

Gaussian-only regenerative stations cannot act as quantum repeatersRyo Namiki,¹ Oleg Gittsovich,¹ Saikat Guha,² and Norbert Lütkenhaus¹¹*Institute for Quantum Computing and Department of Physics and Astronomy, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1*²*Quantum Information Processing Group, Raytheon BBN Technologies, Cambridge, Massachusetts 02138, USA*

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Higher transmission loss diminishes the performance of optical communication—be it the rate at which classical or quantum data can be sent reliably, or the secure key generation rate of quantum key distribution (QKD). Loss compounds with distance—exponentially in an optical fiber, and inverse square with distance for a free-space channel. In order to boost classical communication rates over long distances, it is customary to introduce regenerative relays at intermediate points along the channel. It is therefore natural to speculate whether untended regenerative stations, such as phase-insensitive or phase-sensitive optical amplifiers, could serve as repeaters for long-distance QKD. The primary result of this paper rules out all bosonic Gaussian channels to be useful as QKD repeaters, which include phase-insensitive and phase-sensitive amplifiers as special cases, for any QKD protocol. We also delineate the conditions under which a Gaussian relay renders a lossy channel *entanglement breaking*, which in turn makes the channel useless for QKD.

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I. INTRODUCTION

In recent years, performance of various communication tasks over an optical channel—when limited only by the fundamental noise of quantum mechanical origin—have been extensively studied. A few examples are as follows: finding the communication capacities of the lossy optical channel for transmitting classical information [1], quantum information [2], and that for transmitting both classical and quantum information simultaneously in the presence of a limited amount of pre-shared entanglement [3]. One of the biggest breakthroughs in optical communication using quantum effects was the invention of quantum key distribution (QKD), which is a suite of protocols that can generate information-theoretically secure shared secret keys [4] between two distant parties Alice and Bob over a lossy-noisy optical channel, with the assistance of a two-way authenticated public classical channel. Security of QKD leverages quantum properties of light to ensure the generated shared keys are secure from the most powerful adversary that is physically consistent with the channel noise collectively estimated by Alice and Bob (despite the fact that much of that noise may actually be caused by nonadversarial or natural causes). Various QKD protocols have been proposed in the last three decades [5], some of which have been transitioning to practice [6–8].

For all the communication protocols discussed above, the rates decrease rapidly with channel loss. For the task of classical communication over an ideal pure-loss channel (modeled by a beamsplitter of transmittance η), at any given value of the channel transmittance η , no matter how small, the data rate can in principle be increased without bound by increasing the input power [9]. For QKD, this is not the case. For several well-known QKD protocols (such as BB84 [10] with single photons and BB84 with weak laser light encoding and decoy states, E91 [11] with an ideal entanglement source, and CV-QKD with Gaussian modulation [12,13]), the secret key rate R decays linearly with channel transmittance η in the high-loss ($\eta \ll 1$) regime [14]. Recently, it was shown that this linear rate-transmittance scaling over the lossy bosonic channel—for secure-key generation with two-way public

classical communication assistance—is impossible to improve upon, no matter how one may design a QKD protocol, or how much input power is used [15]. To be specific, the secret key rate of any QKD protocol must be upper bounded by R_{UB} measured in bits/mode and given by

$$R_{UB} = \log_2 \frac{1 + \eta}{1 - \eta}, \quad (1)$$

which equals $R_{UB} \approx 2.88\eta$, for $\eta \ll 1$. This fundamental rate-loss upper bound also applies to the following related tasks: quantum communication (sending qubits noiselessly over a lossy channel), direct secure communication [16], and entanglement generation (where each task may also use assistance of a separate authenticated two-way classical communication channel, in addition to transmissions over the quantum lossy bosonic channel itself) [15].

As we discussed above, for classical communication over an ideal lossy channel, one could in principle increase the input power without bound as the loss increases, to maintain a required data rate. However, an unbounded input power is impractical both from the point of view of the availability of a laser that is powerful enough, and also to avoid hitting up against the fiber’s non-linearity-driven peak power constraint. This is why traditionally, electrical regenerators have been used to compensate for loss in long-haul optical fiber communications, which help restore the signal-to-noise ratio (SNR) of the digitally modulated signals by periodically detecting and regenerating clean optical pulses. Over the last few decades, all-optical amplifiers, such as erbium-doped fiber amplifiers (EDFAs), have become popular in lieu of electrical regenerators, both due to their greater speeds as well as the low noise of modern EDFAs. Caves analyzed the fundamental quantum limits on the noise performance of optical amplifiers [17], for both phase-insensitive (PIA) and phase-sensitive amplifiers (PSA). Loudon analyzed the fundamental limitations on the overall SNR to “chains” of loss segments and optical amplifiers, both in the context of phase-sensitive (coherent detection) receivers, as well as direct detection receivers [18].

For QKD, one way to beat the linear rate-transmittance scaling is to break up the channel into low-loss segments by introducing physically secured center stations; in this approach the overall key rate is still upper bounded by $R \leq \log_2[(1 + \eta')/(1 - \eta')]$ bits/mode, but η' is the transmittance of the longest (lossiest) segment. Quantum repeaters are conceptual devices [19,20], which if supplied at these intermediate stations, can beat the linear rate-transmission scaling without having to physically secure them. There is an approach to build a quantum repeater using one-way communication only [21], so they can act as passive untended devices. However, such structured implementations of those devices require quantum error correction codes operating on blocks of multiple qubits. A recently proposed repeater protocol [22] even eliminates the requirement of a quantum memory, but utilizes photonic cluster states. Building a functional quantum repeater is subject to intensive fundamental research, but is currently far from being a deployable technology. The natural question that thus arises—in analogy to Loudon’s setup for classical optical communication [18]—is whether all-optical amplifiers (PIAs or PSAs), left untended and inserted at regular intervals, might act to some degree as quantum repeaters and thereby help boost the distances over which QKD can be performed over a lossy channel.

The remainder of the paper is organized as follows. First, in Sec. II we summarize the main results derived in this article to put it into perspective. A central finding is a decomposition of a lossy quantum channel with intermediate bosonic Gaussian channel stations into another form without any insertion of middle stations as depicted in Fig. 1. We then continue into the technical part. In Sec. III, we give an overview of bosonic Gaussian states and channels. In Sec. IV, we analyze the scenario when a general multimode Gaussian channel is inserted between two pure-loss segments, and show how one can collect the entire pure loss in the center of the channel by

appropriate modifications to the transmitter and the receiver. In Sec. V, we consider single-mode Gaussian stations, and delineate the conditions for when the Gaussian center station renders the concatenation with the losses on its two sides, an entanglement-breaking channel. The quantum limited stations, the PSA and the PIA, are addressed as special cases. We conclude in Sec. VI with a summary of the main results, and thoughts for future work.

II. OUTLINE OF MAIN RESULTS

In this paper, we show that untended operations represented by Gaussian channels cannot serve as quantum repeater stations when inserted into lossy bosonic channels. Note that by using the word “channel” we automatically imply the action to be trace preserving. Examples of such channels involve beamsplitters, phase shifters, and squeezers, and as a combination thereof also PSA and PIA [23]. We prove our claim by transforming a concatenation of two lossy channel segments with a Gaussian channel in the middle, as a pair of Gaussian channels at the two ends, with the total loss collected in the middle (see Fig. 1). The implication of our no-go result is that if Gaussian channels are employed in center station(s) placed along a lossy channel, the overall QKD key rate, for any QKD protocol, must be upper bounded by $R_{UB} = \log_2[(1 + \eta)/(1 - \eta)]$ bits/mode, with η being the total end-to-end channel transmittance. Simple protocols such as laser-decoy-based BB84, or Gaussian-modulated laser-based CV protocols, with no repeaters, can already attain key rates that have the optimal (linear) rate-transmittance scaling and are only a small constant factor below the general upper bound [5].

For any optical communication protocol over a lossy channel interspersed with Gaussian stations, our result shows there exists *another* protocol with the same performance that does not use any intermediate station, which can be derived from the original protocol by suitably amending the transmitted signals and the receiver measurement. Our result does *not* preclude a Gaussian channel in the middle of the lossy channel to improve the performance of a *given* protocol, *if* the transmitter and receiver are held to be the same. Nor does it preclude the existence of scenarios where it might be technologically easier to implement a protocol with such intermediate stations, as opposed to modifying the transmitter and the receiver per the prescription generated by our analysis. An example of such improvement is the increased range of a QKD protocol with a given level of detector noise (although, any increase in range must be consistent with the $R \sim \eta$ rate-transmission scaling).

Given that the overall rate-transmission scaling cannot be changed, the question remains whether there might be other implementation advantages of Gaussian center stations. It turns out that there are strict conditions on such a scenario. To demonstrate this, we delineate the conditions under which a Gaussian center station causes a lossy channel to become entanglement breaking (EB) [24–26]. It is well known that QKD is not possible on an EB channel, since the output of an EB channel can be simulated quantitatively correctly using a measure-and-prepare scheme [27]. The pure lossy channel is not EB by itself for any nonzero transmittance, $\eta > 0$.

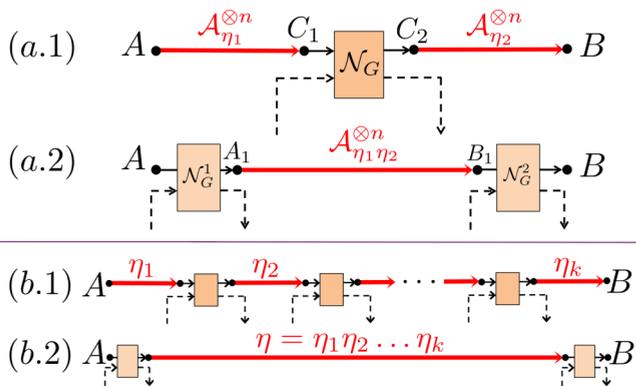


FIG. 1. (Color online) (a) Any n -mode Gaussian channel \mathcal{N}_G sandwiched between two pure-loss channel segments $A_{\eta_1}^{\otimes n}$ and $A_{\eta_2}^{\otimes n}$, respectively, can be decomposed into a single lossy channel $A_{\eta_1 \eta_2}^{\otimes n}$ sandwiched by a pair of Gaussian channels, \mathcal{N}_G^1 and \mathcal{N}_G^2 . The net loss in the channel is the sum (in dB) of the losses of the two individual lossy segments, i.e., $\eta = \eta_1 \eta_2$ and the Gaussian channel at the receiver end \mathcal{N}_G^2 is a Gaussian unitary map. (b) Using this transformation recursively, one can “push” a collection of general Gaussian center stations interspersed through a lossy channel (b.1) to a single Gaussian operation at the input, and a single Gaussian operation at the output (b.2) of the entire loss accumulated in the center.

Let us illustrate our reasoning for the better known case of classical communication over pure-loss bosonic channels. The channel capacity of the lossy bosonic channel (described by single photon transmittance η) using signals with mean photon number \bar{n} per mode, is given by $g(\eta \bar{n}) = (1 + \eta \bar{n}) \log_2(1 + \eta \bar{n}) - \eta \bar{n} \log_2 \eta \bar{n}$ bits per mode [1]. We see that increasing the mean photon number increases the classical communication rate. In practice, it is impractical to keep increasing the mean photon number due to nonlinear effects in the fiber that limit the input power and can distort the signals. For these reasons, one limits the input power and builds optical amplifiers (phase sensitive or phase insensitive) into the fiber. According to our theorems, for the ideal loss-only bosonic channel, the setup of lossy segments with intermediate amplifiers is equivalent to a new transmitter consisting of the old transmitter combined with a very strong amplifier, followed by a transfer through the full distance of the lossy bosonic channel, and then a receiver consisting of a combination of another amplifier and on the original receiver. This replacement protocol corresponds to the situation of using a large input mean photon number, and realizes the classical capacity of the lossy bosonic channel. What we learn is that the intermediate amplifiers do not increase the channel capacity of the lossy bosonic channel, but realize an equivalent protocol that keeps the optical signals—throughout the communication channel—within a peak power level that is sufficiently below the level where nonlinear effects would be encountered.

In QKD, the secrecy capacity of the lossy bosonic channel does not increase unboundedly with the input power of the signals, thus using strong signal pulses pushing into the nonlinear domain of fibers is not important for QKD protocols: The use of equivalent replacement schemes utilizing optical amplifiers would not give any advantage. To the contrary, amplifiers will add additional noise which will eventually be detrimental to the performance of the QKD protocol, with the exception of effects of noisy pre-processing that can increase the secret key rate compared to protocols not using this approach [28]. Note, however, that noisy pre-processing cannot improve on the fundamental secrecy capacity of the lossy bosonic channel, which is solely a function of the channel's end-to-end loss.

Our main result adds to the list of no-go results for Gaussian operations in quantum information protocols, i.e., those that cannot be performed with Gaussian operations and classical processing alone. Some examples are universal quantum computing [29], entanglement distillation of Gaussian states [30–32], optimal cloning of coherent states [33], optimal discrimination of coherent states [34–37], Gaussian quantum error correction [38], and building a joint-detection receiver for classical communication [39].

III. GAUSSIAN STATES AND CHANNELS

In this section, we will provide a basic introduction to the mathematics of Gaussian states and channels, sufficient to develop the results in this paper. For a more detailed account, see Ref. [32]. A quantum state ρ of an n -mode bosonic system is uniquely described by its characteristic function,

$$\chi(\mu) = \text{Tr}[\rho \mathcal{W}(\mu)], \quad (2)$$

where the Weyl operator, $\mathcal{W}(\mu) = \exp[-i\mu^T R]$, with $R = [\hat{x}_1, \dots, \hat{x}_n, \hat{p}_1, \dots, \hat{p}_n]^T$ consisting of field quadrature operators of the n modes satisfying the commutation relations $[\hat{x}_k, \hat{p}_l] = i\delta_{kl}$, with $\mu = [\mu_1, \dots, \mu_{2n}]$ a $2n$ -length real vector. The characteristic function of a Gaussian state ρ is given by

$$\chi_\rho(\mu) = \exp\left[-\frac{i}{4}\mu^T \gamma \mu + i d^T \mu\right], \quad (3)$$

where the $2n \times 2n$ matrix γ is the covariance matrix (CM) and the $2n$ -length vector $d := (\langle \hat{x} \rangle, \langle \hat{p} \rangle)^T$ is the mean, or the displacement vector (DV), of ρ . The Gaussian state ρ can thus be described uniquely by the pair (γ, d) . Due to the canonical uncertainty relation, any CM of physical states has to satisfy

$$\gamma \geq \frac{i}{2}\sigma, \quad (4)$$

where

$$\sigma := \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}. \quad (5)$$

A Gaussian unitary operation U_G transforms a Gaussian state (γ, d) to a Gaussian state (γ', d') as

$$\gamma' = M^T \gamma M, \quad d' = M^T d, \quad (6)$$

where M is a symplectic matrix that satisfies

$$M^T \sigma M = \sigma. \quad (7)$$

A Gaussian channel \mathcal{E} can be described by a triplet (K, m, α) [26]. It transforms a state (γ, d) to the state (γ', d') as

$$\gamma' = K^T \gamma K + \alpha, \quad d' = K^T d + m. \quad (8)$$

From the regularity of CMs in Eq. (4) the physical condition for the pair (K, α) is given by

$$\alpha \geq \frac{i}{2}(\sigma - K^T \sigma K). \quad (9)$$

Composition of two Gaussian channels \mathcal{E}_1 and \mathcal{E}_2 yields another Gaussian channel $\mathcal{E}_{12} = \mathcal{E}_2 \circ \mathcal{E}_1$, where

$$\begin{aligned} K_{12} &= K_1 K_2, \\ m_{12} &= K_2^T m_1 + m_2, \quad \text{and} \\ \alpha_{12} &= K_2^T \alpha_1 K_2 + \alpha_2. \end{aligned} \quad (10)$$

In this paper, we will focus on Gaussian channels with $m = 0$. In Appendix A we show an explicit calculation demonstrating how mean displacement terms can be separated out in any concatenation of Gaussian channels.

In the following subsections, we will delve a little deeper into properties of single-mode Gaussian channels that we use later on.

A. Decomposing a Gaussian unitary operation

The symplectic matrix M in Eq. (6) of a Gaussian unitary can always be decomposed as

$$M = B \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda^{-1} \end{pmatrix} B', \quad (11)$$

where Λ is a positive diagonal matrix, and B, B' are orthogonal symplectic matrices ($B^T = B^{-1}$) [23]. This implies that any n -mode Gaussian unitary operation U_G can be realized by

a passive linear optic circuit B (a circuit involving only beamsplitters and phase shifters [40]), followed by n parallel (tensor-product) single-mode squeezers, followed by another n -mode passive linear optic circuit B' [23]. Therefore a general Gaussian unitary operation can always be decomposed into passive linear optics (beamsplitters and phase shifters), single-mode squeezing and single-mode displacement operations.

Therefore, up to a displacement, a single-mode Gaussian unitary (described by its symplectic matrix M) can be decomposed as

$$M = R_\theta S_G R_\phi, \quad (12)$$

where

$$R_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \quad (13)$$

is the symplectic matrix of a single-mode phase rotation, and

$$S_G = \begin{pmatrix} \sqrt{G} + \sqrt{G-1} & 0 \\ 0 & \sqrt{G} - \sqrt{G-1} \end{pmatrix} \quad (14)$$

is the symplectic matrix of a single-mode (phase-quadrature) squeezer. Note that it is sufficient to restrict the above decomposition to a phase-quadrature squeezer because one can absorb any additional phase in the squeezing operation into R_θ and R_ϕ . This is because the symplectic matrix of a single-mode squeezer with gain G and squeezing angle θ' can be expressed as

$$S_{G,\theta'} = R_{\theta'} S_G R_{\theta'}^\dagger. \quad (15)$$

B. Entanglement breaking channels

An *entanglement breaking* (EB) channel is one whose action on one-half of an entangled state (with an identity map on the other half) always yields a *separable* state. An EB channel can always be written in a *measure-and-prepare* form [24,26]. [See also Eq. (16) below.] Any concatenation of n (not necessarily Gaussian) channels, $\mathcal{E}_n \circ \dots \circ \mathcal{E}_2 \circ \mathcal{E}_1$ is EB if one of channels \mathcal{E}_i is EB. It is instructive to see the argument explicitly for $n = 3$. Consider the serially concatenated channel, $\mathcal{E}_t = \mathcal{E}_3 \circ \mathcal{E}_2 \circ \mathcal{E}_1$, where the center station \mathcal{E}_2 is EB. Supposing its measure-and-prepare form is given by $\mathcal{E}_2(\rho) = \sum_k \text{Tr}(M_k \rho) \sigma_k$, with $M_k \geq 0$ and $\sigma_k \geq 0$, we can write \mathcal{E}_t in a measure-and-prepare form,

$$\mathcal{E}_t(\rho) = \sum_k \text{Tr}[M_k \mathcal{E}_1(\rho)] \mathcal{E}_3(\sigma_k) = \sum_k \text{Tr}[M'_k \rho] \sigma'_k, \quad (16)$$

where $\sigma'_k = \mathcal{E}_3(\sigma_k) \geq 0$, and it is straightforward to show that $M'_k = \sum_i A_i^\dagger M_k A_i \geq 0$, where $\{A_i\}$ represent Kraus operators of \mathcal{E}_1 , i.e., $\mathcal{E}_1(\rho) = \sum_i A_i \rho A_i^\dagger$.

The measure-and-prepare representation of an EB channel implies that the channel's quantum transmission can be seen as transmission of the (probabilistic) classical information obtained as a result of a hard quantum measurement made on the channel's input. This is the intuition behind why such a channel has zero secret-key capacity, and thus cannot be useful for QKD [27]. Because of this reason, when we analyze concatenations of several Gaussian center stations for potential use as repeaters, we will limit our discussion to the case when

all the channels \mathcal{E}_i in the concatenation are non-EB (since this is a necessary condition for QKD). Note, however, that when interspersed with loss segments, even when all center stations are non-EB, the overall input-output map can become EB—a topic that we will discuss in more detail later in Sec. V.

C. Unitary-equivalence classification for single-mode Gaussian channels

Our analysis of general one-mode Gaussian operations will be based on the standard forms of such operations obtained from the unitary equivalence classification of quantum channels developed by Holevo [25,26]. We say that two quantum channels Φ and Φ_S are *unitary equivalent* if there exist unitary operators U_V, U_W such that

$$\Phi_S(\rho) = U_W \Phi(U_V \rho U_V^\dagger) U_W^\dagger. \quad (17)$$

If U_V and U_W above are Gaussian, we say Φ and Φ_S are *Gaussian unitary equivalent*. If a single-mode Gaussian channel $\mathcal{E} \triangleq (K, m, \alpha)$ is not an EB channel, it must be Gaussian unitary equivalent to a channel belonging to one of the following two classes.

(i) *Phase insensitive channel (PIC)*. This class of channels is described by the triplet $(K, 0, \alpha)$, with

$$K = \sqrt{\kappa} \mathbb{1}_2, \quad \text{and} \quad \alpha = (|1 - \kappa|/2 + N) \mathbb{1}_2, \quad (18)$$

where $N \geq 0$ is the excess noise parameter and $\kappa \geq 0$ is a gain parameter. We will denote this channel as \mathcal{A}_κ^N . It acts on the canonical quadratures phase insensitively. When the gain $\kappa \geq 1$, we call it the *phase-insensitive amplifier* (PIA). When $\kappa < 1$, we call it the lossy bosonic channel (with excess thermal noise N). In this case, κ is the channel's transmittance, the fraction of the input photons that appear at the channel's output. We will use the shorthand notation, $\mathcal{A}_\kappa \equiv \mathcal{A}_\kappa^0$ for a quantum-limited phase-insensitive amplifier, or a pure-loss channel, for $\kappa \geq 1$ and $\kappa < 1$, respectively.

It is known that the PIC is EB if and only if [26],

$$N \geq \min(1, \kappa). \quad (19)$$

In our analysis we will assume that the PIC is not EB, i.e., $N \in [0, \min(1, \kappa))$. Furthermore, using the composition rule of Eq. (10), it is easy to see that any single-mode rotation (unitary) \mathcal{R} commutes with a PIC, i.e., $\mathcal{R} \circ \mathcal{A}_\kappa^N = \mathcal{A}_\kappa^N \circ \mathcal{R}$.

(ii) *Additive noise channel (ANC)*. This is a class of *phase-insensitive* Gaussian channels that adds rank-1 noise to the input state, and is described by the triplet $(K, 0, \alpha)$, with

$$K = \mathbb{1}_2, \quad \text{and} \quad \alpha = \frac{1}{2} \text{diag}(0, \epsilon), \quad (20)$$

where the noise parameter $\epsilon > 0$. We will denote this channel as \mathcal{I}^ϵ , and will call it the additive noise channel (ANC).

IV. GAUSSIAN REGENERATIVE STATIONS IN A LOSSY CHANNEL

In this section we investigate lossy bosonic channels that have intermediate Gaussian channels inserted at some intervals. We will show that such an arrangement is still equivalent (up to Gaussian operations at the entrance and the exit) to a lossy bosonic channel with the total loss of the original loss segments. As a consequence, insertion of

Gaussian channels cannot increase the secrecy capacity of the lossy bosonic channel.

The setup for the main result of this paper is schematically depicted in Fig. 1. Consider a pure-loss optical channel \mathcal{A}_η with a given amount of total end-to-end (A to B) transmittance $\eta \in (0, 1]$. Let us place a Gaussian center station—a quantum channel, or a trace-preserving completely positive map, $\mathcal{N}_G^{C_1 \rightarrow C_2}$ —somewhere in the middle, thereby splitting \mathcal{A}_η into two pure-loss segments: a pure-loss channel with transmittance η_1 , \mathcal{A}_{η_1} (A to C_1), and a pure-loss channel with transmittance η_2 , \mathcal{A}_{η_2} (C_2 to B), such that $\eta_1 \eta_2 = \eta$. We show that the overall channel action from A to B is unaffected by the transformation shown in Fig. 1(a), which replaces the Gaussian center station $\mathcal{N}_G^{C_1 \rightarrow C_2}$ by a Gaussian operation $\mathcal{N}_G^{1 A \rightarrow A_1}$ at the input of the channel and a Gaussian operation $\mathcal{N}_G^{2 B_1 \rightarrow B}$ at the output of the channel. By applying this transformation recursively, it is easy to see that one can replace any number of Gaussian center stations interspersed through the lossy channel \mathcal{A}_η into two Gaussian operations, at the input and the output, respectively.

Let us consider an n -mode lossy bosonic channel $\mathcal{A}_\eta^{\otimes n} \triangleq (K_0, 0, \alpha_0)$ with

$$K_0 = \sqrt{\eta} \mathbb{1}_{2n}, \quad \alpha_0 = \frac{1 - \eta}{2} \mathbb{1}_{2n}. \quad (21)$$

Let $\mathcal{N}_G \triangleq (K, 0, \alpha)$ denote an n -mode Gaussian channel, which we consider as the candidate for a center station. Note that this Gaussian center station could act collectively on n spatial and/or temporal modes of the propagating field. The main result of this section is the proof of the following proposition, also depicted schematically in Fig. 1.

Proposition 1. For any n -mode Gaussian channel \mathcal{N}_G there exists a Gaussian channel \mathcal{N}_G^1 and a Gaussian unitary channel \mathcal{N}_G^2 that satisfy

$$\mathcal{A}_{\eta_2}^{\otimes n} \circ \mathcal{N}_G \circ \mathcal{A}_{\eta_1}^{\otimes n} = \mathcal{N}_G^2 \circ \mathcal{A}_{\eta_1 \eta_2}^{\otimes n} \circ \mathcal{N}_G^1. \quad (22)$$

Proof. Our goal is to find a pair of Gaussian channels \mathcal{N}_G^1 and \mathcal{N}_G^2 that satisfies the physical condition Eq. (9). From the composition rule of Eq. (10) we find the total channel action $\Phi_t := \mathcal{A}_{\eta_2}^{\otimes n} \circ \mathcal{N}_G \circ \mathcal{A}_{\eta_1}^{\otimes n}$ can be described by $\Phi_t \triangleq (K_t, 0, \alpha_t)$ with

$$\begin{aligned} K_t &= \sqrt{\eta_1 \eta_2} K, \\ \alpha_t &= \eta_2 \left(\frac{1 - \eta_1}{2} K^T K + \alpha \right) + \frac{1 - \eta_2}{2} \mathbb{1}_{2n}. \end{aligned} \quad (23)$$

We will prove the proposition by constructing the required Gaussian channels using a symplectic matrix denoted by M . The properties of this matrix and its existence are the subject of the following theorem.

Theorem 2. For a given α_t in Eq. (23), there exists a CM matrix γ' and a symplectic matrix M such that

$$\alpha_t = \eta_1 \eta_2 \alpha + (1 - \eta_1 \eta_2) \gamma', \quad (24)$$

and

$$M^T \gamma' M \geq \frac{1}{2} \mathbb{1}_{2n}. \quad (25)$$

Proof. From the physical condition of a Gaussian channel in Eq. (9), we have

$$\frac{i\sigma}{2} \leq \frac{i\sigma}{2} + \frac{1}{2} K^T (\mathbb{1}_{2n} - i\sigma) K \quad (26)$$

$$= \frac{1}{2} [K^T K + i(\sigma - K^T \sigma K)] \quad (27)$$

$$\leq \frac{1}{2} (K^T K + 2\alpha), \quad (28)$$

where we used in the first line that the matrix $\mathbb{1}_{2n} - i\sigma$ is positive semidefinite and in the last line that \mathcal{N}_G is a physical channel. Our calculation implies $\gamma := \frac{1}{2}(K^T K + 2\alpha)$ is a CM of an n -mode Gaussian state due to Eq. (4). Consider now the convex combination of this CM with the CM of the n -mode vacuum state $\gamma' := p\gamma + (1-p)\frac{1}{2}\mathbb{1}_{2n}$ with mixing probability $p = \eta_2(1 - \eta_1)/(1 - \eta_1 \eta_2) \in [0, 1]$. A straightforward calculation verifies that $\alpha_t = \eta_1 \eta_2 \alpha + (1 - \eta_1 \eta_2) \gamma'$. As γ' is a valid CM, there exists a symplectic matrix M such that one obtains a diagonal form $M^T \gamma' M \geq \frac{1}{2} \mathbb{1}_{2n}$, which corresponds to a product of thermal states. ■

We are now in a position to define the Gaussian channels $\mathcal{N}_G^1 \triangleq (\tilde{K}, 0, \tilde{\alpha})$ and $\mathcal{N}_G^2 \triangleq (M^{-1}, 0, 0)$ with the help of

$$\tilde{K} = \frac{1}{\sqrt{\eta_1 \eta_2}} K_t M \quad (\equiv K M), \quad (29)$$

$$\tilde{\alpha} = \frac{1}{\eta_1 \eta_2} \left(M^T \alpha_t M - \frac{1 - \eta_1 \eta_2}{2} \mathbb{1}_{2n} \right). \quad (30)$$

To show that the channels are proper physical channels, we can concentrate on \mathcal{N}_G^1 since \mathcal{N}_G^2 corresponds to a unitary Gaussian channel. To prove that \mathcal{N}_G^1 is physical, we use in a first step the results of theorem 2, and then the physicality constraints on the channel \mathcal{N}_G , followed by a rewriting of the variables. These steps allow us to obtain

$$\begin{aligned} \tilde{\alpha} &\geq M^T \alpha M \\ &\geq M^T \frac{i}{2} (\sigma - K^T \sigma K) M \\ &= \frac{i}{2} (\sigma - \tilde{K}^T \sigma \tilde{K}), \end{aligned} \quad (31)$$

Hence, $\mathcal{N}_G^1 \triangleq (\tilde{K}, 0, \tilde{\alpha})$ is a valid Gaussian channel. It is again straightforward to verify that $\Phi_t = \mathcal{N}_G^2 \circ \mathcal{A}_{\eta_1 \eta_2}^{\otimes n} \circ \mathcal{N}_G^1$. This proves proposition 1. ■

Overall, we showed the equivalence of a bosonic Gaussian channel sandwiched between two lossy bosonic channels to a single lossy bosonic channel, bearing the total loss of the the original bosonic channels, and now sandwiched between two Gaussian channels. This corresponds to the conversion of (a.1) into (a.2) of Fig. 1. A simple iteration of this result shows that any pattern of Gaussian channels interspersed between loss segments can be rearranged into a lossy bosonic channel sandwiched between Gaussian channels [see Fig. 1(b)].

As the initial Gaussian channel can be combined with the state preparation, and the final Gaussian channel can be combined with the detection setup, it is evident that the total secret key rate of this arrangement is still bound by R_{UB} of Eq. (1).

V. ENTANGLEMENT-BREAKING CONDITIONS FOR SINGLE-MODE CENTER STATIONS

As discussed in the introduction, there might be practical reasons one want to use interspersed intermediate stations, even if the resulting key rate is still limited by the bound of Eq. (1). In this section we will demonstrate severe restrictions on the situations where such an advantage may exist. To do so, we will investigate when such a sequence of lossy channels and Gaussian center stations becomes entanglement breaking (EB) so that its secrecy capacity goes to zero [27]. In the following part of this article we execute the central first step of such an investigation and focus on single-mode Gaussian channels.

A pure-loss channel is not EB by itself, but increasing loss could make the channel progressively more fragile and susceptible to being EB when concatenated with other Gaussian operations, such as amplifiers. In the following subsections, we show the explicit conditions on the parameters of a Gaussian non-EB center station \mathcal{N}_G , such that the composition $\Phi_0 \equiv \mathcal{A}_{\eta_2} \circ \mathcal{N}_G \circ \mathcal{A}_{\eta_1}$ is EB, and specialize the conditions to the cases when \mathcal{N}_G is either a PSA or a PIA.

A. General non-EB center stations

There is no point in considering EB center stations \mathcal{N}_G as they would trivially render Φ_0 EB. Any single-mode Gaussian non-EB station \mathcal{N}_G is unitary-equivalent to either a phase insensitive channel (PIC) or an additive noise channel (ANC) (see Sec. III C). In order to evaluate EB conditions, we go deeper into decomposing Φ_0 , as depicted in Fig. 2. The two branches of Fig. 2 consider decompositions when \mathcal{N}_G is unitary equivalent to a PIC or an ANC, respectively.

(1) \mathcal{N}_G unitary equivalent to a PIC \mathcal{A}_g^N . Since phase rotations commute with PICs, it is straightforward to see that the concatenated channel $\Phi_0 \equiv \mathcal{A}_{\eta_2} \circ \mathcal{N}_G \circ \mathcal{A}_{\eta_1}$ is unitary equivalent to a channel $\Phi \equiv \mathcal{A}_{\eta_2} \circ \mathcal{S}_{G_2, \theta} \circ \mathcal{A}_g^N \circ \mathcal{S}_{G_1} \circ \mathcal{A}_{\eta_1}$ [see Fig. 2(b), lines 1.a and 1.b], where \mathcal{S}_{G_1} denotes a phase-quadrature squeezer (PSA) with gain G_1 , and \mathcal{S}_{G_2} is another PSA with gain G_2 and squeezing angle θ . It is easy to deduce the parameters for the channel Φ as

$$K_{\text{PIC}} = \sqrt{g\eta_1\eta_2} S_1 R_\theta S_2 R_\theta^\dagger, \quad (32)$$

$$\alpha_{\text{PIC}} = \frac{1}{2} [\eta_2(1 - \eta_1)g K_\theta^T S_1^2 K_\theta + \eta_2(|g - 1| + 2N)K_\theta^T K_\theta + (1 - \eta_2)\mathbb{1}_2], \quad (33)$$

respectively, where $K_\theta = R_\theta S_2 R_\theta^\dagger$, and

$$S_i = \begin{pmatrix} \sqrt{G_i} + \sqrt{G_i - 1} & 0 \\ 0 & \sqrt{G_i} - \sqrt{G_i - 1} \end{pmatrix}, \quad (34)$$

$$R_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}. \quad (35)$$

The following theorem shows that the total unitary equivalent channel Φ is further unitary equivalent to a PIC Φ_s , as shown in line 1.c of Fig. 2(b).

Theorem 3. Φ is unitary equivalent to a channel Φ_s that is a PIC $\mathcal{A}_{\eta_s}^N$, whose descriptive parameters (K_s, α_s) are given by

$$K_s = V K W = \sqrt{g\eta_1\eta_2} \mathbb{1}_2, \text{ and} \\ \alpha_s = W^T \alpha_{\text{PIC}} W = \sqrt{\det(\alpha_{\text{PIC}})} \mathbb{1}_2, \quad (36)$$

where V and W are Gaussian unitaries.

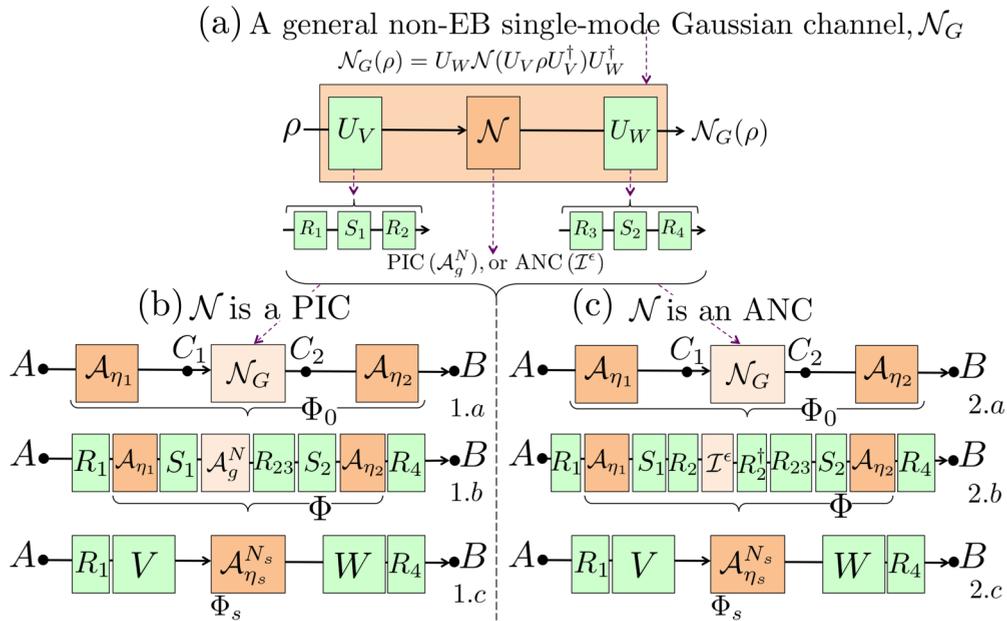


FIG. 2. (Color online) A general single-mode non-EB Gaussian channel \mathcal{N}_G is unitary equivalent to a Gaussian channel \mathcal{N} , which can be one of two forms, a *phase-insensitive channel* (PIC), \mathcal{A}_g^N , or a phase-sensitive *additive noise channel* (ANC) \mathcal{T}^c . For both cases of the center stations sandwiched between two lossy segments \mathcal{A}_{η_1} and \mathcal{A}_{η_2} , the total channel action $\Phi_0 \equiv \mathcal{A}_{\eta_2} \circ \mathcal{N}_G \circ \mathcal{A}_{\eta_1}$ is shown to be unitary equivalent to a PIC channel $\mathcal{A}_{\eta_s}^N$ as in (b)1.c and (c)2.c. Green-shaded boxes denote single-mode unitary (reversible) operations, whereas red-shaded boxes denote (in-general irreversible) actions of a single-mode quantum channel—a trace-preserving completely positive map.

Proof. See Appendix B. ■

Comparing Eq. (36) with Eq. (18), it is easy to see that Φ_s is in fact a PIC, $\mathcal{A}_{\eta_s}^{N_s}$, with

$$\begin{aligned} \eta_s &= g\eta_1\eta_2, \text{ and} \\ N_s &= \sqrt{\det(\alpha_{\text{PIC}})} - \frac{|1 - \eta_s|}{2}. \end{aligned} \quad (37)$$

The condition under which the Gaussian center station \mathcal{N}_G causes the lossy channel to be an EB channel is determined by applying Eq. (19) to the parameters of Φ_s in Eq. (37) since it is unitary equivalent to a PIC. Therefore, the channel Φ_0 is EB if

$$\sqrt{\det(\alpha_{\text{PIC}})} \geq \frac{1}{2}(1 + g\eta_1\eta_2). \quad (38)$$

(2) \mathcal{N}_G unitary equivalent to an ANC \mathcal{I}^ϵ . It is straightforward to deduce [see line 2.b in Fig. 2(c)] that, $\Phi_0 \equiv \mathcal{A}_{\eta_2} \circ \mathcal{N}_G \circ \mathcal{A}_{\eta_1}$ is unitary equivalent to a channel $\Phi = \mathcal{A}_{\eta_2} \circ \mathcal{S}_{G_2, \theta} \circ (\mathcal{R}_{-\phi} \circ \mathcal{I}^\epsilon \circ \mathcal{R}_\phi) \circ \mathcal{S}_{G_1} \circ \mathcal{A}_{\eta_1}$, whose parameters are given by

$$\begin{aligned} K_{\text{ANC}} &= \sqrt{\eta_1\eta_2} S_1 K_\theta, \quad \text{and} \\ \alpha_{\text{ANC}} &= \frac{1}{2}[\eta_2(1 - \eta_1)K_\theta^T S_1^T K_\theta + \eta_2 K_\theta^T \epsilon' K_\theta + (1 - \eta_2)\mathbb{1}_2], \end{aligned} \quad (39)$$

$$(40)$$

where \mathcal{R}_ϕ is a phase rotation, $K_\theta = R_\theta S_2 R_\theta^\dagger$, and

$$\epsilon' := \frac{1}{2} R_\phi \begin{pmatrix} 0 & 0 \\ 0 & \epsilon \end{pmatrix} R_{-\phi}^\dagger. \quad (41)$$

Next we prove that Φ is unitary equivalent to a PIC Φ_s [see line 2.c of Fig. 2(b)].

Theorem 4. Φ is unitary equivalent to a PIC Φ_s described by

$$\begin{aligned} K_s &= \sqrt{\eta_1\eta_2} \mathbb{1}_2, \quad \text{and} \\ \alpha_s &= \sqrt{\det(\alpha_{\text{ANC}})} \mathbb{1}_2. \end{aligned} \quad (42)$$

The excess noise parameter is given by

$$N_s = \sqrt{\det(\alpha_{\text{ANC}})} - \frac{|1 - \eta_1\eta_2|}{2}. \quad (43)$$

Proof. As in the proof of Theorem 3 (see Appendix B), we can simultaneously diagonalize K and α . This follows Eq. (42). Comparing Eqs. (42) and (18), we can determine the parameter N_s in Eq. (43). ■

The condition under which the Gaussian center channel \mathcal{N}_G causes the lossy channel to be an EB channel is determined by applying Eq. (19) to the parameters of Φ_s in Eqs. (42) and (43) since it is unitary equivalent to a PIC. The condition for Φ_0 to be EB, translates to

$$\sqrt{\det(\alpha_{\text{ANC}})} \geq \frac{1}{2}(1 + \eta_1\eta_2). \quad (44)$$

B. Explicit examples of a Gaussian center station: optical amplifiers

In this subsection we illustrate our results in Sec. V A with the important example of optical amplifiers used as center stations. We will consider the cases of a phase-sensitive amplifier (PSA) and phase-insensitive amplifier (PIA). Detailed proofs of the results will be deferred to Appendix C. If the center station \mathcal{N}_G is a PSA of gain G_{PSA} , then the composition

$\Phi_0 \equiv \mathcal{A}_{\eta_2} \circ \mathcal{N}_G \circ \mathcal{A}_{\eta_1}$ becomes EB if the gain G_{PSA} exceeds a threshold $G_{\text{PSA}}^{\text{thres}}$ as

$$G_{\text{PSA}} \geq G_{\text{PSA}}^{\text{thres}} := 1 + \frac{\eta_1}{(1 - \eta_1)(1 - \eta_2)}. \quad (45)$$

(See Appendix C 1 for proof.) If \mathcal{N}_G is a PIA of gain G_{PIA} , then Φ_0 becomes EB if the Gain G_{PIA} exceeds a threshold value $G_{\text{PIA}}^{\text{thres}}$ (see Appendix C 2 for proof):

$$G_{\text{PIA}} \geq G_{\text{PIA}}^{\text{thres}} := \frac{1}{1 - \eta_1}. \quad (46)$$

Note that the transmittance η_2 of the loss segment *after* the PIA does not play a role in determining when Φ_0 becomes EB. The expression for the threshold shows that when the channel transmittance η_1 of the initial is low, an amplifier with even a small amount of gain can render the lossy channel EB. Finally, the concatenation of a chain of PSA center stations, $\eta_1 \rightarrow \text{PSA}(G_1) \rightarrow \eta_2 \rightarrow \text{PSA}(G_2) \rightarrow \dots \rightarrow \text{PSA}(G_k) \rightarrow \eta_{k+1}$, can be decomposed as $\mathcal{N}_G^2 \circ \mathcal{A}_{\eta_1\eta_2\dots\eta_{k+1}} \circ \mathcal{N}_G^1$, where \mathcal{N}_G^1 is a PSA at the channel input (of an appropriate gain and squeezing angle) followed by classical thermal noise addition, $\mathcal{A}_{\eta_1\eta_2\dots\eta_{k+1}}$ is the entire channel loss collected in the middle, and \mathcal{N}_G^2 is a PSA at the channel output. For expressions of the gain and phase parameters of the PSAs at the transmitter and the receiver, see Appendix C 3.

We note here that PIAs can improve the signal-to-noise ratio (SNR) of a sub-unity-efficiency optical heterodyne detection receiver, albeit up to 3 dB of the quantum limited SNR, when preceding the receiver. PSAs on the other hand have been proposed for use in optical imaging [41] and secure-key generation [42], to boost the effective detection efficiency of homodyne detection receivers, in principle pushing the receiver's performance all the way to the quantum limited SNR, by preceding the receiver with a PSA whose gain quadrature is phase matched to the homodyne detector's local oscillator. Despite these practical uses of optical amplifiers, our results in the earlier sections show that these amplifiers cannot increase the secret key capacity, and the results in the current section show that it is unlikely that they will help to realize the given secret key capacity in a practical implementation.

VI. CONCLUSIONS

It was recently shown [15] that for QKD (secure key generation), along with a few other optical quantum communication tasks such as quantum (qubit) communication, entanglement generation, and direct-secure communication (each with two-way authenticated classical communication assistance), the rates are upper bounded by $R_{\text{UB}} = \log_2[(1 + \eta)/(1 - \eta)]$ bits per mode over a pure-loss optical channel of transmittance η . This upper bound reads $R_{\text{UB}} \approx 2.88\eta$ when $\eta \ll 1$ (high loss), which translates to an exponential decay of rate with distance L in fiber ($\eta \propto e^{-\alpha L}$), and an inverse-square decay with L in free-space ($\eta \propto 1/L^2$) [14]. Quantum repeaters are conceptual devices that, when inserted along the lossy channel, can help circumvent this rate-loss trade-off.

In this paper, we have proven the inefficacy of bosonic Gaussian channels—optical processes that can be assembled using passive linear optics (beam splitters and phase shifters) and squeezers (phase-sensitive amplifiers, and the interaction

of parametric downconversion)—to be used as quantum repeaters. Note that our result leaves open the interesting question whether there are other simple types of untended repeater stations that can be readily implemented and achieve the same goal as quantum repeaters based on one-way communication. We prove our result by showing that any concatenation of such untended Gaussian operations along a lossy channel can be simulated by one Gaussian operation at the channel input and one at the channel output, where the entire loss in the channel is collected in the middle. We thereby argue that any communication protocol that uses such a chain of Gaussian center stations can be replaced by another protocol of the same performance without those stations, the transmitter and receiver of which are slightly modified versions of those used by the original protocol. As a consequence, the upper bound R_{UB} , as shown above, still applies. Note, however, that our formulation is entirely based on the property of Gaussian channels and does not preclude the possibility that a trace-decreasing Gaussian operation [32] could serve as a quantum repeater.

It would be possible that intermediate trace-preserving Gaussian operations could be of practical advantage, while the same performance of any protocol working with such middle stations is in principle achievable *without* middle stations. In order to demonstrate practical restrictions for use of conventional Gaussian stations, we separately analyzed the case of a general single-mode Gaussian channel sandwiched between lossy channels. We derived the conditions that the center station renders the end-to-end lossy channel entanglement breaking, and hence useless for QKD. From special cases for quantum-limited optical amplifiers as center stations, we found that in a high-loss regime, even modest amplification gains will render the overall channel entanglement breaking.

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APPENDIX A: EXTRACTING ONE MEAN DISPLACEMENT IN A CONCATENATED GAUSSIAN OPERATION

The action of the concatenation of n Gaussian channels $\mathcal{E}_n \circ \cdots \circ \mathcal{E}_2 \circ \mathcal{E}_1$, where $\mathcal{E}_i \triangleq (K_i, m_i, \alpha_i)$ can always be mimicked by a concatenation $\mathcal{E}'_n \circ \cdots \circ \mathcal{E}'_2 \circ \mathcal{E}'_1$, where all the displacement terms are pushed to the n th channel, i.e., $\mathcal{E}'_i \triangleq (K_i, 0, \alpha_i)$, $1 \leq i \leq n-1$, and $\mathcal{E}'_n \triangleq (K_n, m_t, \alpha_n)$, where m_t is a function of $\{m_1, \dots, m_n\}$, and $\{K_1, \dots, K_n\}$. To see this, consider the composition of n Gaussian channels:

$$\mathcal{E}_{123\dots n} = \mathcal{E}_n \circ \cdots \circ \mathcal{E}_3 \circ \mathcal{E}_2 \circ \mathcal{E}_1, \quad (\text{A1})$$

for which we may write

$$\begin{aligned} d_{123\dots n} &:= K_{123\dots n}^T d + m_{12\dots n} \\ &= K_n^T K_{123\dots n-1}^T d + K_n^T m_{123\dots n-1} + m_n \\ &\vdots \\ &= K_{1 \rightarrow n}^T d + \sum_{j=2}^n K_{j \rightarrow n}^T m_{j-1} + m_n \\ &= K_t d + m_t, \end{aligned} \quad (\text{A2})$$

where

$$\begin{aligned} K_t &:= K_{1 \rightarrow n}, \\ m_t &:= \sum_{j=2}^n K_{j \rightarrow n}^T m_{j-1} + m_n, \\ K_{j \rightarrow n} &:= \begin{cases} K_j K_{j+1} K_{j+2} \cdots K_{n-1} K_n & j < n-1 \\ K_n & j = n. \end{cases} \end{aligned} \quad (\text{A3})$$

From Eq. (A2) we can confirm that the total change in the first moment d is given by $K_t = K_{1 \rightarrow n}$ and a constant shift $m_t = \sum_{j=2}^n K_{j \rightarrow n}^T m_{j-1} + m_n$. Hence, we can obtain the same transformation of d , for example, by setting $m_1 = m_2 = \cdots = m_{n-1} = 0$ and $m_n = m_t$, while leaving the gain terms K_j of each channel \mathcal{E}_j as they are. To be precise, the following two channels equivalently act on (γ, d) .

$$E := \underbrace{\mathcal{E}_n}_{(K_n, m_n, \alpha_n)} \circ \cdots \circ \underbrace{\mathcal{E}_2}_{(K_2, m_2, \alpha_2)} \circ \underbrace{\mathcal{E}_1}_{(K_1, m_1, \alpha_1)}, \quad (\text{A4})$$

$$E' := \underbrace{\mathcal{E}_n}_{(K_n, m_t, \alpha_n)} \circ \cdots \circ \underbrace{\mathcal{E}_2}_{(K_2, 0, \alpha_2)} \circ \underbrace{\mathcal{E}_1}_{(K_1, 0, \alpha_1)}. \quad (\text{A5})$$

Therefore, mean displacement terms can be absorbed into the final Gaussian operation, and their effect can be treated separately in the analysis of the sequential channel action. In this manner, one can usually discuss Gaussian channel properties by assuming $m_j = 0$ for all j without loss of generality, and taking into account the effect of m 's, if at all needed, at once.

APPENDIX B: PROOF OF THEOREM 3

We restate Theorem 3 below for completeness.

Theorem. Φ is unitary equivalent to a channel Φ_s that is a PIC $\mathcal{A}_{\eta_s}^{N_s}$, whose descriptive parameters (K_s, α_s) are given by

$$\begin{aligned} K_s &= V K W = \sqrt{g \eta_1 \eta_2} \mathbb{1}_2, \quad \text{and} \\ \alpha_s &= W^T \alpha W = \sqrt{\det(\alpha)} \mathbb{1}_2, \end{aligned} \quad (\text{B1})$$

where V and W are Gaussian unitaries.

Proof. Let W_0 be an orthogonal matrix that diagonalizes α so that $W_0^T \alpha W_0 = \text{diag}(\lambda_1, \lambda_2)$. We then have $\det(\alpha) = \lambda_1 \lambda_2$. It is then easy to see that the expression for α_s in Eq. (B1) can be obtained by choosing

$$W = (\lambda_1 \lambda_2)^{-1/4} W_0 \sqrt{\text{diag}(\lambda_2, \lambda_1)}. \quad (\text{B2})$$

Given this W , we can choose $V = W^{-1} K_\theta^{-1} S_1^{-1}$ to obtain the expression for K_s in Eq. (B1). Decomposing Φ into Φ_s , sandwiched between unitaries V and W is depicted in line 1.c of Fig. 2(b). ■

APPENDIX C: ANALYSIS OF OPTICAL AMPLIFIERS AS REGENERATIVE STATIONS

In this Appendix, we prove the entanglement breaking conditions stated in Sec. VB, for when an optical amplifier, either phase-insensitive (PIA) or phase-sensitive (PSA), is used as a center station, sandwiched between two pure-loss channel segments. We will evaluate these conditions by applying our general results from Sec. VA.

The decomposition shown in line 1.b of Fig. 2(b), with the excess noise parameter N of \mathcal{A}_g^N set to zero, includes the quantum-noise-limited PSA and PIA as special cases. For $g \geq 1$ and $N = 0$, we obtain a simple expression of the determinant of α in Eq. (33):

$$\begin{aligned} \det(\alpha) &= \frac{1}{4} \{ (1 - g\eta_1\eta_2)^2 - 4\eta_2 [G_2(1 - g\eta_1)(1 - \eta_2) \\ &\quad + gG_1(1 - \eta_1)(1 - g\eta_2) \\ &\quad - 2g\sqrt{G_1G_2}(1 - \eta_1)(1 - \eta_2) \\ &\quad \times (\sqrt{G_1G_2} + \sqrt{(G_1 - 1)(G_2 - 1)} \cos 2\theta)] \}. \end{aligned} \quad (\text{C1})$$

1. PSA sandwiched by two lossy channels

For the case of the quantum-limited PSA, by setting $G_2 = g = 1$ in Eq. (C1) one obtains

$$\det(\alpha) = \frac{(1 - \eta_1\eta_2)^2}{4} + \eta_2(G_1 - 1)(1 - \eta_1)(1 - \eta_2). \quad (\text{C2})$$

From this relation and with $\eta_s = \eta_1\eta_2$ the EB condition of Eq. (38) reads

$$G_1 \geq 1 + \frac{\eta_1}{(1 - \eta_1)(1 - \eta_2)}. \quad (\text{C3})$$

Remark 5. A pure loss channel \mathcal{A}_η , $\eta \leq 1$, is not EB. A quantum-limited PSA (which is a squeezer, and hence a unitary) is not EB. This is consistent with the observation that setting either η_1 or η_2 close to 1 requires the PSA gain G_1 to go to infinity in order for the composition (loss-PSA-loss) to be EB. It is interesting that even though pure-loss channels and quantum-limited PSA are not EB by themselves, composing them can yield an EB channel if the gain and transmittances satisfy the condition in Eq. (C3).

2. PIA sandwiched by two lossy channels

For the case of the quantum-limited PIA by setting $G_1 = G_2 = 1$ in Eq. (C1) one obtains

$$\det(\alpha) = \frac{[1 + 2(g - 1)\eta_2 - g\eta_1\eta_2]^2}{4}. \quad (\text{C4})$$

From this relation and with $\eta_s = g\eta_1\eta_2$, the EB condition of Eq. (38) now reads

$$g \geq \frac{1}{1 - \eta_1}. \quad (\text{C5})$$

Remark 6. One notable point is that the transmittance η_2 of the lossy channel that appears *after* the PIA, does not play a role in the EB condition in Eq. (C5).

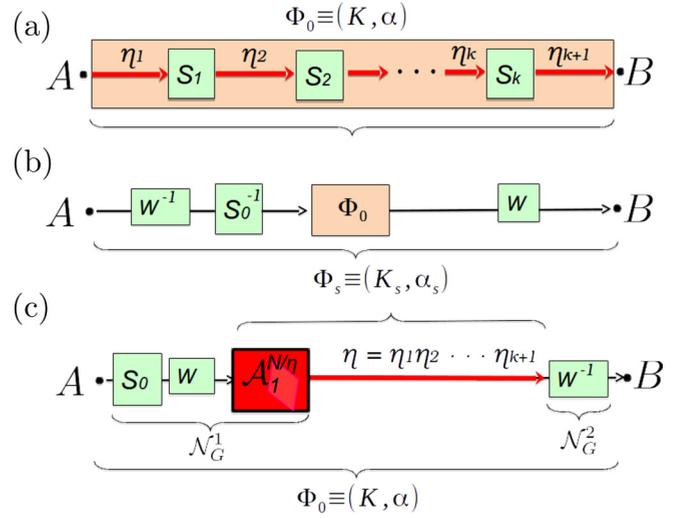


FIG. 3. (Color online) (a) A sequence of PSAs S_i with $i = 1, 2, \dots, k$ connected by the lossy segments of transmission η_i with $i = 1, 2, \dots, k + 1$. The total loss is given by $\eta = \prod_{i=1}^{k+1} \eta_i$. (b) The PSA-loss chain Φ_0 can be transformed to the standard form of a PIC Φ_s by using squeezing unitary operations W and S_0 . (c) The standard form Φ_s is decomposed into a thermal noise channel $\mathcal{A}_1^{N/\eta}$ and a transmission- η pure lossy segment. Then, the origin channel Φ_0 can be simulated by adding unitary operators to cancel out the unitary operators in (b) at the input end and output end. This turns Φ_0 into the form with original loss sandwiched by the operation \mathcal{N}_G^1 and the output-end operation \mathcal{N}_G^2 —an explicit example of our main result explained in Fig. 1.

3. Analysis of a chain of PSA center stations

Let us consider a chain of PSA center stations, interspersed between the number of $k + 1$ lossy segments with transmission $\{\eta_i\}_{i=1,2,\dots,k+1}$ as in Fig. 3(a). The channel action is formally written as

$$\begin{aligned} \Phi_0 &= \mathcal{A}_{\eta_{k+1}} \circ \mathcal{S}_{G_k} \circ \mathcal{A}_{\eta_k} \circ \dots \\ &\quad \dots \circ \mathcal{A}_{\eta_3} \circ \mathcal{S}_{G_2} \circ \mathcal{A}_{\eta_2} \circ \mathcal{S}_{G_1} \circ \mathcal{A}_{\eta_1}, \end{aligned} \quad (\text{C6})$$

where the action of PSAs \mathcal{S}_{G_i} with the amplification gain of $\{G_i\}_{i=1,2,\dots,k}$ can be described by Eq. (34).

By repeatedly using the composition rule of Eq. (10) we can write the channel parameters $\Phi_0 \triangleq (K, 0, \alpha)$ as follows:

$$K = \sqrt{\eta_{k+1}\eta_k\eta_{k-1}\dots\eta_1} S_1 S_2 \dots S_k = \sqrt{\eta} S_0, \quad (\text{C7})$$

$$\begin{aligned} \alpha &= \frac{1}{2} \left[\bar{\eta}_{k+1} \mathbb{1}_2 + \eta_{k+1} \left\{ \bar{\eta}_k S_k^T S_k + \eta_k \bar{\eta}_{k-1} S_k^T S_{k-1}^T S_{k-1} S_k \right. \right. \\ &\quad \left. \left. + \eta_k \eta_{k-1} \bar{\eta}_{k-2} S_k^T S_{k-1}^T S_{k-2}^T S_{k-2} S_{k-1} S_k + \dots \right\} \right] \\ &= \frac{1}{2} \left[\bar{\eta}_{k+1} \mathbb{1}_2 + \eta_{k+1} \left\{ \sum_{n=0}^{k-1} \frac{\bar{\eta}_{k-n}}{\eta_{k-n}} \prod_{j=0}^n \eta_{k-j} (S_{k-j})^2 \right\} \right] \\ &= \begin{pmatrix} \alpha^{(+)} & 0 \\ 0 & \alpha^{(-)} \end{pmatrix}, \end{aligned} \quad (\text{C8})$$

where

$$\eta := \eta_{k+1}\eta_k\eta_{k-1}\cdots\eta_1, \quad (\text{C9})$$

$$\bar{\eta}_i := 1 - \eta_i, \quad (\text{C10})$$

$$S_0 := S_1S_2\cdots S_k, \quad (\text{C11})$$

$$\alpha^{(\pm)} := \frac{1}{2} \left[\bar{\eta}_{k+1} + \eta_{k+1} \times \left\{ \sum_{n=0}^{k-1} \frac{\bar{\eta}_{k-n}}{\eta_{k-n}} \prod_{j=0}^n \eta_{k-j} (\sqrt{G_{k-j}} \pm \sqrt{G_{k-j}-1})^2 \right\} \right]. \quad (\text{C12})$$

Let us define a squeezer,

$$W := (\alpha^{(+)}\alpha^{(-)})^{-1/2} \sqrt{\text{diag}[\alpha^{(-)}, \alpha^{(+)}]}, \quad (\text{C13})$$

that symmetrizes α as $W^T\alpha W \propto \mathbb{1}_2$ and set $V = W^{-1}S_0^{-1}$ similarly to the proof of Theorem 3 in Appendix B [see

Fig. 3(b)]. Then, we can convert Φ_0 to the standard form of a PIC $\Phi_S \triangleq (K_S, 0, \alpha_S)$ with

$$K_S = \sqrt{\eta}\mathbb{1}_2, \quad (\text{C14})$$

$$\alpha_S = \sqrt{\det(\alpha)}\mathbb{1}_2 = \left[\frac{1}{2}(1 - \eta) + N \right] \mathbb{1}_2. \quad (\text{C15})$$

This implies the EB condition due to Eq. (19):

$$\sqrt{\det(\alpha)} \geq \frac{1}{2}(1 + \eta). \quad (\text{C16})$$

Let us now explicitly show the decomposition of the PSA chain into a pair of Gaussian operations at the input and the output. See Fig. 3 for a pictorial depiction. From the standard form we can split out the pure lossy segment by using the relation $\Phi_S \equiv \mathcal{A}_\eta^N = \mathcal{A}_\eta \circ \mathcal{A}_1^{N/\eta}$, which can be confirmed easily from the composition rule of Eq. (10). We can retrieve the original channel Φ_0 by canceling the unitary operators W , S_0 , and W^{-1} as in Fig. 3(c). To be specific, we can write the original channel in the sandwiched form $\Phi_0 = \mathcal{N}_G^2 \circ \mathcal{A}_\eta \circ \mathcal{N}_G^1$ with the two Gaussian channels $\mathcal{N}_G^1 = \mathcal{A}_1^{N/\eta} \circ \mathcal{W} \circ S_0$ and $\mathcal{N}_G^2 = \mathcal{W}^{-1}$. Note that the channel at the receiver \mathcal{N}_G^2 is a squeezing unitary, which is consistent with Proposition 1.

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design the forward error correcting code used for the protocol. For example, the error correcting code for a direct-secure-communication protocol over a lossy optical channel may need to assume the loss and noise in the channel in order to realize private communication.

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