_2[\pm l/2+0,s] - \tilde{B}_2[\pm l/2-0,s] = i\sqrt{\frac{\gamma}{c}}\tilde{A}_{2(1)}[s]e^{\pm ik_0l/2}, \quad (18)$$

where the subscript in parentheses corresponds to the lower sign. The values of the functions $\tilde{B}_{1,2}$ at the singular points $x = \pm l/2$ are defined by the following smooth regularization:

$$\tilde{B}_{1,2}[\pm l/2,s] = \frac{\tilde{B}_{1,2}[\pm l/2+0,s] + \tilde{B}_{1,2}[\pm l/2-0,s]}{2}. \quad (19)$$

Applying the matching conditions, (17) and (18), to expressions (15) (starting from the domain $x < -l/2$) and (16) (starting from the domain $x > l/2$), we express all the coefficients in Eqs. (15) and (16) in terms of the functions $\tilde{A}_{1,2}[s]$. Then using the regularization, (19), we find $\tilde{B}_{1,2}[x,s]$ at the points $x = \pm l/2$:

$$\begin{aligned} \tilde{B}_1[-l/2,s] &= -\frac{i}{2}\sqrt{\frac{\gamma}{c}}\tilde{A}_1[s]e^{ik_0l/2}, \\ \tilde{B}_1[l/2,s] &= -\frac{i}{2}\sqrt{\frac{\gamma}{c}}(2\tilde{A}_1[s]e^{ik_0l/2-sl/c} + \tilde{A}_2[s]e^{-ik_0l/2}), \\ \tilde{B}_2[-l/2,s] &= -\frac{i}{2}\sqrt{\frac{\gamma}{c}}(\tilde{A}_1[s]e^{-ik_0l/2} + 2\tilde{A}_2[s]e^{ik_0l/2-sl/c}), \\ \tilde{B}_2[l/2,s] &= -\frac{i}{2}\sqrt{\frac{\gamma}{c}}\tilde{A}_2[s]e^{ik_0l/2}. \end{aligned} \quad (20)$$

Substituting these expressions into the first two equations in system (11), we get a closed system for functions $\tilde{A}_{1,2}$. Its solution is

$$\tilde{A}_1[s] = \frac{C_1(s+\gamma-i\Omega/2) - C_2\gamma e^{ik_0l-sl/c}}{(s+\gamma)^2 + \Omega^2/4 - \gamma^2 e^{2ik_0l-2sl/c}}, \quad (21)$$

$$\tilde{A}_2[s] = \frac{C_2(s+\gamma+i\Omega/2) - C_1\gamma e^{ik_0l-sl/c}}{(s+\gamma)^2 + \Omega^2/4 - \gamma^2 e^{2ik_0l-2sl/c}}. \quad (22)$$

The time evolution of the amplitudes $A_{1(2)}(t)$ is given by the inverse Laplace transform

$$A_{1(2)}(t) = \int_{-i\infty+0}^{+i\infty+0} \frac{ds}{2\pi i}\tilde{A}_{1(2)}[s]e^{st}. \quad (23)$$

The integration cannot be performed analytically and we restrict the analysis to a long-time regime. When the TLAs are located at arbitrary positions, the typical long-time regime corresponds to the rapid decay (at the rate of the order of γ) of both amplitudes $A_{1(2)}(t)$. However, if the atomic transition frequencies coincide (i.e., $\Omega = 0$) and the distance l between the two TLAs takes one of the special values such that $e^{ik_0l} = 1$ (we refer to this case as the "resonant" one) or $e^{ik_0l} = -1$ (referred to as the "antiresonant" case), the amplitudes $A_{1(2)}(t)$ may have parts which remain finite in the limit $t \to \infty$ [15]. Indeed, considering, for instance, the resonant case, one sees that the denominator $\mathcal{D}[s]$ of expressions (21) and (22) has the root $s = 0$:

$$\begin{aligned} &\mathcal{D}[s;\Omega=0,l=l_{\text{res}}] \\ &\quad = [s+\gamma(1-e^{-sl/c})][s+\gamma(1+e^{-sl/c})] \\ &\quad = 2\gamma s(1+\gamma l/c)[1+O(sl/c,s/\gamma)]. \end{aligned} \quad (24)$$

The contribution of this root to the contour integral, (23), determines the nonvanishing (at $t \to \infty$) parts of the amplitudes:

$$A_1(t\to\infty) = \frac{C_1 - C_2}{2(1+\gamma l/c)} = -A_2(t\to\infty). \quad (25)$$

This result shows that only the symmetric initial atomic state, (1), with $C_1 = C_2 = 1/\sqrt{2}$, can decay completely, while its counterpart, the antisymmetric atomic state with $C_1 = -C_2 = 1/\sqrt{2}$, decays only partially, so that there remains a finite probability of finding the atomic system in the excited state at long times. Similarly, when the distance l takes an antiresonant value (so that $e^{ik_0l} = -1$), only the antisymmetric initial atomic state, (1), with $C_1 = -C_2 = 1/\sqrt{2}$, can decay completely, while the symmetric atomic state with $C_1 = C_2 = 1/\sqrt{2}$ remains finite at $t \to \infty$.

These stable states become metastable in the presence of a transition frequency detuning ($\Omega \neq 0$) or of a deviation of the interatomic distance from a resonant (antiresonant) value. Our next task is to calculate the decay rate of the metastable states in such situations.

We start our consideration with the case where there is no frequency detuning ($\Omega = 0$) but the interatomic distance l slightly deviates from a resonant position, $k_0l = k_0l_{\text{res}} + \delta$, where we have introduced the phase parameter δ ($\delta < \pi$), so that $e^{ik_0l} = e^{i\delta}$. In what follows we assume a small deviation, so that $\delta \ll 1$.

Now the denominator in Eqs. (21) and (22) does not vanish at $s = 0$, which means that no part $A_{1(2)}(t)$ remains finite at $t \to \infty$, i.e., the antisymmetric ($C_1 = -C_2 = 1/\sqrt{2}$) superposition, (1), is no more stable. For the antisymmetric state, the expressions for the amplitudes, (21) and (22), can be rewritten in the form

$$\tilde{A}_1[s] = -\tilde{A}_2[s] = \frac{1}{\sqrt{2}(s+\gamma-\gamma e^{i\delta-sl/c})}. \quad (26)$$

The long-time behavior of the amplitudes is determined by the domain of small real part s' in the integral, (23), over the complex variable $s = s' + is''$. We assume (and find the necessary conditions later) that the function, (26), has a complex pole $s_p = s'_p + is''_p$ such that $s'_p l/c \ll 1$. Keeping only the first-order terms in $s'_p l/c$ in the denominator of function (26) we obtain the system of equations

$$s'_p\left[1+\frac{\gamma l}{c}\cos\left(\delta-\frac{s''_p l}{c}\right)\right] = -\gamma\left[1-\cos\left(\delta-\frac{s''_p l}{c}\right)\right], \tag{27}$$

$$s''_p - \gamma\left(1-\frac{s'_p l}{c}\right)\sin\left(\delta-\frac{s''_p l}{c}\right) = 0. \tag{28}$$

Interested in the case of slow decay (compared to spontaneous decay) of the metastable state, we assume $s'_p \ll \gamma$. To provide this inequality the cosine term on the right-hand side of (27) should be close to 1, hence $|\sin(\delta - s''_p l/c)| \ll 1$. This means that the argument of the sinus function is either small or very close to πn, $n \neq 0$. The second variant would mean that $s'' l/c \approx \pi n$; according to (28) this can be realized only at very large values of the parameter $\gamma l/c \gtrsim 1/\delta$. Restricting ourselves to small or moderately large ($1 \lesssim \gamma l/c \ll 1/\delta$) values of this parameter, we deal only with the first variant, when the argument of the sinus function is small. Replacing the sinus term in Eq. (28) with its argument and neglecting the term $s'_p l/c$ compared to unity, we find $s''_p = \gamma\delta/(1+\gamma l/c)$ and, as a consequence, $s'_p = -\gamma\delta^2/[2(1+\gamma l/c)^3]$. In the vicinity of s_p the denominator in Eq. (26) takes the form $\sqrt{2}(1+\gamma l/c)(s - s_p)$, so the pole contribution to the integral, (23), takes the form

$$\begin{aligned} A_1(t\to\infty) &= -A_2(t\to\infty) \\ &= \frac{1}{\sqrt{2}(1+\gamma l/c)} \\ &\quad\times \exp\left(\left[i\frac{\gamma\delta}{1+\gamma l/c} - \frac{\gamma\delta^2}{2(1+\gamma l/c)^3}\right]t\right). \end{aligned} \tag{29}$$

Thus, the probability of finding the system in the metastable (antisymmetric for the case of an almost-"resonant" distance l) state behaves as $e^{-\Gamma_\delta t}$, where the decay rate Γ_δ is given by

$$\Gamma_\delta = \frac{\gamma\delta^2}{(1+\gamma l/c)^3}. \tag{30}$$

In the case $\gamma l/c \ll 1$ this expression reduces to $\gamma\delta^2$ obtained in Ref. [15]. The assumptions used in the derivation of (30)—$|\delta - s''_p l/c| \ll 1$, $s'_p l/c \ll 1$, and $s'_p \ll \gamma$—mean that $\delta/(1+\gamma l/c) \ll 1$, $(\gamma l\delta^2/c)/(1+\gamma l/c)^3 \ll 1$, and $\delta^2/(1+\gamma l/c)^3 \ll 1$. The strongest of these inequalities is the first one, $\delta/(1+\gamma l/c) \ll 1$, which is fulfilled by the above assumption, $\delta \ll 1$. Note that for large interatomic distances l (when $\gamma l/c \gg 1$) the amplitude of a slowly decaying part, (29), diminishes with the increase in $\gamma l/c$. Finally, we indicate the obvious connection between δ and the corresponding deviation $\Delta l = l - l_{\text{res}}$ of the interatomic distance l from the resonant value l_{res}: $\Delta l = \delta/k_0 = 2\pi\delta\lambda$.

Now we move to the second situation: when there is no deviation of the interatomic distance from the resonant one (i.e., $e^{ik_0 l} = 1$) but there is a small but finite frequency detuning Ω. We assume that $\Omega \ll \gamma$. Then expressions (21) and (22) for the metastable antisymmetric state take the form

$$\tilde{A}_1[s] = -\tilde{A}_2[s] = \frac{1}{\sqrt{2}}\frac{\gamma[1+e^{-sl/c}]}{(s+\gamma)^2+\Omega^2/4-\gamma^2 e^{-2sl/c}}. \tag{31}$$

Searching for a small root of the denominator, such that $|sl/c| \ll 1$, we find straightforwardly $s = -\Omega^2/[8\gamma(1+\gamma l/c)]$, with the corresponding restriction on the interatomic distance $\Omega^2 l/[c\gamma(1+\gamma l/c)] \ll 1$. This results in the exponential decay of the antisymmetric state amplitude:

$$A_1(t) = -A_2(t) = \frac{1}{\sqrt{2}}\frac{1}{(1+\gamma l/c)}\exp\left[-\frac{\Omega^2 t}{8\gamma(1+\gamma l)}\right]. \tag{32}$$

Hence the probability of finding the system in the metastable antisymmetric (for resonant distance l) state behaves as $e^{-\Gamma_\Omega t}$, where the decay rate Γ_Ω is given by

$$\Gamma_\Omega = \frac{\Omega^2}{4\gamma(1+\gamma l/c)}. \tag{33}$$

The obtained results, (30) and (33), for the decay rates of the *antisymmetric* ($C_1 = -C_2$) metastable state of a singly excited state, (1), of the two-TLA system separated by the almost-resonant distance l ($e^{ik_0 l} = 1$), remain valid for the decay of the *symmetric* metastable state ($C_1 = C_2$) of TLAs separated by an antiresonant distance ($e^{ik_0 l} = -1$).

Expressed in terms of the single TLA probability decay rate $\Gamma = 2\gamma$ [see (7)] expressions (30) and (33) coincide with the results for a singly excited state of two TLAs reported in Sec. I. In the next sections we consider the decay of the doubly excited state, (2).

IV. DOUBLY EXCITED SYSTEM OF TWO-LEVEL ATOMS: WEYL BASIS

Here we study the decay of the doubly excited state, (2). The question we address is whether this decay can result in the formation of a stable singly excited state of a system of TLAs. In the limiting case where two identical TLAs are located "at the same point" (i.e., at a distance small compared to the resonance wavelength λ) and the short-ranged dipole interaction is not accounted for, the answer is negative: the interaction of TLAs with light [$\sim S^+A(0) + \text{H.c.}$] is governed by the total "spin" $S^+ = S_1^+ + S_2^+$, thus the decay of the completely (doubly in this case) excited TLA system goes down along the ladder of the collective Dicke states generated by subsequent actions of the operator S^- on the excited state [9]. In the considered case of two TLAs this corresponds to rapid decays to the lower—symmetric—singly excited state, which, in turn, decays rapidly to the ground state [26]. The answer is not obvious when the distance l between TLAs coupled to a 1D waveguide is much greater than λ.

Looking for possible stable states we assume the absence of frequency detuning ($\Omega = 0$). There are various decay channels for a doubly excited system of two TLAs. To simplify calculations we use the symmetry of the model and introduce new photon operators (referred to as the Weyl basis [27]):

$$b_S(x) = \frac{1}{\sqrt{2}}[a_R(x) + a_L(-x)], \tag{34}$$

$$b_A(x) = \frac{1}{\sqrt{2}}[a_R(x) - a_L(-x)], \tag{35}$$

which obey the usual boson commutation relations:

$$[b_S(x),b_S^\dagger(x')] = \delta(x-x') = [b_A(x),b_A^\dagger(x')], \tag{36}$$

$$[b_S(x),b_A^\dagger(x)] = 0. \tag{37}$$

The system Hamiltonian, (6), in this basis is given by

$$\begin{aligned}\hat{H}_W = &-ic\int dx b_S^\dagger(x)\partial_x b_S(x) - ic\int dx b_A^\dagger(x)\partial_x b_A(x)\\ &+\sqrt{\gamma c}\{S_S^+[b_S(l/2)e^{ik_0l/2} + b_S(-l/2)e^{-ik_0l/2}]\\ &+S_A^+[b_A(l/2)e^{ik_0l/2} - b_A(l/2)e^{-ik_0l/2}] + \text{H.c.}\}.\end{aligned} \tag{38}$$

As shown by the first line, both b_S and b_A photons are right-moving, which considerably simplifies the problem. In Eq. (38) we have introduced new operators:

$$\vec{S}_S = \frac{1}{\sqrt{2}}(\vec{S}_1 + \vec{S}_2), \quad \vec{S}_A = \frac{1}{\sqrt{2}}(\vec{S}_2 - \vec{S}_1). \tag{39}$$

The action of the operators S_S^- and S_A^- on the doubly excited state $|e\rangle = |e_1\rangle \otimes |e_2\rangle$, (2), of two TLAs creates singly excited states, (1), symmetric $|S\rangle$ (with $C_1 = C_2 = 1/\sqrt{2}$) and antisymmetric $|A\rangle$ (with $C_1 = -C_2 = 1/\sqrt{2}$), respectively: $|S\rangle = S_S^-|e\rangle$ and $|A\rangle = S_A^-|e\rangle$. These states are generated also by the action of S_S^+ and S_A^+ on the ground state $|g\rangle = |g_1\rangle \otimes |g_2\rangle$ of the two-TLA system: $|S\rangle = S_S^+|g\rangle$ and $|A\rangle = -S_A^+|g\rangle$. Further action of S_S^+ and S_A^+ on these states is given by

$$S_S^+|S\rangle = S_A^+|A\rangle = |e\rangle, \quad S_S^+|A\rangle = S_A^+|S\rangle = 0\,. \tag{40}$$

The omitted frequency detuning term $\Omega(S_1^z - S_2^z)$ would cause the mutual transformation of the states $|S\rangle \leftrightarrow |A\rangle$. In the absence of detuning (i.e., $\Omega = 0$), the doubly excited state, (2), decays along two channels: through the singly excited symmetric state $|S\rangle$ with the emission of a b_S photon or through the singly excited antisymmetric state $|A\rangle$ with the emission of a b_A photon; in general, these intermediate states decay further with the emission of either a b_S or a b_A photon, respectively. However, it may be that one of the intermediate states corresponds to one of the metastable states studied in the previous section. Our task is to find the probability of the formation of such states.

V. TIME EVOLUTION OF THE DOUBLY EXCITED SYSTEM

A. System of equations for amplitudes

We search the time-dependent state of the doubly excited system in the form

$$\begin{aligned}|\Psi(t)\rangle = &A(t)|e\rangle \otimes |0\rangle_{\text{ph}} + |S\rangle \otimes \int dx B_S(x,t)b_S^\dagger(x)|0\rangle_{\text{ph}}\\ &+|A\rangle \otimes \int dx B_A(x,t)b_A^\dagger(x)|0\rangle_{\text{ph}}\\ &+|g\rangle \otimes \int dx_1dx_2 C_S(x_1,x_2,t)b_S^\dagger(x_1)b_S^\dagger(x_2)|0\rangle_{\text{ph}}\\ &+|g\rangle \otimes \int dx_1dx_2 C_A(x_1,x_2,t)b_A^\dagger(x_1)b_A^\dagger(x_2)|0\rangle_{\text{ph}},\end{aligned} \tag{41}$$

where $|0\rangle_{\text{ph}}$ denotes the vacuum state of the photon subsystem. The initial state, (2), corresponds to the initial condition $A(t=0) = 1$, while all the other amplitudes at $t = 0$ vanish.

In the general case, i.e., for an arbitrary interatomic distance l, the amplitudes $A(t)$ and $B_{S(A)}(t)$ tend to 0 at $t \to \infty$; i.e., the system evolves to a state with atoms in their ground states and two emitted photons. However, if the atoms are in one of the special, "resonant" or "antiresonant" positions (so that $e^{ik_0l} = 1$ or -1), one may expect that one of the intermediate states—with only one emitted photon [i.e., described by the amplitude $B_A(x,t)$ or $B_S(x,t)$]—does not vanish in the long-time limit. To be more precise, one may expect that the probability of finding the system in one of these "intermediate" states,

$$P_{S(A)}(t) = \int_{-\infty}^{\infty} |B_{S(A)}(x,t)|^2 dx, \tag{42}$$

remains finite in the limit $t \to \infty$. Our task is to calculate these quantities. In what follows we describe the method of finding the state, (41), for an arbitrary case (having in mind possible applications beyond the problem studied in the present paper) and later we concentrate on the particular case of interest.

After the Laplace transformation, (9), the Schrödinger equation for (41) takes the form

$$s\tilde{A}[s] - 1 = -i\sqrt{\gamma c}\{\tilde{B}_S[l/2,s]e^{ik_0l/2} + \tilde{B}_S[-l/2,s]e^{-ik_0l/2} + \tilde{B}_A[l/2,s]e^{ik_0l/2} + \tilde{B}_A[-l/2,s]e^{-ik_0l/2}\}, \tag{43}$$

$$\begin{aligned}s\tilde{B}_S[x,s] = &-c\partial_x\tilde{B}_S[x,s] - i\sqrt{\gamma c}\tilde{A}[s][\delta(x+l/2)e^{ik_0l/2} + \delta(x-l/2)e^{-ik_0l/2}]\\ &-2i\sqrt{\gamma c}\{\tilde{C}_S[l/2,x,s]e^{ik_0l/2} + \tilde{C}_S[-l/2,x,s]e^{-ik_0l/2}\},\end{aligned} \tag{44}$$

$$\begin{aligned}s\tilde{B}_A[x,s] = &-c\partial_x\tilde{B}_A[x,s] - i\sqrt{\gamma c}\tilde{A}[s][\delta(x-l/2)e^{-ik_0l/2} - \delta(x+l/2)e^{ik_0l/2}]\\ &+2i\sqrt{\gamma c}\{\tilde{C}_A[l/2,x,s]e^{ik_0l/2} - \tilde{C}_A[-l/2,x,s]e^{-ik_0l/2}\},\end{aligned} \tag{45}$$

$$\begin{aligned}s\tilde{C}_S[x_1,x_2,s] = &-c\partial_{x_1}\tilde{C}_S[x_1,x_2,s] - c\partial_{x_2}\tilde{C}_S[x_1,x_2,s] - i\frac{\sqrt{\gamma c}}{2}\tilde{B}_S[x_2,s][\delta(x_1-l/2)e^{-ik_0l/2} + \delta(x_1+l/2)e^{ik_0l/2}]\\ &-i\frac{\sqrt{\gamma c}}{2}\tilde{B}_S[x_1,s][\delta(x_2-l/2)e^{-ik_0l/2} + \delta(x_2+l/2)e^{ik_0l/2}],\end{aligned} \tag{46}$$

$$\begin{aligned}s\tilde{C}_A[x_1,x_2,s] = &-c\partial_{x_1}\tilde{C}_A[x_1,x_2,s] - c\partial_{x_2}\tilde{C}_A[x_1,x_2,s] + i\frac{\sqrt{\gamma c}}{2}\tilde{B}_A[x_2,s][\delta(x_1-l/2)e^{-ik_0l/2} - \delta(x_1+l/2)e^{ik_0l/2}]\\ &+i\frac{\sqrt{\gamma c}}{2}\tilde{B}_A[x_1,s][\delta(x_2-l/2)e^{-ik_0l/2} - \delta(x_2+l/2)e^{ik_0l/2}].\end{aligned} \tag{47}$$

As mentioned after Eq. (38), b_S and b_A photons are right-moving. As there were no photons in the initial state, (2), no photons can appear in the domain $x < -l/2$ at any time. This means that

$$\tilde{B}_{A(S)}[x,s] = 0 \quad \text{if} \quad x < -l/2; \tag{48}$$

$$\tilde{C}_{A(S)}[x_1,x_2,s] = 0 \quad \text{if} \quad x_1 < -l/2 \quad \text{or} \quad x_2 < -l/2. \tag{49}$$

These relations provide the boundary condition for the system of equations (43)–(47). We solve this system following the "down-up" direction; i.e., first, using (46) and (47), we express the amplitudes $\tilde{C}_{S(A)}[x_1,x_2,s]$ in terms of the amplitudes $\tilde{B}_{S(A)}$, which, in turn, is expressed via $\tilde{A}$ with the use of (44) and (45). Next the amplitude $\tilde{A}$ is obtained from (43). This program is realized in the following subsections for the general case of an arbitrary interatomic distance l. Later we consider a particular (resonant) case and derive explicit expressions for the probability (42).

B. Amplitudes $\tilde{C}_{S(A)}[x_1,x_2,s]$

We start to solve this system by finding the amplitudes $\tilde{C}_S[x_1,x_2,s]$ and $\tilde{C}_A[x_1,x_2,s]$. These functions have discontinuities at the points $\pm l/2$. For instance, integrating (46) over an infinitesimal interval around the singular point $x_2 = -l/2$, we find

$$\tilde{C}_S[x_1,-l/2+0,s] - \tilde{C}_S[x_1,-l/2-0,s] = -\frac{i}{2}\sqrt{\frac{\gamma}{c}}\tilde{B}_S[x_1,s]e^{ik_0l/2}. \tag{50}$$

Due to the symmetry $\tilde{C}_{S(A)}[x_1,x_2,s] = \tilde{C}_{S(A)}[x_2,x_1,s]$ it is sufficient to consider only the sector $x_1 > x_2$. This sector is split into several subsectors (see Fig. 2), where the amplitudes are continuous and obey the simple equation

$$s\tilde{C}_{S(A)}[x_1,x_2,s] = -c[\partial_{x_1} + \partial_{x_2}]\tilde{C}_{S(A)}[x_1,x_2,s] = -c\partial_\xi \tilde{C}_{S(A)}[x_1 = \xi+\eta/2, x_2 = \xi-\eta/2,s], \tag{51}$$

where the new coordinates $\xi = (x_1+x_2)/2$ and $\eta = x_1 - x_2$ have been introduced.

Its general solution is

$$\tilde{C}_{S(A)}[x_1,x_2,s] = e^{-s(x_1+x_2)/(2c)} f_{S(A)}(x_1 - x_2), \tag{52}$$

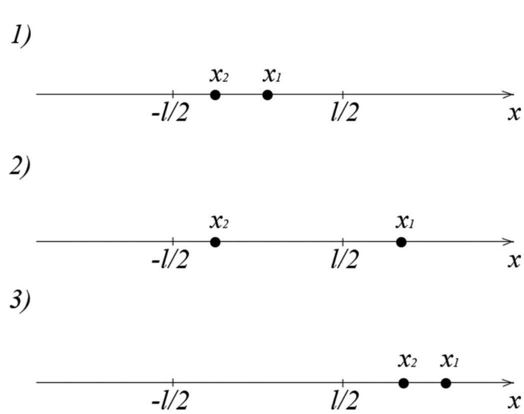

FIG. 2. Possible arrangements of photon coordinates.

where the functions $f_{S(A)}(x_1 - x_2)$ depend on the considered subsector. Actually, for our purposes we need to find the solution only in the first two subsectors in Fig. 2. The functions $f_{S(A)}(x_1 - x_2)$ are easily determined step by step by using the matching conditions at the boundaries of the subsectors. For instance, consider the matching condition, (50), at $x_2 = -l/2$, while $-l/2 < x_1 < l/2$. Having in mind that $C_S[x_1,x_2,s] = 0$ at $x_2 < -l/2$, we immediately obtain from (50) the expression for $C_S[x_1,-l/2+0,s]$ in terms of the function $\tilde{B}_S[x_1,s]$. Now we use the fact that when the variable x_1 varies within the interval $-l/2 < x_1 < l/2$ (and $x_2 = -l/2+0$ lies in the same interval), Eq. (46) is free of singularities and reduces to (51) with the general solution (52). Using (52) at $x_2 = -l/2+0$ we rewrite (50) in the form

$$f_S(x_1 + l/2)e^{-s(x_1/2-l/4)} = -\frac{i}{2}\sqrt{\frac{\gamma}{c}}\tilde{B}_S[x_1,s]e^{ik_0l/2}. \tag{53}$$

Replacing here $x_1 \to x_1 - x_2 - l/2$, we find the sought function $f_S(x_1 - x_2)$ in the subsector $-l/2 < x_2 < x_1 < l/2$,

$$f_S(x_1 - x_2) = -\frac{i}{2}\sqrt{\frac{\gamma}{c}}\tilde{B}_S[x_1 - x_2 - l/2,s]e^{s(x_1-x_2-l)/(2c)}e^{ik_0l/2}. \tag{54}$$

Therefore, the function $C_S[x_1,x_2,s]$, (52), in the subsector $-l/2 < x_2 < x_1 < l/2$ is given by

$$C_S[x_1,x_2,s] = -\frac{i}{2}\sqrt{\frac{\gamma}{c}}\tilde{B}_S[x_1 - x_2 - l/2,s]e^{-s(x_2+l/2)/c}e^{ik_0l/2}. \tag{55}$$

Continuing this procedure, we find the function $C_S[x_1,x_2,s]$ in the sector $-l/2 < x_2 < l/2 < x_1$. To this end we use the matching condition at the point $x_1 = l/2$,

$$\tilde{C}_S[l/2+0,x_2,s] = \tilde{C}_S[l/2-0,x_2,s] - \frac{i}{2}\sqrt{\frac{\gamma}{c}}\tilde{B}_S[x_2,s]e^{-ik_0l/2}, \tag{56}$$

which follows from integrating (46) over an infinitesimal interval around the point $x_1 = l/2$. For the term $\tilde{C}_S[l/2-0,x_2,s]$ on the right-hand side of (56) we use the earlier obtained expression, (55), at $x_1 = l/2 - 0$. The arguments of the function $\tilde{C}_S[l/2+0,x_2,s]$ belong to the interval $-l/2 < x_2 < l/2 < x_1$, where Eq. (46) is free of singularities, so that its general solution is (52) with an unknown function f_S. Putting (52) (at $x_1 = l/2+0$) into (56) we get

$$e^{-(l/2+x_2)s/(2c)} f_S(l/2 - x_2) = -\frac{i}{2}\sqrt{\frac{\gamma}{c}}\{\tilde{B}_S[-x_2,s]e^{ik_0l/2-s(l/2+x_2)/c} + \tilde{B}_S[x_2,s]e^{-ik_0l/2}\}. \tag{57}$$

Shifting here the variable $x_2 \to x_2 - x_1 + l/2$ we find the function $f_S(l/2 - x_2)$ and then, according to (52), the function $\tilde{C}_S[x_1,x_2,s]$ in the sector $-l/2 < x_2 < l/2 < x_1$:

$$\tilde{C}_S[x_1,x_2,s] = -\frac{i}{2}\sqrt{\frac{\gamma}{c}}\{\tilde{B}_S[x_1 - x_2 - l/2,s]e^{ik_0l/2-s(l/2+x_2)/c} + \tilde{B}_S[l/2 - x_1 + x_2,s]e^{-ik_0l/2+s(l/2-x_1)/c}\}. \tag{58}$$

In a similar way we obtain the amplitude $C_A[x_1,x_2,s]$ in the subsector $-l/2 < x_2 < x_1 < l/2$,

$$\tilde{C}_A[x_1,x_2,s] = -\frac{i}{2}\sqrt{\frac{\gamma}{c}}\tilde{B}_A[x_1 - x_2 - l/2,s]e^{ik_0l/2-s(l/2+x_2)/c}, \quad (59)$$

and in the subsector $-l/2 < x_2 < l/2 < x_1$,

$$\tilde{C}_A[x_1,x_2,s] = -\frac{i}{2}\sqrt{\frac{\gamma}{c}}\{\tilde{B}_A[x_1 - x_2 - l/2,s] \times e^{ik_0l/2-s(l/2+x_2)/c} - \tilde{B}_A[l/2 - x_1 + x_2,s] \times e^{-ik_0l/2+s(l/2-x_1)/c}\}. \quad (60)$$

These expressions allow one to obtain the amplitudes $\tilde{C}_{S(A)}[x_1,x_2,s]$ in the case where one of the coordinates equals $\pm l/2$. Using a regularization similar to (19) we find

$$\tilde{C}_S[x,-l/2,s] = -\frac{i}{4}\sqrt{\frac{\gamma}{c}}\tilde{B}_S[x,s]e^{ik_0l/2} \quad (61)$$

and

$$\tilde{C}_S[l/2,x,s] = -\frac{i}{2}\sqrt{\frac{\gamma}{c}}\left\{\tilde{B}_S[-x,s]e^{ik_0l/2-s(l/2+x)/c} + \frac{1}{2}\tilde{B}_S[x,s]e^{-ik_0l/2}\right\}. \quad (62)$$

Similarly,

$$\tilde{C}_A[x,-l/2,s] = -\frac{i}{4}\sqrt{\frac{\gamma}{c}}\tilde{B}_A[x,s]e^{ik_0l/2} \quad (63)$$

and

$$\tilde{C}_A[l/2,x,s] = -\frac{i}{2}\sqrt{\frac{\gamma}{c}}\left\{\tilde{B}_A[-x,s]e^{ik_0l/2-s(l/2+x)/c} - \frac{1}{2}\tilde{B}_A[x,s]e^{-ik_0l/2}\right\}. \quad (64)$$

The argument x in Eqs. (61)–(64) belongs to the interval $(-l/2,l/2)$.

C. Amplitudes $\tilde{B}_{S(A)}[x,s]$ in the interval $-l/2 < x < l/2$

Putting expressions (61)–(64) into Eqs. (44) and (45), we can rewrite these equations in the form

$$[s + \gamma + c\partial_x]\tilde{B}_S[x,s] = -\gamma\tilde{B}_S[-x,s]e^{ik_0l-s(l/2+x)} - i\sqrt{\gamma c}[\delta(x + l/2)e^{ik_0l/2} + \delta(x - l/2)e^{-ik_0l/2}]\tilde{A}[s], \quad (65)$$

$$[s + \gamma + c\partial_x]\tilde{B}_A[x,s] = \gamma\tilde{B}_A[-x,s]e^{ik_0l-s(l/2+x)} - i\sqrt{\gamma c}[\delta(x - l/2)e^{-ik_0l/2} - \delta(x + l/2)e^{ik_0l/2}]\tilde{A}[s]. \quad (66)$$

First, consider these equations in the interval $-l/2 < x < l/2$. As they contain simultaneously the functions $\tilde{B}_{S(A)}[x,s]$ and $\tilde{B}_{S(A)}[-x,s]$, we add another couple of equations which are obtained from (65) and (66) by the inversion $x \to -x$. For instance, the functions $\tilde{B}_S[x,s]$ and $\tilde{B}_S[-x,s]$ obey the system of equations

$$[c\partial_x + \gamma + s/2](\tilde{B}_S[x,s]e^{sx/(2c)}) = -\gamma(\tilde{B}_S[-x,s]e^{-sx/(2c)})e^{ik_0l-sl/(2c)}, \quad (67)$$

$$[c\partial_x - \gamma - s/(2c)](\tilde{B}_S[-x,s]e^{-sx/(2c)}) = \gamma(\tilde{B}_S[x,s]e^{sx/(2c)})e^{ik_0l-sl/(2c)}, \quad (68)$$

which is the system of ordinary differential equations with constant coefficients for the functions $\tilde{B}_S[x,s]e^{sx/(2c)}$ and $\tilde{B}_S[-x,s]e^{-sx/(2c)}$. A similar system can be constructed for the function $\tilde{B}_A[x,s]$. The solutions to these systems of equations in the interval $-l/2 < x < l/2$ have the form

$$\tilde{B}_S[x,s] = e^{-s(x/2+l/4)/c}\tilde{B}_S[-l/2+0,s]D_S[x,s], \quad (69)$$

$$\tilde{B}_A[x,s] = e^{-s(x/2+l/4)/c}\tilde{B}_A[-l/2+0,s]D_A[x,s], \quad (70)$$

where the functions $D_{S(A)}[x,s]$ are given by

$$D_S[x,s] = \frac{Q_- \cosh(qx) - Q_+ \sinh(qx)}{Q_- \cosh(ql/2) + Q_+ \sinh(ql/2)}, \quad (71)$$

$$D_A[x,s] = \frac{Q_+ \cosh(qx) - Q_- \sinh(qx)}{Q_+ \cosh(ql/2) + Q_- \sinh(ql/2)}. \quad (72)$$

Here we have introduced the notation

$$Q_\pm = \sqrt{\gamma + s/2 \pm \gamma e^{ik_0l-sl/(2c)}}, \quad (73)$$

$$q = \frac{1}{c}\sqrt{(\gamma + s/2)^2 - \gamma^2 e^{2ik_0l-sl/c}} = \frac{Q_+Q_-}{c}. \quad (74)$$

At $x \to l/2+0$, the functions $D_{S(A)}[x,s] \to 1$, thus justifying representations (69) and (70) for the solutions. The functions $\tilde{B}_{S(A)}[-l/2+0,s]$ which determine these solutions are fixed by the jump of the functions $\tilde{B}_{S(A)}[x,s]$ at the singular point $x = -l/2$ of Eqs. (44) and (45). Having in mind condition (48) (no right-moving photons at $x < -l/2$), we immediately find, from (44) and (45),

$$\tilde{B}_S[-l/2+0,s] = -i\sqrt{\frac{\gamma}{c}}e^{ik_0l/2}\tilde{A}[s], \quad (75)$$

$$\tilde{B}_A[-l/2+0,s] = i\sqrt{\frac{\gamma}{c}}e^{ik_0l/2}\tilde{A}[s]. \quad (76)$$

Thus the functions $\tilde{B}_{S(A)}[x,s]$ are expressed in terms of the amplitude $A[s]$, still unknown. This amplitude is derived and discussed in the next subsection.

D. Amplitude $\tilde{A}[s]$

The amplitude $\tilde{A}[s]$ of finding the atomic system in the doubly excited state is determined by Eq. (43), which includes the amplitudes $\tilde{B}_{S(A)}[x,s]$ at points $x = \pm l/2$. As the functions $\tilde{B}_{S(A)}[x,s]$ are discontinuous at these points, the values $\tilde{B}_{S(A)}[\pm l/2,s]$ are defined by a smooth regularization similar to (19). In this way we find, at point $x = -l/2$,

$$\tilde{B}_{S(A)}[-l/2,s] = \mp\frac{i}{2}\sqrt{\frac{\gamma}{c}}e^{ik_0l/2}\tilde{A}[s]. \quad (77)$$

To find the functions $\tilde{B}_{S(A)}[x,s]$ at point $x = l/2$, we use expressions (69) and (70) at $x = l/2 + 0$ together with the jump values at point $x = l/2$ [see Eqs. (44) and (45)],

$$\tilde{B}_{S(A)}[l/2+0,s] - \tilde{B}_{S(A)}[l/2-0,s] = -i\sqrt{\frac{\gamma}{c}}e^{-ik_0 l/2}\tilde{A}[s]. \tag{78}$$

As a result, we get

$$\tilde{B}_S[l/2,s] = -\frac{i}{2}\sqrt{\frac{\gamma}{c}}[e^{-sl/(2c)}e^{ik_0 l/2}D_S[l/2,s] + e^{-ik_0 l/2}]\tilde{A}[s], \tag{79}$$

$$\tilde{B}_A[l/2,s] = \frac{i}{2}\sqrt{\frac{\gamma}{c}}[e^{-sl/(2c)}e^{ik_0 l/2}D_A[l/2,s] - e^{-ik_0 l/2}]\tilde{A}[s]. \tag{80}$$

Putting these expressions into (43) we get a closed equation for $\tilde{A}[s]$ with the solution

$$\tilde{A}[s] = \frac{1}{s + 2\gamma + \gamma e^{-sl/(2c)+ik_0 l}\Delta D[l/2,s]}, \tag{81}$$

where $\Delta D[l/2,s] \equiv D_S[l/2,s] - D_A[l/2,s]$ and the functions $D_{S(A)}[x,s]$ are given by (71) and (72). Being put in the earlier obtained expressions for the amplitudes $\tilde{B}_{S(A)}[x,s]$ and $\tilde{C}_{S(A)}[x_1,x_2;s]$, the amplitude $\tilde{A}[s]$ determines these amplitudes in the considered spatial intervals. In particular, the expressions for $\tilde{B}_{S(A)}[x,s]$ have been derived for the interval $(-l/2,l/2)$. However, the quantity of our interest, the probability, (42), of finding the system in one of the intermediate states, we need to know the amplitudes $\tilde{B}_{S(A)}[x,s]$ also at $x > l/2$. Therefore, we postpone the analysis of particular physical situations and proceed, in the following subsection, with the derivation of general expressions for these amplitudes.

E. Amplitudes $\tilde{B}_{S(A)}[x,s]$ in the interval $l/2 < x$

For brevity, we present an explicit calculation only for the amplitude of the symmetric mode $\tilde{B}_S[x,s]$. It obeys Eq. (44), which contains the functions $\tilde{C}_S[x,-l/2,s]$ and $\tilde{C}_S[x,l/2,s]$ with $x > l/2$. To find them we exploit the derived equation, (58), for the interval $(-l/2 < x_2 < l/2 < x_1)$. Taking there the limit $x_2 \to -l/2+0$ and using condition (49) for $x_2 = -l/2-0$, we find the regularized [similar to (19)] amplitude

$$\tilde{C}_S[x,-l/2,s] = -\frac{i}{4}\sqrt{\frac{\gamma}{c}}\{\tilde{B}_S[x,s]e^{ik_0 l/2} + \tilde{B}_S[-x,s]e^{-ik_0 l/2+s(l/2-x)/c}\}. \tag{82}$$

Similarly, taking the limit in Eq. (58), we get $x_2 \to l/2-0$ $\tilde{C}_S[x_1,x_2 = l/2-0,s]$; then we integrate (46) over an infinitesimal interval around $x_2 = l/2$ to obtain

$$\tilde{C}_S[x_1,x_2 = l/2+0,s] = \tilde{C}_S[x_1,x_2 = l/2-0,s] - \frac{i}{2c}\sqrt{\frac{\gamma}{c}}\tilde{B}_S[x_1,s]e^{-ik_0 l/2}. \tag{83}$$

Finally, we find $\tilde{C}_S[x_1,x_2 = l/2,s]$, defined [similar to (19)] as the half-sum of the left (at $x_2 = l/2-0$) and right (at $x_2 = l/2+0$) values:

$$\tilde{C}_S[x,l/2,s] = -\frac{i}{2c}\sqrt{\frac{\gamma}{c}}\left\{\tilde{B}_S[x-l,s]e^{ik_0 l/2 - sl/c} + \tilde{B}_S[l-x,s]e^{-ik_0 l/2+s(l/2-x)/c} + \frac{1}{2}\tilde{B}_S[x,s]e^{-ik_0 l/2}\right\}. \tag{84}$$

Putting (82) and (84) into (44), we obtain the equation for $\tilde{B}_S[x,s]$ in the interval $x > l/2$:

$$(c\partial_x + s + \gamma)\tilde{B}_S[x,s] = -\gamma e^{(ik_0 - s/c)l}\tilde{B}_S[x-l,s] - \gamma e^{(l/2-x)s/c}\tilde{B}_S[l-x,s]. \tag{85}$$

Note that the term $\tilde{B}_S[-x,s]$ in Eq. (82) vanishes at $x > l/2$ according to the boundary condition, (48).

Equation (85) is a nonlocal (functional) equation, which relates the unknown function to different values of the argument. To get its formal solution we introduce a new function,

$$F[z,s] = \int_{l/2}^{\infty} e^{-zx}\tilde{B}_S[x,s]dx, \tag{86}$$

which is a Laplace transform of $\tilde{B}_S[x,s]$ over the variable x. Since $\partial_x \tilde{B}_S[x,s] \to zF[z,s] - \tilde{B}_S[l/2+0,s]$, the equation for $F[z,s]$ has the form

$$[cz + s + \gamma + \gamma e^{(ik_0 - s/c - z)l}]F[z,s] = \tilde{B}_S[l/2+0,s] - \gamma e^{(ik_0 - s/c - z)l}G[z] - \gamma e^{-[s/(2c)+z]l}G[-z-s/c], \tag{87}$$

where the function $G[z]$ is defined by

$$G[z] = \int_{-l/2}^{l/2} e^{-zx}\tilde{B}_S[x,s]dx. \tag{88}$$

This function is determined by the amplitude $\tilde{B}_S[x,s]$ in the interval $-l/2 < x < l/2$, which has been calculated in Sec. V C (with $\tilde{A}[s]$ determined in Sec. V D). The solution to (87) reads

$$F[z,s] = \frac{c\tilde{B}_S[l/2+0,s] - \gamma e^{-zl}(e^{(ik_0 - s/c)l}G[z] + e^{-sl/(2c)}G[-z-s/c])}{cz + s + \gamma + \gamma e^{(ik_0 - s/c - z)l}}. \tag{89}$$

Finally, we make the inverse Laplace transform and find the amplitude of interest, $\tilde{B}_S[x,s]$, for $x > l/2$:

$$\tilde{B}_S[x,s] = \int_{-i\infty+0}^{+i\infty+0}\frac{dz}{2\pi i}F[z,s]e^{zx}. \tag{90}$$

This expression together with the results in Sec. V C completely determine the amplitude $\tilde{B}_S[x,s]$ and thus allow one to calculate the probability $P_S(t)$, (42). This is done for a particular case of interest in the following section.

VI. PROBABILITY OF REACHING A METASTABLE DARK STATE

The general formalism developed in the previous section completely describes the dynamics of the doubly excited system of TLAs in the case of an arbitrary interatomic distance. However, on one hand, this dynamics is so sophisticated that it is hardly possible to describe it with explicit expressions. On the other hand, the general case is of restricted interest because the TLA system merely decays to the ground state with two photons emitted. Albeit the formalism allows one, in principle, to calculate the correlation functions and entanglement of these photons, we do not pursue this objective in the current paper. Instead, we concentrate on answering the question whether the system can decay only partially, with the emission of only a single photon and the formation of a dark metastable singly excited state. Apparently this is not a general but an exclusive situation which can be realized at special interatomic distances l. Namely, to form a symmetric (S) dark state the two atoms should be separated by the distance l such that $e^{ik_0 l} = -1$, while an antisymmetric state may occur when $e^{ik_0 l} = 1$. Here we consider the first configurations with $e^{ik_0 l} = -1$ and calculate the probability $P_S(t \to \infty)$, (42), of finding the system in the dark S state. For simplicity we restrict our analysis to the case $\gamma l/c \ll 1$ (but at the same time $k_0 l \gg 1$).

First, let us analyze the time behavior of the amplitude $A(t)$ of finding the doubly excited system in its initial state. The Laplace transform $\tilde{A}[s]$, (81), remains finite, $\tilde{A}[s] \approx 1/(2\gamma)$, when $s \to 0$, hence $A(t) \to 0$ in the limit $t \to \infty$. This (quite expectably) means that the doubly excited state of two TLAs completely decays. As shown below, characteristic values of the Laplace variable s, which determine the fast decay rate of the amplitude $A(t)$, are of the order of γ; i.e., $sl/c \sim \gamma l/c \ll 1$. Indeed, estimating the functions $D_{S(A)}[l/2,s]$ determined by (71) and (72), at $sl/c, \gamma l/c \ll 1$, we find

$$D_S[l/2,s] \approx 1 - \frac{sl}{2c}, \quad D_A[l/2,s] \approx 1 - \frac{4\gamma + s}{2c}, \tag{91}$$

$$\Delta D[l/2,s] = D_S[l/2,s] - D_A[l/2,s] \approx 2\gamma l/c \ll 1. \tag{92}$$

Using (92) we arrive at a simplified expression for (81) at small $\gamma l/c$ and sl/c:

$$\tilde{A}[s] \approx \frac{1}{s + 2\gamma}. \tag{93}$$

From this expression we see that the characteristic values of the Laplace variable s, which determine the decay of the amplitude $A(t)$, are of the order of γ, which confirms self-consistently the expansion in $sl/c \sim \gamma l/c \ll 1$. Expression (93) corresponds to the "superradiant" decay [9,26], $|A(t)|^2 \approx e^{-4\gamma t} = e^{-2\Gamma t}$.

In contrast to the simple decay law for the amplitude $A(t)$ of the upper excited state $|e\rangle$, the time evolution of the intermediate states of the decaying system is less trivial. Consider the amplitude $B_S(x,t)$, which describes one of the intermediate states, namely, the atomic system in a singly excited symmetric state with a photon at position x. In the limit of interest $t \to \infty$ expression (42) reduces to

$$P_S(t \to \infty) = \int_{l/2}^{\infty} |B_S(x, t \to \infty)|^2 dx, \tag{94}$$

while the omitted integral from $-l/2$ to $l/2$ decays exponentially with time. This decay is caused by the simple fact that the emitted photon escapes from the interatomic interval $(-l/2, l/2)$.

The term $\tilde{B}_S[l/2 + 0, s]$ in Eq. (89) is determined by Eqs. (78), (69), and (75) and reads explicitly

$$\tilde{B}_S[l/2+0,s] = -i\sqrt{\frac{\gamma}{c}} e^{-ik_0 l/2} \tilde{A}[s][1 + e^{ik_0 l} e^{-sl/(2c)} D_S[l/2,s]]. \tag{95}$$

In the considered particular case of the "antiresonant" interatomic distance ($e^{ik_0 l} = -1$) and $\gamma l/c \ll 1$, expression (91) gives $D_S[l/2,s] \approx 1 - sl/(2c)$, so that

$$\tilde{B}_S[l/2+0,s] \approx -i\sqrt{\frac{\gamma}{c}} e^{-ik_0 l/2} sl \tilde{A}[s], \tag{96}$$

where it is taken into account that the characteristic values of s are of the order of γ; i.e., $sl/c \ll 1$. The other terms in the numerator of (89) are parametrically smaller than (96). Indeed, for typical values $z \sim \gamma/c$ (see below) these terms are proportional to the difference $G[z] - e^{-sl/(2c)} G[-z - s/c]$ of two slightly distinct integrals, (88), each of them already being proportional to the small interval length l. Thus, for the considered antiresonant case expression (89) reduces to

$$F[z,s] \approx -i\sqrt{\frac{\gamma}{c}} \frac{sl \tilde{A}[s] e^{-ik_0 l/2}}{cz + s + \gamma[1 - e^{-(z+s/c)l}]}. \tag{97}$$

This function has a pole at $z = -s/c$ so the leading contribution to (90) is

$$\tilde{B}_S[x,s] = -\frac{i\sqrt{\gamma}}{c^{3/2}} e^{-ik_0 l/2} \frac{sl}{s + 2\gamma} e^{-sx/c}, \tag{98}$$

where expression (93) for $\tilde{A}[s]$ was used. Now, using the inverse Laplace transformation [similar to (23)], we return to the time-dependent amplitude

$$\tilde{B}_S[x,t] = 2i e^{-ik_0 l/2} \left(\frac{\gamma}{c}\right)^{3/2} l e^{-2\gamma(t - x/c)} \theta(ct - x), \tag{99}$$

where the θ function provides vanishing of the amplitude outside the "light cone" $x = ct$. Putting (99) into integral (94) we find the desired expression for the probability of getting to a dark (symmetric) state:

$$P_S(t \to \infty) = \left(\frac{\gamma l}{c}\right)^2 = \frac{1}{4}\left(\frac{\Gamma l}{c}\right)^2. \tag{100}$$

Thus, we have arrived at a remarkable conclusion: when the interatomic distance l between two TLAs corresponds to an "antiresonance" (i.e., $e^{ik_0 l} = -1$), the doubly excited state, (2), of the atomic subsystem does not necessarily decay to the ground state (with two photons emitted). Instead, there is a *finite* probability of reaching a stable dark state with only one emitted photon. This stable state for the antiresonant location of atoms corresponds to the symmetric combination ($C_1 = C_2$) of singly excited states, (1). Similarly, there is a finite probability of the doubly excited system's reaching the stable antisymmetric state, if the interatomic distance takes a "resonant" value (i.e., $e^{ik_0 l} = 1$).

Of course, as discussed in Sec. III, if the distance l deviates from the antiresonant one or the atomic transition frequencies are not equal, the dark state becomes only metastable and slowly (for a small deviation) decays to the ground state. Nevertheless, the found finiteness of the transition probability from the doubly excited state, (2), to a (meta)stable singly excited state, (1), make sense and can be observed because the typical time of this transition, $\sim 1/\gamma$ [to be more precise, $1/(4\gamma)$], is much shorter than the subsequent decay time of the metastable state.

The probability, (100), has been obtained for the simplified case $\gamma l/c \ll 1$, where P_S is small. However, the simplification has been chosen only for obtaining the explicit analytical result. When $\gamma l/c \sim 1$ one may expect this probability to become appreciable ($\lesssim 1$). But this situation requires a more sophisticated analysis. Larger values of $\gamma l/c$ ($\gamma l/c \ll 1$) are not favorable for the formation of the dark state, as its probability is reduced in accordance with (25).

VII. CONCLUSION

We have studied a system of two qubits (TLAs) coupled to a 1D waveguide. We have described the time evolution of the initial state of the system; this initial state corresponds to a singly or doubly excited atomic subsystem in the absence of waveguide photons. The evolution is rather complicated, because the system possesses both dissipative (due to the possibility of photons flying away from the atoms) and dynamical (due to the coherent photon exchange between the atoms) features.

Our first goal was to calculate the decay rates of metastable (dark) states, which can be realized in a singly excited system of two TLAs when the interatomic distance l takes special ("resonant" or "antiresonant") values. The decay of these metastable states may be caused by deviations of the interatomic distance from the special ones or by frequency detuning (i.e., unequal frequencies of the transition of the two TLAs). The corresponding decay rates for both decay mechanisms are given by expressions (30) and (33), respectively. For a small deviation or detuning, the decay rate of metastable states can be much lower than the usual decay rate $\Gamma = 2\gamma$ of a single excited TLA.

Our second goal was to answer the question whether the doubly excited state $|\Psi_{\text{in}}^{(2)}\rangle$ of two identical TLAs can evolve to one of the metastable configurations. We have found that for an arbitrary distance l between the atoms, the atomic subsystem excitation decays rapidly (with a decay rate on the scale of Γ) to the ground state with the emission of two photons. However, when l coincides with one of the special positions, the system decays rapidly (with the emission of a single photon) to a superposition of a bright and a dark (metastable) state of type $|\Psi_{\text{in}}^{(1)}\rangle$. The bright component of this superposition decays rapidly to the ground atomic state with the second photon emitted, while the dark state remains stable (for ideal resonance conditions) or decays slowly in accordance with the results of the previous point. Remarkably, the probability of the formation of the dark metastable state is finite even for well-separated TLAs. For the case $\Gamma l/c \ll 1$, we have been able to find an explicit expression, (100), for this probability. In the considered limit this probability is low $[\Gamma l/(2c)]^2$ but increases quadratically with an increase in l as long as $\Gamma l/c \lesssim 1$. Larger values of $\Gamma l/c$ ($\Gamma l/c \ll 1$) are not favorable for the formation of the dark state, as its probability is reduced in accordance with (25). Thus, the most favorable condition for observation of metastable dark states formed upon the decay of the doubly excited two-TLA system corresponds to the parameter values $\Gamma l/c1$.

There are several reasons for interest in the metastable states of the two qubits. An academic one is due to the somewhat counterintuitive stability of some excited states of an open multiparticle system. On the other hand, such metastable states of pairs of qubits may find application as memory storage elements in information processing. They can be used also as detectors sensitive to external perturbations, e.g., to those which cause a detuning of the TLA frequencies and thus initiate the decay of the metastable state.

As mentioned in Sec. I, a system of qubits talking through a 1D waveguide can be realized not only with resonant atoms interacting with optical waveguide photons but also with quantum dots connected through a plasmon-carrying wire, Josephson junctions, etc. One of the challenging experimental problems connected with realizations of such systems remains the suppression of parasitic emission of photons to the surrounding space. When the rate of these losses becomes lower than the single-atom decay rate due to the emission of waveguide photons, the effects studied in the present paper may become observable.

Finally, we note that the formalism used in this paper is based on the direct solution of equations for amplitudes of quantum states, without the approximate exclusion of the photon degrees of freedom in favor of the reduced density matrix of the atomic subsystem. Therefore the explored formalism allows one to describe also correlation functions of emitted photons (entanglement, etc.).

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