Self-induced mode mixing of ultraintense lasers in vacuum

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We study the effects of the quantum vacuum on the propagation of a Gaussian laser beam in vacuum. By means of a double perturbative expansion in paraxiality and quantum vacuum terms, we provide analytical expressions for the self-induced transverse mode mixing, rotation of polarization, and third harmonic generarion. We discuss the possibility of searching for the self-induced, spatially dependent phase shift of a multipetawatt laser pulse, which may allow the testing of quantum electrodynamics and new physics models, such as Born-Infeld theory and models involving new minicharged or axion-like particles, in parametric regions that have not yet been explored in laboratory experiments.

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I. INTRODUCTION

Ultraintense lasers are important tools for nuclear fusion, ultrafast microscopy, or particle acceleration [1]. In addition, the unprecedented photon concentrations at beam foci may allow us to probe the quantum vacuum (QV) [2] in a regime of extremely high luminosity and low single-particle energy, complementary to the high energy and moderate luminosity of charged particle pulses in conventional colliders. Even for pulses comprising the electric-field amplitude below the Schwinger limit $E_S = m_e^2 c^3/(e\hbar) \approx 1.3 \times 10^{18}$ V/m [3], quantum electrodynamics (QED) and several new physics models, such as Born-Infeld (BI) theory [4] and models involving minicharged or axion-like particles [5,6], predict the existence of nonlinear corrections to Maxwell equations in vacuum. Stringent laboratory constraints on the parameters driving these corrections have been obtained from the searches for vacuum birefringence on a low-intensity laser beam propagating in a strong external magnetic field [7]. However, they are still several orders of magnitude above the values predicted by QED. Therefore, the search for the effects of the QV on the propagation of optical beams is of great interest both to test QED in the regime of low-energy photons and to search for signatures of new physics. Although QV effects in the interaction of lasers with matter have already been observed [8], all-optical experiments are interesting for probing the behavior of light in a different, matterless regime.

Several new petawatt and multipetawatt facilities are being projected around the globe for the near-future. The growing peak intensities that they will provide can hopefully be used to test the QV [9]. This may be done by searching for harmonic generation in an ultraintense standing wave [10] or using frequency upshifting to increase the photon-photon scattering cross section [11]. Another option is to study the collision of two laser beams [6,12] or to devise a photon collider [13]. Setups in which three beams coincide have also been considered since they would allow for a clear signature [14]. An array of intense laser beams can be used to create a Bragg grating to deflect a probe pulse [15]. It has also been argued that photon-photon scattering in vacuum may affect the propagation of a finite-size pulse in a waveguide [16] or in the presence of background fields [17].

In this paper, we study the propagation and self-interaction of a single ultraintense pulse in vacuum, which is of great relevance since it occurs in any ultraintense laser facility and will become increasingly important as higher intensities are achieved. We explicitly compute the QV corrections to the propagation of the beam under assumptions that are accurate for the intensities that will be available in the not-too-distant future. All the results are derived analytically for an incoming Gaussian beam and are presented in a closed and rather simple form, without the need for any numerical integrations. Remarkably, we find that the nonlinear QV effects drive the mixing of different transverse modes, resulting in a self-induced modulated shift of the phase of the pulse. We argue that this is the dominant effect under general conditions, in principle, being observable at intensities lower than those needed to measure other QV effects on the propagation of a pulse [18–20]. All effects decrease for increasing waists, as expected for the non-self-interaction of plane waves. We finally discuss the possibility of searching for such self-induced phase shift of the pulse in future multipetawatt facilities.

II. MATHEMATICAL FORMALISM

We use conventions in terms of the electromagnetic tensor and its potential $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Greek indices μ and ν run from 0 to 3, with $\partial_0 = c^{-1}\partial_t$, $\partial_1 = \partial_x$, etc. Roman indices will take values $i = 1,2,3$. Einstein summation convention is used. We take a mostly minus metric $g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ and $\epsilon^{0123} = 1$ for the Levi-Civita tensor. The relation to the electric and magnetic fields is

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix}, \tag{1}$$

and therefore

$$F_{\mu\nu}F^{\mu\nu} = 2\left(-\frac{\vec{E}^2}{c^2} + \vec{B}^2\right), \quad \epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = -\frac{8}{c}\vec{E}\cdot\vec{B}. \tag{2}$$

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Assuming parity, Lorentz, and gauge symmetry, the most general effective Lagrangian density for the electromagnetic field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, up to $O(F^4)$ without derivatives [38] of F is

$$\mathcal{L} = \mathcal{L}_0 + \xi_L \mathcal{L}_0^2 + \xi_T \mathcal{L}_T, \tag{3}$$

where ξ_L and ξ_T are real parameters and $\mathcal{L}_{0,T}$ are given by

$$\begin{aligned} \mathcal{L}_0 &= -\frac{1}{4}\epsilon_0 c^2 F_{\mu\nu}F^{\mu\nu} = \frac{\epsilon_0}{2}(\vec{E}^2 - c^2\vec{B}^2), \\ \mathcal{L}_T &= \frac{7}{256}\epsilon_0^2 c^4 (\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma})^2 = \frac{7}{4}\epsilon_0^2 c^2 (\vec{E}\cdot\vec{B})^2. \end{aligned} \tag{4}$$

This effective action neglects the possibility of the creation of *real* particles [8,21] and is valid for $|\vec{E}| \ll E_S$, $\hbar\omega \ll m_e c^2$. In QED, the nonlinear terms come from loops of virtual electron-positron pairs, yielding the Euler-Heisenberg Lagrangian [22,23], in which the parameters take the values

$$\xi_L^{\rm QED} = \xi_T^{\rm QED} \equiv \xi = \frac{8\alpha}{45(4\pi\epsilon_0)E_S^2} \simeq 6.7 \times 10^{-30}\ \frac{\rm m^3}{\rm J}, \tag{5}$$

where $\alpha \simeq 1/137$ is the fine-structure constant. In theories beyond the standard model, such as the BI theory [4] or models involving axion-like or minicharged particles [5], the values of ξ_L and ξ_T can be different [6]. The best laboratory constraints on the ξ_L and ξ_T parameters have been obtained by the PVLAS collaboration [7]. Their failure to detect nonlinear QV effects implies that

$$\frac{|7\xi_T - 4\xi_L|}{3} < 1.7 \times 10^{-26}\ \frac{\rm m^3}{\rm J} = 2.5 \times 10^3 \xi. \tag{6}$$

In the particular case where $\xi_L = \xi_T$, this limit is three orders of magnitude larger than the QED prediction, thus leaving room for the possible emergence of new physics. Note that PVLAS does not bound BI theory, for which $4\xi_L = 7\xi_T$. In this article, we put forward an alternative setup to search for this kind of QV effects. We show that our proposal can explore regions of the ξ_L, ξ_T parameter space that have not been constrained by PVLAS, e.g., allowing us to test BI theory. Furthermore, if the pulse is powerful enough, it can also improve the sensitivity of PVLAS for any value of the ratio ξ_L/ξ_T and, possibly, lead to the detection of QV polarization effects due to QED or new physics.

The Euler-Lagrange equations read

$$\begin{aligned} &-\partial_\mu F^{\mu\alpha} + \tfrac{1}{2}(\xi_L\epsilon_0 c^2)\partial_\mu[(F_{\rho\sigma}F^{\rho\sigma})F^{\mu\alpha}] \\ &\quad + \tfrac{7}{32}(\xi_T\epsilon_0 c^2)\epsilon^{\beta\gamma\delta\rho}\epsilon^{\mu\nu\tau\alpha}F_{\mu\nu}\partial_\tau[F_{\beta\gamma}F_{\delta\rho}] = 0. \end{aligned} \tag{7}$$

The nonlinear terms are small compared to the leading one in situations conceivable for near-future facilities. Therefore Eq. (3) can be solved using a perturbative expansion. Choosing Lorenz gauge

$$\partial_\mu A^\mu = 0, \tag{8}$$

we get

$$A^\mu = A^\mu_{\rm lin} + (\xi_L\epsilon_0 c^2)\mathcal{A}^\mu_L + (\xi_T\epsilon_0 c^2)\mathcal{A}^\mu_T + O(\xi^2), \tag{9}$$

where $A^\mu_{\rm lin}$ solves $\Box A^\mu_{\rm lin} = 0$, and $\mathcal{A}_L$ and $\mathcal{A}_T$ encode the leading nonlinear corrections and satisfy

$$\begin{aligned} \Box\mathcal{A}^\alpha_L &= J^\alpha_L = \tfrac{1}{2}\partial_\mu[(F_{\rho\sigma}F^{\rho\sigma})F^{\mu\alpha}], \\ \Box\mathcal{A}^\alpha_T &= J^\alpha_T = \tfrac{7}{32}\epsilon^{\beta\gamma\delta\rho}\epsilon^{\mu\nu\tau\alpha}F_{\mu\nu}\partial_\tau[F_{\beta\gamma}F_{\delta\rho}]. \end{aligned} \tag{10}$$

The currents on the right-hand side, linked to the QV polarization, are computed using $A^\mu_{\rm lin}$ (first Born approximation). The electric field is $E_i = -\partial_t A^i - c\partial_i A^0$ and we define

$$E_i = E_{i,\rm lin} + (\xi_L\epsilon_0 c^2)\mathcal{E}_{i,L} + (\xi_T\epsilon_0 c^2)\mathcal{E}_{i,T} + O(\xi^2) \tag{11}$$

and write

$$\Box\mathcal{E}_{i,\rm LT} = j^i_{\rm LT} \equiv -\partial_t J^i_{\rm LT} - c\partial_i J^0_{\rm LT}. \tag{12}$$

The paraxial approximation considers a beam of wavelength $\lambda = 2\pi/k$ propagating in the z direction $f(x,y,z)e^{i(\omega t - kz)}$ with a slowly varying envelope, $\partial_z f \ll k f$, $\partial_z^2 f \ll k\,\partial_z f$. Hereafter, we require all the equations to be satisfied at leading order in the paraxial limit. A solution of the linear equations in this approximation, linearly polarized along x, is

$$A^\mu_{\rm lin} = {\rm Re}\left[e^{i(\omega t - kz)}\left(0, f, 0, -i\frac{\partial_x f}{k}\right)\right], \tag{13}$$

with f satisfying

$$\mathcal{P}_\omega(f) \equiv 2ik\,\partial_z f - \partial^2_{xx} f - \partial^2_{yy} f = 0. \tag{14}$$

A paraxial beam can be written as a linear combination of the Hermite-Gauss transverse modes [24],

$$\begin{aligned} u_{mn}(x,y,z) = &\frac{i\sqrt{k z_0}(z - i z_0)^{m/2+n/2}}{\sqrt{\pi\, 2^{m+n} m! n!}(z + i z_0)^{1+m/2+n/2}} \\ &\times H_m\left(\frac{\sqrt{k/z_0}\,x}{\sqrt{1+(z/z_0)^2}}\right) \\ &\times H_n\left(\frac{\sqrt{k/z_0}\,y}{\sqrt{1+(z/z_0)^2}}\right) e^{\left(-\frac{ik(x^2+y^2)}{2(z+iz_0)}\right)}, \end{aligned} \tag{15}$$

where the $H_{m,n}$ are Hermite polynomials and z_0 is an integration constant (the Rayleigh length). These functions solve Eq. (14) and form a complete orthonormal basis,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u^*_{mn}u_{pq}\,dx\,dy = \delta_{mp}\delta_{nq}. \tag{16}$$

Since in the considered system there is third harmonic generation, let us also introduce the paraxial operator for frequency 3ω,

$$\mathcal{P}_{3\omega}(f) \equiv 6ik\partial_z f - \partial^2_{xx} f - \partial^2_{yy} f, \tag{17}$$

and the Hermite-Gauss modes $v_{mn}(x,y,z)$, which are just the u_{mn} with $k \to 3k$.

The $j^\alpha_{L,T}$ in (10) and (12) include terms involving frequencies ω and 3ω. Let us split the terms as

$$j^i_{\rm LT} = {\rm Re}\big[e^{i(\omega t - kz)}\hat{j}^i_{\rm LT} + e^{3i(\omega t - kz)}\tilde{j}^i_{\rm LT}\big]. \tag{18}$$

Similarly, for the electric field we define

$$\mathcal{E}_{i,\rm LT} = {\rm Re}[e^{i(\omega t - kz)}\hat{\mathcal{E}}_{i,\rm LT} + e^{3i(\omega t - kz)}\tilde{\mathcal{E}}_{i,\rm LT}]. \tag{19}$$

In the paraxial approximation, Eqs. (12) can be written as

$$\mathcal{P}_\omega(\hat{\mathcal{E}}_{i,\rm LT}) = \hat{j}^i_{\rm LT}, \quad \mathcal{P}_{3\omega}(\tilde{\mathcal{E}}_{i,\rm LT}) = \tilde{j}^i_{\rm LT}. \tag{20}$$

Given the function f that describes the incident beam, (13), we can compute $j^i_{\rm LT}$ using (10) and (12) and then solve (20) to find the corrections to the electric field.

We deal with these equations by expanding in modes. Let us introduce some notation for the terms of frequency ω; analogous definitions can be made for the third harmonic terms. Consider the expansion,

$$\hat{j}^i_{\rm LT} = \sum_{m,n} \gamma^{i,{\rm LT}}_{mn}(z) u_{mn}, \tag{21}$$

where

$$\gamma^{i,{\rm LT}}_{mn}(z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u^*_{mn}\hat{j}^i_{L,T}\,dx\,dy. \tag{22}$$

Accordingly,

$$\hat{\mathcal{E}}_{i,{\rm LT}} = \sum_{m,n} \beta^{i,{\rm LT}}_{mn}(z) u_{mn}, \tag{23}$$

with

$$2i\,k\,\partial_z \beta^{i,{\rm LT}}_{mn}(z) = \gamma^{i,{\rm LT}}_{mn}(z). \tag{24}$$

The correction to each component is found by solving the corresponding set of ordinary differential equations with the initial condition $\lim_{z\to-\infty}\beta^{i,{\rm LT}}_{mn}(z) = 0$. The outgoing wave is parametrized by

$$\lim_{z\to\infty}\beta^{i,{\rm LT}}_{mn}(z) = \frac{1}{2ik}\int_{-\infty}^{\infty}\gamma^{i,{\rm LT}}_{mn}(z)dz. \tag{25}$$

III. CORRECTIONS FOR A GAUSSIAN BEAM

The Gaussian beam is the lowest order mode, $f = A\,u_{00}$, where A is a real constant related to the power and energy density of the beam at its focus $x = y = z = 0$ as $P = \frac{1}{2}\epsilon_0\,c\,\omega^2 A^2$ and $\rho_0 = \frac{2P}{\pi\,w_0^2 c}$, where w_0 is the waist radius and $z_0 = k\,w_0^2/2$ is the Rayleigh length. The paraxial approximation can be ultimately considered as an expansion controlled by the small parameter $(k\,w_0)^{-1}$.

The differential equations for the electric field correction, (20), of frequency ω read

$$\begin{aligned}
\mathcal{P}_\omega(\hat{\mathcal{E}}_{x,L}) &= 2\mathcal{M}\big\{k^2z_0^2(3x^4 - 2x^2y^2 - y^4) + \big(z^2+z_0^2\big)^2 \\
&\quad + 2k\big(z^2+z_0^2\big)[i\,x^2(z+3i\,z_0) + y^2(z_0 - i\,z)]\big\},\\
\mathcal{P}_\omega(\hat{\mathcal{E}}_{x,T}) &= \tfrac{7}{2}\mathcal{M}[z^2 + z_0(-2k\,x^2 + z_0)]\\
&\quad \times [z^2 + z_0(-2k\,y^2 + z_0)],\\
\mathcal{P}_\omega(\hat{\mathcal{E}}_{y,L}) &= 4\mathcal{M}k^2z_0^2 x\,y\,(x^2 - y^2),\\
\mathcal{P}_\omega(\hat{\mathcal{E}}_{y,T}) &= 7\mathcal{M}kxy\big\{2\,i\,z^3 - 4z^2z_0 + 2i\,z\,z_0^2\\
&\quad + [k(x^2+3y^2) - 4z_0]z_0^2\big\},
\end{aligned} \tag{26}$$

where we have defined the common factor

$$\mathcal{M} = \frac{A^3\pi^{-\frac{3}{2}}c\,k^{\frac{9}{2}}z_0^{\frac{7}{2}}}{(z - i\,z_0)^5(z+i\,z_0)^6}\exp\left[-\frac{k(x^2+y^2)}{2\big(z^2+z_0^2\big)}(i\,z + 3z_0)\right]. \tag{27}$$

For each of the four $\hat{j}^i_{\rm LT}$, there is an infinite number of $\gamma_{mn}(z)$ nontrivial terms, (22). They can be found analytically since the integration over the x-y plane only involves products of Gaussians and polynomials. Therefore, the $\gamma_{mn}(z)$ are quotients of complex polynomials which can also be explicitly integrated to find $\beta_{mn}(z)$ in (24). To illustrate this point, let us

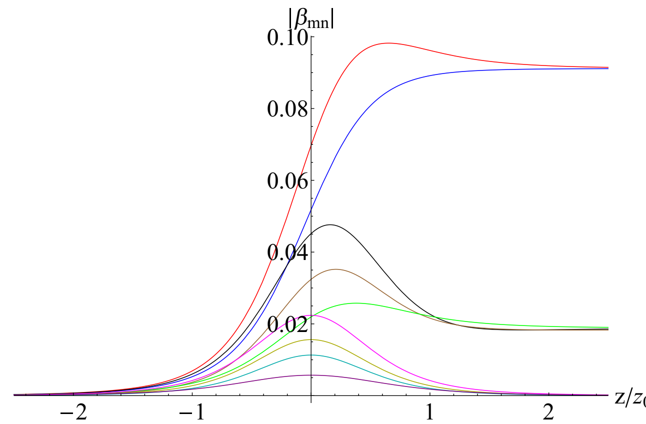

FIG. 1. (Color online) Absolute value of the coefficients of the higher transverse modes $0 < m + n \leqslant 6$ as a function of z, for x-polarized modes. $\xi_L = \xi_T$ is assumed and the vertical axis is in units of $\frac{A^3k^3c}{z_0^2}$. Curves, from top to bottom: u_{20}, u_{02}, u_{40}, u_{22}, u_{04}, u_{60}, u_{42}, u_{24}, u_{06}.

explicitly write the result for the correction due to ξ_L to the x polarization for the 00 mode,

$$\begin{aligned}
\gamma^{x,L}_{00} &= -i\frac{A^3c\,k^4z_0^3}{4\pi\big(z^2+z_0^2\big)^3},\\
\beta^{x,L}_{00} &= -\frac{A^3c\,k^3}{64z_0^3}\left[\frac{5z_0^3z + 3z_0z^3}{\big(z^2+z_0^2\big)^2} + 3\left(\frac{\pi}{2} + \arctan\frac{z}{z_0}\right)\right],
\end{aligned}$$

giving $\beta^{x,L}_{00}(\infty) = -\frac{3A^3c\,k^3}{64\,z_0^2}$. As a second example, consider the 60 mode,

$$\begin{aligned}
\gamma^{x,L}_{60} &= -i\frac{\sqrt{5}A^3c\,k^4z_0^2(31z_0 - 12i\,z)}{128\pi(z_0+i\,z)^6},\\
\beta^{x,L}_{60} &= -\frac{A^3c\,k^3z_0^2(28z_0 + 15i\,z)}{256\sqrt{5}\pi(z_0+i\,z)^5},
\end{aligned}$$

which yields $\beta^{x,L}_{60}(\infty) = 0$.

This energy transfer to higher transverse modes is depicted in Figs. 1 and 2, where we plot $|\beta_{mn}(z)|$ for modes with $0 < m + n \leqslant 6$ assuming $\xi_L = \xi_T$.

Remarkably, only a few terms $\beta_{mn}(\infty)$ are nonvanishing, allowing us to write explicitly the corrections to the outgoing wave,

$$\begin{aligned}
&\epsilon_0c^2(\xi_L\hat{\mathcal{E}}_{x,L} + \xi_T\hat{\mathcal{E}}_{x,T})\big|_{z\to\infty}\\
&\quad = \frac{-3\,A^3k^3\epsilon_0c^3}{64\,z_0^2}\bigg[\left(\xi_L + \frac{7}{4}\xi_T\right)\left(u_{00} + \frac{u_{20}+u_{02}}{\sqrt{2}}\right)\\
&\qquad + \left(\xi_L + \frac{21}{4}\xi_T\right)\frac{u_{22}}{16}\\
&\qquad + \sqrt{\frac{3}{2}}\frac{7}{48}\left(\xi_L + \frac{5}{4}\xi_T\right)(u_{40}+u_{04})\bigg],
\end{aligned} \tag{28}$$

$$\begin{aligned}
&\epsilon_0c^2(\xi_L\hat{\mathcal{E}}_{y,L} + \xi_T\hat{\mathcal{E}}_{y,T})\big|_{z\to\infty}\\
&\quad = \frac{\sqrt{3}\,A^3k^3\epsilon_0c^3}{512\sqrt{2}\,z_0^2}\left(\xi_L - \frac{7}{4}\xi_T\right)(u_{13} - u_{31}).
\end{aligned} \tag{29}$$

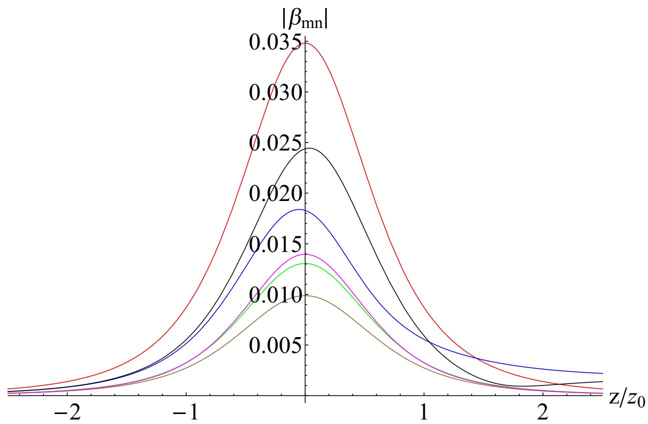

FIG. 2. (Color online) Absolute value of the coefficients of the higher transverse modes $m+n \leqslant 6$ as a function of z, for y-polarized modes. $\xi_L = \xi_T$ is assumed and the vertical axis is in units of $\frac{A^3 k^3 c}{z_0^2}$. Curves, from top to bottom: u_{11}, u_{13}, u_{31}, u_{15}, u_{33}, u_{51}.

Therefore, the QV corrections generate higher transverse modes of both polarizations. As expected, all corrections vanish as $z_0 \to \infty$; see the discussion in Sec. IV. We now study the physical consequences in more detail.

A. Spatially dependent phase shift

For an x-polarized beam of frequency ω at $z \to \infty$, we can write $E_x(z\to\infty) \approx \mathrm{Re}[-i\,\omega\, A\, u_{00} e^{i(\omega t - kz - \phi)}]$, where

$$\phi = -\frac{\epsilon_0 c^2}{\omega\, A\, u_{00}}(\xi_L \hat{\mathcal{E}}_{x,L} + \xi_T \hat{\mathcal{E}}_{x,T}). \tag{30}$$

Using (28) and the explicit form of the modes u_{mn} given in Eq. (16), we obtain

$$\phi = \frac{\mathcal{K}}{4}\bigg[\xi_L\left(1 + 2\bar{x}^2 + 2\bar{y}^2 + \frac{1}{2}\bar{x}^2\bar{y}^2 + \frac{7}{12}\bar{x}^4 + \frac{7}{12}\bar{y}^4\right) + \frac{7}{4}\xi_T\left(1 + 2\bar{x}^2 + 2\bar{y}^2 + \frac{3}{2}\bar{x}^2\bar{y}^2 + \frac{5}{12}\bar{x}^4 + \frac{5}{12}\bar{y}^4\right)\bigg], \tag{31}$$

where

$$(\bar{x},\bar{y}) = \frac{\sqrt{k\, z_0}}{z}(x,y) \tag{32}$$

and

$$\mathcal{K} = \frac{3A^2k^2\epsilon_0 c^2}{64 z_0^2} = \frac{3\pi}{16}(k\, w_0)^{-2}\rho_0. \tag{33}$$

The heuristic interpretation is that the intense radiation effectively changes the refractive index to a value slightly larger than unity, thus increasing the optical path length followed by the beam. Since this effective refractive index depends on the local intensity and therefore on the position, the wavefront is distorted, yielding a position-dependent phase shift analogous to that caused by the Kerr effect.

By detecting the on-axis ($\bar{x} = \bar{y} = 0$) self-induced phase shift when an ultraintense pulse propagates in vacuum, a

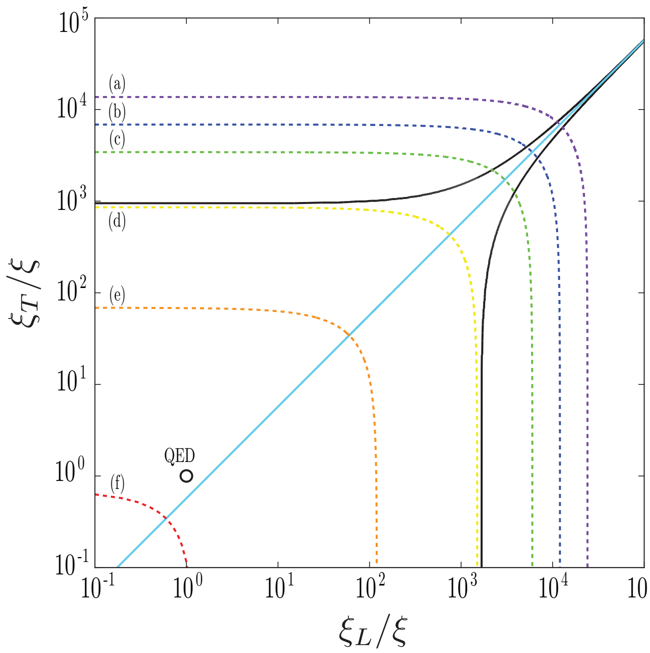

FIG. 3. (Color online) Predicted sensitivity of ξ_L and ξ_T (on a logarithmic scale) from the search for the on-axis self-induced phase shift of a laser pulse in vacuum. The region outside the solid black lines has been excluded by PVLAS experiment. The diagonal (blue) solid line corresponds to the relation $7\xi_T - 4\xi_L = 0$, predicted by BI theory. Regions outside the dashed lines can be tested with ultraintense lasers of increasing power. Line (a) corresponds to the limit achievable at a 5-PW laser facility; (b), to 10 PW; (c), to 20 PW; (d), to 80 PW; (e), to 10^3 PW; and (f), to 10^5 PW.

combination of the ξ_L and ξ_T can be measured:

$$\frac{4\xi_L + 7\xi_T}{\xi} = \left(\frac{\phi}{10^{-8}\ \mathrm{rad}}\right)\left(\frac{w_0}{\lambda}\right)^4\left(\frac{\lambda}{800\ \mathrm{nm}}\right)^2 \times \frac{4.8\times 10^{20}\ \mathrm{W}}{P}. \tag{34}$$

Figure 3 shows the resulting discovery potentials for this QV effect in the ξ_L-ξ_T parameter space for several values of the peak power P of the pulse, considering a diffraction-limited $w_0 \approx \lambda$ beam with the typical Ti:sapphire wavelength $\lambda \approx$ 800 nm. Even if the paraxial approximation breaks down near the diffraction limit, Eq. (34) yields limiting benchmark values. For the plot, we assume that the phase shift can be measured down to the level $\phi \approx 10^{-8}$ rad, as for lower intensity pulses [6,25–27]. This is certainly a technological challenge for ultraintense lasers and it is not obvious which experimental technique would give the best precision. On the other hand, the sensitivity can be noticeably enhanced with an off-axis measurement, since the phase shift increases according to Eq. (31). In an actual experiment, this fact may compensate a noise level on ϕ higher than 10^{-8} rad.

The search for self-induced phase shifts in vacuum can improve the sensitivity on the measurement of ξ_L and ξ_T for laser powers exceeding 80 PW. Even for lower powers, this kind of experiment could be used to explore the parameter region around the solid (blue) diagonal line, representing BI

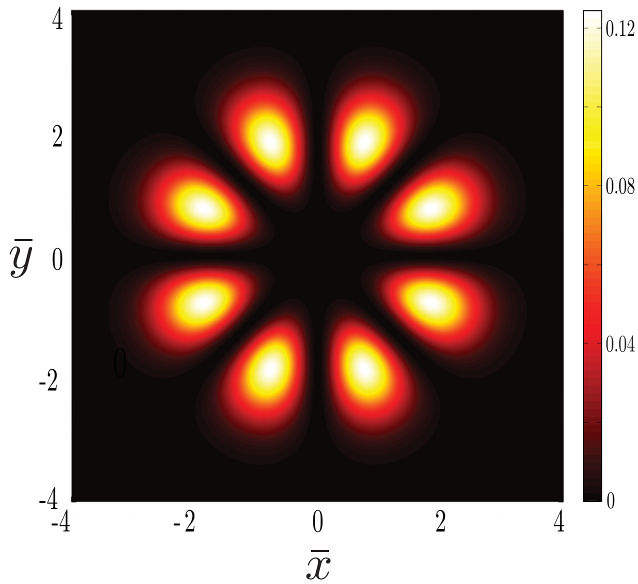

FIG. 4. (Color online) Normalized angular distribution of y-polarized photons at $z \gg z_0$, given by $\frac{1}{2}|u_{13} - u_{31}|^2 \approx \frac{4}{3\pi} e^{-(\bar{x}^2+\bar{y}^2)} \bar{x}^2\, \bar{y}^2 (\bar{y}^2 - \bar{x}^2)^2$.

theory. QED can only be tested with $P \sim 10^5$ PW, which is probably beyond the reach of near-future facilities.

B. Rotation of polarization

We now discuss the generation of y-polarized photons, orthogonal to the initial polarization of the beam; see also [18]. From Eqs. (29), we can compute the fraction of the incident power that goes into this transverse polarization:

$$\frac{P_y}{P} = \frac{3\pi^2}{2^{14}} (k\, w_0)^{-4} \left[\left(\xi_L - \frac{7}{4}\xi_T \right) \rho_0 \right]^2 . \tag{35}$$

The distribution of y-polarized photons in the transverse plane is depicted in Fig. 4.

For a possible experimental realization, not only is it necessary to produce a detectable number of y-polarized photons, but also an extremely high purity of the linear polarization of the laser source is mandatory in order to avoid undesired background. For state-of-art PW-class lasers worldwide, the so-called extinction ratio of the outgoing radiation, defined as $10 \cdot \log_{10}(P_y^{\rm laser}/P)$ dB, is typically in the range of -20 to -30 dB. However, recent developments have achieved extinction ratios $\sim\!-100$ dB for the polarization purity of x-ray sources [28]. Assuming that these improvements can be transferred to the optical domain, we can envisage a situation with $P_y^{\rm laser}/P \sim 10^{-10}$. We can compare directly the predicted sensitivity derived from searching for the generation of the orthogonal polarization P_y with the current limits provided by PVLAS, since they depend on the same combination of the parameters:

$$\frac{4\xi_L - 7\xi_T}{|4\xi_L - 7\xi_T|_{\rm PVLAS}} = \left(\frac{w_0}{\lambda}\right)^4 \left(\frac{\lambda}{800\ \rm nm}\right)^2 \times \left(\frac{P_y^{\rm laser}/P}{10^{-10}}\right)^{1/2} \frac{2.3 \times 10^{20}\ \rm W}{P}. \tag{36}$$

Taking $\lambda \approx 800$ nm, $w_0 = \lambda$, and $P_y^{\rm laser}/P \approx 10^{-10}$, we find that searching for self-induced rotation of polarization can improve the PVLAS sensitivity for $P \sim 3 \times 10^5$ PW.

C. Third harmonic

The consequences of the terms of frequency 3ω [19] can also be studied by expanding in modes. Curiously, if we insert the Gaussian beam ansatz, $f = A\, u_{00}$, the currents $\tilde{j}^{\mu}_{L,T}$ defined in (18) vanish. This nontrivial cancellation happens for any polarization of the Gaussian beam. Thus, there are two options for having third harmonic generation: either we go to the next paraxial order in the Gaussian beam description or we consider the incoming beam as a different transverse mode. We examine these possibilities in turn.

1. Gaussian beam, subleading paraxial term

The classical corrections associated with the paraxial expansion for the propagation of a Gaussian beam were discussed long ago, in [29]; see also [30] and [31]. We add to (13) the first correction term,

$$A^{\mu}_{\rm lin} = {\rm Re}\left[e^{i(\omega t - k z)} \left(0, f + g, 0, -i\frac{f_{,x}}{k} - i\frac{g_{,x}}{k} - \frac{1}{k^2} f_{,xz} \right) \right]. \tag{37}$$

The function g must satisfy $\mathcal{P}_{\omega}(g) = \frac{\partial^2 f}{\partial z^2}$. Taking the Gaussian beam $f = A\, u_{00}$, g can be written as [29]

$$g = \frac{A\, z\, (\pi\, k\, z_0)^{-\frac{1}{2}}}{8(z_0 - i\, z)^3} \left[\left(\frac{k(x^2+y^2)}{(z_0 - i\, z)} \right)^2 - \frac{8k(x^2+y^2)}{(z_0 - i\, z)} + 8 \right] \times \exp\left[-\frac{k(x^2+y^2)}{2(z_0 - i\, z)} \right]. \tag{38}$$

We can now insert (37) into (10) and (12) to compute the $\tilde{j}^i_{\rm LT}$ as defined in (18). Unlike the case of frequency ω, the currents can be written explicitly in terms of a few Hermite-Gaussian modes. The computation yields

$$\mathcal{P}_{3\omega}(\tilde{\mathcal{E}}_{x,L}) = C\left[\frac{11z + 10\, i\, z_0}{(z + i\, z_0)^6} v_{00} - \frac{\sqrt{2}}{(z + i\, z_0)^5} v_{02} \right],$$

$$\mathcal{P}_{3\omega}(\tilde{\mathcal{E}}_{x,T}) = C\left[-\frac{7(z + 2\, i\, z_0)}{4(z + i\, z_0)^6} v_{00} - \frac{7\sqrt{2}}{4(z + i\, z_0)^5} v_{20} \right],$$

$$\mathcal{P}_{3\omega}(\tilde{\mathcal{E}}_{y,L}) = C\left[-\frac{1}{(z + i\, z_0)^5} v_{11} \right],$$

$$\mathcal{P}_{3\omega}(\tilde{\mathcal{E}}_{y,T}) = C\left[\frac{7}{4(z + i\, z_0)^5} v_{11} \right],$$

where $C = \frac{A^3 c k^3 z_0}{6\sqrt{3}\pi}$. These expressions can be integrated to find the third harmonic Fourier component of the electric field as a function of (x,y,z). We get

$$\begin{aligned}\epsilon_0 c^2(\xi_L \tilde{\mathcal{E}}_{x,L} + \xi_T \tilde{\mathcal{E}}_{x,T}) &= \frac{A^3 k^2 z_0 \epsilon_0 c^3}{144\sqrt{3}\pi}\left\{\left[\frac{55z + 51i\, z_0}{5(z_0 - i\, z)^5}\xi_L - \frac{7(5z + 9i\, z_0)}{20(z_0 - i\, z)^5}\xi_T\right]v_{00}\right.\\ &\quad \left. - \frac{i\sqrt{2}}{(z_0 - i\, z)^4}\left(\xi_L v_{02} + \frac{7}{4}\xi_T v_{20}\right)\right\},\\ \epsilon_0 c^2(\xi_L \tilde{\mathcal{E}}_{y,L} + \xi_T \tilde{\mathcal{E}}_{y,T}) &= \frac{A^3 k^2 z_0 \epsilon_0 c^3}{144\sqrt{3}\pi}\left(-\xi_L + \frac{7}{4}\xi_T\right)\frac{i}{(z_0 - i\, z)^4}v_{11}.\end{aligned}$$

Note that all the coefficients vanish as $z \to \infty$. Therefore, even if third harmonic radiation is generated in the vicinity of the beam focus, it fades away due to destructive interference in the outgoing wave. This is rather similar to what happens in a non-phase-matched material. In fact, one can think of this result in the following way: in vacuum, energy-momentum conservation, in a nonlinear process in which three ω photons are converted to a single 3ω photon, requires the initial photons to be parallel. But parallel photons do not interact via QV corrections: for instance, the Lorentz invariants $F_{\mu\nu}F^{\mu\nu}$ and $\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$ vanish for a plane wave canceling the J^μ currents defined in (10). Thus, it is not possible to have phase matching and, so, efficient nonlinear effects allowing for the mentioned process. However, this argument does not straightforwardly apply to a laser beam in which $N+3$ of the ω photons could be converted to N ω photons and one 3ω photon for some integer N, because the N extra photons might potentially absorb the energy-momentum excess. Thus, we expect to find third harmonic generation to the next order in the Euler-Heisenberg expansion, which includes six-photon interactions, allowing $N = 1$ in the above argument. This is in agreement with the results reported in [19], where a number of the order of $(\xi\rho_0)^4$ 3ω photons were predicted by using a different modelization of the incoming beam.

Despite the neutralization of the outgoing wave, it is interesting to discuss the fraction of the power which is converted to third harmonic as a function of z:

$$\frac{P_{3\omega}}{P_{\rm in}} = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(|\tilde{E}_x|^2 + |\tilde{E}_y|^2)dx\,dy}{\omega^2 A^2}. \tag{39}$$

This is a bell-shaped function with a width of the order of the Rayleigh length. It reaches its maximum at $z = 0$. If we assume that $\xi_L = \xi_T = \xi$, we find

$$\left.\frac{P_{3\omega}}{P_{\rm in}}\right|_{z=0,\xi_L=\xi_T} = 0.06(\rho_0\,\xi)^2(w_0 k)^{-8}, \tag{40}$$

where ρ_0 is the energy density at the beam focus $\rho_0 = \frac{2P_{\rm in}}{\pi w_0^2 c}$.

2. *Higher transverse modes for the incoming beam*

Let us consider the TE_{01} mode, namely, take $f = A\, u_{01}$ instead of $f = A\, u_{00}$. We can compute the third harmonic wave at leading paraxial order, which turns out to be x polarized and stemming only from the ξ_L term,

$$\epsilon_0 c^2 \xi_L \tilde{\mathcal{E}}_{x,L} = \frac{i\, A^3 \epsilon_0 \omega^3 z_0^2 \xi_L}{36\pi(z_0 - i\, z)^4}v_{01}. \tag{41}$$

Again, the third harmonic wave vanishes as $z \to \infty$. The fraction of power of the incoming wave transformed to third harmonic in the focal plane $z = 0$ is

$$\left.\frac{P_{3\omega}}{P_{\rm in}}\right|_{z=0} = \frac{1}{81}\left(\frac{2P_{\rm in}\xi_L}{\pi\, w_0^2 c}\right)^2(w_0 k)^{-4}. \tag{42}$$

Note that we have expressed the right-hand side in terms of $P_{\rm in}$ instead of ρ_0, since ρ_0 was defined for the TE_{00} mode as the energy density of the incoming beam at focus $\vec{r} = 0$, a quantity that vanishes for the TE_{01} mode. Note also the different power of $w_0 k$ compared to (40), related to the order in the paraxial expansion.

Similar considerations apply if the incoming beam enters in any particular transverse mode or any linear combination of them, including vortices. The QV corrections generate a wave of frequency 3ω around the focal plane. This wave appears at leading paraxial order except for the particular TE_{00} case discussed above. In all cases, there is destructive interference and the outgoing third harmonic wave is canceled out. This fact implies that the experimental verification of the existence of third harmonic is challenging, if not impossible, as long as the paraxial approximation holds.

D. Self-steepening

The self-induced phase shift experienced by ultraintense lasers in vacuum depends on the intensity profile of the laser itself; see Eq. (31). Because of this, we can establish a formal analogy between the QV and the Kerr nonlinear media in which $n = n_0 + n_2 I$, where n_0 stands for the linear refractive index of the medium and the coefficient n_2 describes the strength of the nonlinear correction because of the presence of light. Using the results presented above, it is straightforward to show that the effective nonlinear coefficient of the QV is $n_2^{\rm vac} \approx \frac{\xi}{c}(kw_0)^{-4}$.

Bearing this analogy in mind, in the limit of very short laser pulses other higher order nonlinear effects such as optical pulse self-steepening may also take place [32]. The latter effect arises from the intensity dependence of the group velocity and typically becomes important for pulses shorter than 100 fs. The strength of self-steepening can be quantified using the dimensionless parameter $s = 1/\omega t_0$, where ω is the central angular frequency of the pulse and t_0 is the pulse duration. Remarkably, this effect implies that different parts of the pulse will experience different temporal displacements upon propagation, depending on the local intensity. In particular, the higher the intensity, the more slowly the pulse will evolve. This fact indeed leads to an interesting limiting case in which the trailing edge of the pulse catches the central part, thus creating a shock front similar to that observed in material waves. Interestingly, the characteristic distance z_s at which the shock occurs (in the absence of any dispersion or attenuation) can be estimated to be [32]

$$z_s \approx \frac{0.39 L_{\rm nl}}{s}, \tag{43}$$

where the numerical coefficient in the numerator accounts for the influence of the specific pulse shape (e.g., for a "sech"-shaped pulse it would indeed be different) and $L_{\rm nl} = c/(n_2 \omega I)$ is the so-called characteristic nonlinear distance [33], i.e., the typical distance at which nonlinear effects become dominant in the evolution of ultrashort pulses. We recall that z_0 should be comparable to z_s for the shock effect to be observed, since the Rayleigh length z_0 delimits the spatial interaction region of the ultraintense pulse with the QV.

Let us now consider a set of parameters related to some realistic experimental schemes that could be accomplished in the near-future. For typical PW-class laser systems, $P = 1$ PW, $t_0 = 30$ fs, and $\lambda = 800$ nm. Thus, assuming that one can focus the PW laser beam close to the diffraction limit, we would end up with maximum peak intensities of $\sim 10^{23}$ W/cm^2, which would then correspond to $z_s \approx 3 \times 10^4$ km, far beyond the Rayleigh length, $z_0 \approx 1\ \mu$m. Pushing to the limit, $z_s \approx 30$ m for the Schwinger intensity $I_S \approx 2.3 \times 10^{29}$ W/cm^2. Thus, this rough calculation suggests that the effect of self-steepening could not be observed at optical frequencies even in the vicinity of the Schwinger limit.

IV. DISCUSSION

As early as in 1952, it was noted that plane waves are unaffected by QV polaritzation [34]. Accordingly, all effects computed above vanish as $w_0 \to \infty$ and grow for decreasing waists, for which a Gaussian beam can be reinterpreted as a collection of plane waves crossing at increasing angles. With a single beam, these "crossing angles" are limited. The largest possible angles and therefore the largest signatures can only be achieved y making several pulses collide [12,14]. Nevertheless, synchronizing and making beams collide head-to-head near their focus is rather nontrivial. Thus, it is worthwhile to consider the present setup as a complementary possibility.

A relevant question is whether an imperfect vacuum may spoil a faint signal as a phase shift, $\phi \approx 10^{-8}$ rad. If a few molecules remain in the vacuum chamber, their leading effect on the beam comes from nonlinear Thomson scattering [35] (for high intensities, the system is in the barrier suppression regime and electrons can be considered free). In order to roughly estimate the effect on the phase, we write the electric field as a sum of incoming and scattered waves, $A\, e^{i\,\omega t} + i\,\epsilon\, e^{i\,\omega t + \phi_{\rm sc}} \approx A\, e^{i\,(\omega t + \epsilon/A)}$, leading to a phase correction of the order of $|\epsilon/A| = \sqrt{n_{\gamma,\rm sc}/n_{\gamma,i}}$, where $n_{\gamma,\rm sc}$ ($n_{\gamma,i}$) is the number of scattered (incoming) photons. To make it smaller than 10^{-8}, we need $n_{\gamma,\rm sc}/n_{\gamma,i} < 10^{-16}$. The quotient can be estimated using the results of [36] and is of the order of $\frac{p\,h\,w_0}{k_B T}\sqrt{\frac{P\,r_0^3}{m_e^3 c^5}}$, where p is the pressure and r_0 the classical electron radius. Taking, e.g., $P = 100$ PW, $T = 300$ K, $w_0 = 1\ \mu$m, we get $p < p_{\rm lim} \approx 10^{-8}$ Pa, an ultrahigh vacuum achievable with present-day technology. We remark that the pressure may also be gauged with the ultraintense laser itself [36,37].

A more constraining problem comes from the incoming pulse profile. Even if it is not a Gaussian beam, an expression similar to Eq. (31) can be found as long as the pulse can be written in terms of the modes. In any case, a spatially modulated phase shift of the order of $\mathcal{K}\,\xi$ will appear. Since the distortion of the wavefront depends on the pulse intensity, one can, in principle, think of measuring how it changes by tuning the peak power of the pulse, in a version of the P-scan technique customarily used in the determination of nonlinear optical properties of materials. However, with present-day technology, the temporal and spatial profile of ultraintense pulses is only poorly known and fluctuates from shot to shot. Precisely measuring the subtle effects discussed in this paper would be rather challenging if a better control of ultraintense pulse profiles is not developed.

V. CONCLUSION

We have discussed how QV polarization affects the propagation of ultraintense laser pulses, giving analytical expressions for the leading corrections. These effects are ubiquitous and will become increasingly important as facilities with higher peak powers are built. We have shown that the first effect that may be measured is a wavefront distortion resulting in a spatially dependent phase shift. If upgrades regarding peak power, beam quality, and precise phase measurements are met, it would be possible to search for new physics in yet unexplored parametric regions. These requirements do not seem too far-fetched for next-generation lasers, although nontrivial technological advances are necessary.

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