Heisenberg uncertainty relation for three canonical observables

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(Received 18 July 2014; published 15 December 2014)

Uncertainty relations provide fundamental limits on what can be said about the properties of quantum systems. For a quantum particle, the commutation relation of position and momentum observables entails Heisenberg's uncertainty relation. A third observable is presented which satisfies canonical commutation relations with both position and momentum. The resulting triple of pairwise canonical observables gives rise to a Heisenberg uncertainty relation for the product of three standard deviations. We derive the smallest possible value of this bound and determine the specific squeezed state which saturates the triple uncertainty relation. Quantum optical experiments are proposed to verify our findings.

DOI: 10.1103/PhysRevA.90.062118 PACS number(s): 03.65.Ta, 03.67.−a

I. INTRODUCTION

In quantum theory, two observables $\hat{p}$ and $\hat{q}$ are canonical if they satisfy the commutation relation

$$[\hat{p},\hat{q}] = \frac{\hbar}{i}, \tag{1}$$

with the momentum and position of a particle being a well-known and important example. The nonvanishing commutator expresses the incompatibility of the Schrödinger pair $(\hat{p},\hat{q})$ of observables since it imposes a lower limit on the product of their standard deviations, namely

$$\Delta q \Delta p \geqslant \frac{\hbar}{2}. \tag{2}$$

In 1927, Heisenberg [1] analyzed the hypothetical observation of an individual electron with photons and concluded that the product of the measurement errors should be governed by a relation of the form (2). His proposal inspired Kennard [2] and Weyl [3] to mathematically derive Heisenberg's uncertainty relation, thereby turning it into a constraint on measurement outcomes for an ensemble of identically prepared systems. Schrödinger's [4] generalization of (2) included a correlation term, and Robertson [5,6] derived a similar relation for any two noncommuting Hermitian operators. Recently claimed violations of (2) do not refer to Kennard and Weyl's *preparation* uncertainty relation but to Heisenberg's *error-disturbance* relation (cf. [7–9]). However, these claims have been criticized strongly [10,11].

Uncertainty relations are now understood to provide fundamental limits on what can be said about the properties of quantum systems. Imagine measuring the standard deviations Δp and Δq separately on two ensembles prepared in the same quantum state. Then, the bound (2) does not allow one to simultaneously attribute definite values to the observables $\hat{p}$ and $\hat{q}$.

In this paper, we will consider a *Schrödinger triple* $(\hat{p},\hat{q},\hat{r})$ consisting of *three* pairwise canonical observables [12], i.e.,

$$[\hat{p},\hat{q}] = [\hat{q},\hat{r}] = [\hat{r},\hat{p}] = \frac{\hbar}{i}, \tag{3}$$

and derive a *triple* uncertainty relation. In a system of units where both $\hat{p}$ and $\hat{q}$ carry physical dimensions of $\sqrt{\hbar}$, the observable $\hat{r}$ is given by

$$\hat{r} = -\hat{q} - \hat{p}, \tag{4}$$

which corresponds to a suitably *rotated* and *rescaled* position operator $\hat{q}$. It is important to point out that any Schrödinger triple for a quantum system with one degree of freedom is unitarily equivalent to $(\hat{p},\hat{q},\hat{r})$; furthermore, any such triple is maximal in the sense that there are no four observables that equicommute to $\hbar/i$ [13]. Therefore, the algebraic structure defined by a Schrödinger triple $(\hat{p},\hat{q},\hat{r})$ is unique up to unitary transformations.

Given that (1) implies Heisenberg's uncertainty relation (2), we wish to determine the consequences of the commutation relations (3) on the product of the *three* uncertainties associated with a Schrödinger triple $(\hat{p},\hat{q},\hat{r})$.

II. RESULTS

We will establish the *triple uncertainty relation*

$$\Delta p \Delta q \Delta r \geqslant \left(\tau \frac{\hbar}{2}\right)^{3/2}, \tag{5}$$

where the number τ is the *triple constant* with value

$$\tau = \csc\left(\frac{2\pi}{3}\right) \equiv \sqrt{\frac{4}{3}} \simeq 1.16. \tag{6}$$

The bound (5) is found to be *tight*; the state of minimal triple uncertainty is found to be a generalized squeezed state,

$$|\Xi_0\rangle = \hat{S}_{\frac{i}{4}\ln 3}|0\rangle, \tag{7}$$

being *unique except for rigid translations in phase space.* The operator $\hat{S}_{\frac{i}{4}\ln 3}$, defined in Eq. (22) is a generalized squeezing operator: it generates the state $|\Xi_0\rangle$ by *contracting* the standard coherent state $|0\rangle$ (i.e., the ground state of a harmonic oscillator with unit mass and unit frequency) along the main diagonal in phase space by an amount characterized by $\ln \sqrt[4]{3} < 1$, at the expense of a *dilation* along the minor diagonal.

To visualize this result, let us determine the Wigner function of the state $|\Xi_0\rangle$ with position representation (cf. [14])

$$\langle q|\Xi_0\rangle = \frac{1}{\sqrt[4]{\tau\pi}} \exp\left(-\frac{1}{2}e^{-i\frac{\pi}{6}}q^2\right). \tag{8}$$

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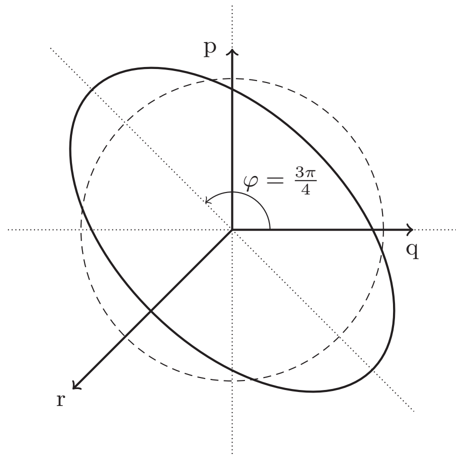

FIG. 1. Phase-space contour lines of the Wigner functions associated with the states $|\Xi_0\rangle$ (full line) and a standard coherent state $|0\rangle$ (dashed), respectively; both lines enclose the same area.

Thus its Wigner function associated with the state $|\Xi_0\rangle$ minimizing the triple uncertainty relation is found to be

$$W_{\Xi_0}(q,p) = \frac{1}{\pi} \exp\left(-\frac{\tau}{\hbar}(q^2 + p^2 + qp)\right), \tag{9}$$

which is positive. Its phase-space contour line enclosing an area of size $\hbar$, shown in Fig. 1, confirms that we deal with a squeezed state aligned with the minor diagonal.

To appreciate the bound (5), let us evaluate the triple uncertainty $\Delta p \Delta q \Delta r$ in two instructive cases. (i) Since the pairs $(\hat{p},\hat{q})$, $(\hat{q},\hat{r})$, and $(\hat{r},\hat{p})$ are canonical, the inequality (2)—as well as its generalization due to Robertson and Schrödinger—applies to each of them implying the lower bound

$$\Delta p \Delta q \Delta r \geqslant \left(\frac{\hbar}{2}\right)^{3/2}. \tag{10}$$

However, it remains open whether there is a state in which the triple uncertainty saturates this bound. Our main result (5) reveals that *no* such state exists. (ii) In the vacuum $|0\rangle$, a coherent state with minimal pair uncertainty, the *triple* uncertainty takes the value

$$\Delta p \Delta q \Delta r = \sqrt{2}\left(\frac{\hbar}{2}\right)^{3/2}. \tag{11}$$

The factor of $\sqrt{2}$ in comparison with (10) has an intuitive explanation: while the vacuum state $|0\rangle$ successfully minimizes the product $\Delta p \Delta q$, it does not simultaneously minimize the uncertainty associated with the pairs $(\hat{q},\hat{r})$ and $(\hat{r},\hat{p})$. Thus the minimum of the inequality (5) cannot be achieved by coherent states.

The observations (i) and (ii) suggest that the bound (5) on the triple uncertainty is not an immediate consequence of Heisenberg's inequality for canonical *pairs*, Eq. (2). Furthermore, the invariance groups of the triple uncertainty relation, of Heisenberg's uncertainty relation, and of the inequality by Schrödinger and Robertson are different, because they depend on two, three, and four (cf. [15]) continuous parameters, respectively.

III. THREEFOLD SYMMETRY

The commutation relations (3) are invariant under the cyclic shift $\hat{p} \to \hat{q} \to \hat{r} \to \hat{p}$, implemented by a unitary operator $\hat{Z}$,

$$\hat{Z}\hat{p}\hat{Z}^\dagger = \hat{q}, \quad \hat{Z}\hat{q}\hat{Z}^\dagger = \hat{r}, \quad \hat{Z}\hat{r}\hat{Z}^\dagger = \hat{p}. \tag{12}$$

Note that the third equation follows from the other two equations. The third power of $\hat{Z}$ obviously commutes with both $\hat{p}$ and $\hat{q}$ so it must be a scalar multiple of the identity, $\hat{Z}^3 \propto \hat{\mathbb{I}}$.

To determine the operator $\hat{Z}$ we first note that its action displayed in (12) is achieved by a clockwise rotation by $\pi/2$ in phase space followed by a gauge transformation in the position basis:

$$\hat{Z} = \exp\left(-\frac{i}{2\hbar}\hat{q}^2\right)\exp\left(-\frac{i\pi}{4\hbar}(\hat{p}^2 + \hat{q}^2)\right). \tag{13}$$

A Baker-Campbell-Hausdorff (BCH) calculation reexpresses this product in terms of a single exponential:

$$\hat{Z} = \exp\left(-i\frac{\pi}{3\hbar\sqrt{3}}(\hat{p}^2 + \hat{q}^2 + \hat{r}^2)\right). \tag{14}$$

The operator $\hat{Z}$ cycles the elements of the Schrödinger triple $(\hat{p},\hat{q},\hat{r})$ just as a Fourier transform operator swaps position and momentum of the Schrödinger pair $(\hat{p},\hat{q})$ (apart from a sign). If one introduces a unitarily equivalent symmetric form of the Schrödinger triple with operators $(\hat{P},\hat{Q},\hat{R})$ associated with an equilateral triangle in phase space, the metaplectic operator $\hat{Z}$ simply acts as a rotation by $2\pi/3$, i.e., as a fractional Fourier transform.

Furthermore, denoting the factors of $\hat{Z}$ in (13) by $\hat{A}$ and $\hat{B}$ (with suitably chosen phase factors), respectively, we find that $\hat{B}^2 = \hat{\mathbb{I}}$ and $(\hat{A}\hat{B})^3 \equiv \hat{Z}^3 = \hat{\mathbb{I}}$. These relations establish a direct link between the threefold symmetry of the Schrödinger triple $(\hat{p},\hat{q},\hat{r})$ and the *modular group* $SL_2(\mathbb{Z})/\{\pm 1\}$ which $\hat{A}$ and $\hat{B}$ generate [16].

IV. EXPERIMENTS

To experimentally confirm the triple uncertainty relation (5), we propose an approach based on optical homodyne detection. We exploit the fact that the state $|\Xi_0\rangle$ is a *generalized coherent state*, also known as a *correlated coherent state* [17]: such a state is obtained by squeezing the vacuum state $|0\rangle$ along the momentum axis followed by a suitable rotation in phase space.

The basic scheme for homodyne detection consists of a beam splitter, photodetectors, and a reference beam, called the *local oscillator*, with which the signal is mixed; by adjusting the phase of the local oscillator one can probe different directions in phase space. If θ is the phase of the local oscillator, a homodyne detector measures the probablity distribution of the observable

$$\hat{x}(\theta) = \frac{1}{\sqrt{2}}(a^\dagger e^{i\theta} + a e^{-i\theta}) = \hat{q}\,\cos\theta + \hat{p}\,\sin\theta \tag{15}$$

along a line in phase space defined by the angle θ; here $\hat{q}$ and $\hat{p}$ denote the quadratures of the photon field while the operators $a^\dagger$ and a create and annihilate single photons [18]; note that $\hat{r} \equiv \sqrt{2}\hat{x}(5\pi/4)$.

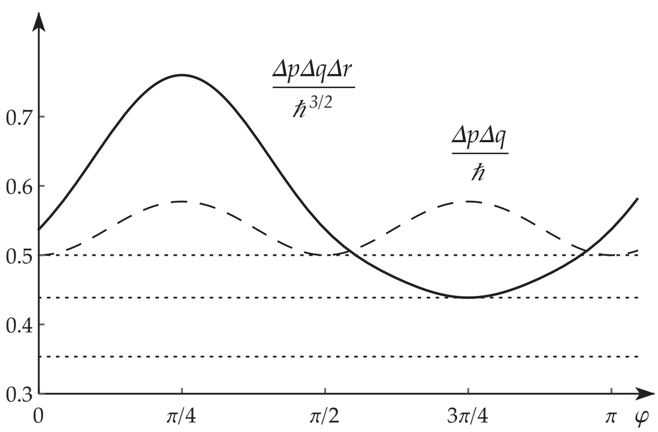

FIG. 2. Dimensionless pair and triple uncertainties for squeezed states with $\gamma = \ln \sqrt[4]{3}$, rotated away from the position axis by an angle $\varphi \in [0,\pi]$. The pair uncertainty $\Delta p \Delta q$ starts out at its minimum value of $1/2$ which is achieved again for $\varphi = \pi/2$ and $\varphi = \pi$ (dashed line). The triple uncertainty has period π, reaching its minimum for $\varphi = 3\pi/4$ for the state $|\Xi_0\rangle$ (full line). The dotted lines (top to bottom) represent the bounds (2), (5), and (10), with values $1/2$, $(\tau/2)^{3/2}$, and $(1/2)^{3/2}$.

The probability distributions of the observables $\hat{q}$, $\hat{p}$, and $\hat{r}$, corresponding to the angles $\theta = 0$, $\pi/2$, and $5\pi/4$, can be measured upon preparing a large ensemble of the state $|\Xi_0\rangle$. The resulting product of their variances may then be compared with the value of the tight bound given in Eq. (5). Under rigid phase-space rotations of the triple $(\hat{q},\hat{p},\hat{r})$ by an angle φ the triple uncertainty will vary as predicted in Fig. 2 (full line). A related experiment has been carried out successfully in order to directly verify other Heisenberg- and Schrödinger-Robertson-type uncertainty relations [19,20].

V. MINIMAL TRIPLE UNCERTAINTY

To determine the states which minimize the left-hand-side of Eq. (5), we need to evaluate it for all normalized states $|\psi\rangle \in \mathcal{H}$ of a quantum particle. To this end we introduce the *uncertainty functional* (cf. [21]),

$$J_\lambda[\psi] = \Delta_p[\psi]\Delta_q[\psi]\Delta_r[\psi] - \lambda(\langle\psi|\psi\rangle - 1), \tag{16}$$

using the standard deviations $\Delta_x[\psi] \equiv \Delta x \equiv (\langle\psi|\hat{x}^2|\psi\rangle - \langle\psi|\hat{x}|\psi\rangle^2)^{1/2}$, $x = p,q,r$, while the term with Lagrange multiplier λ takes care of normalization. In a first step, we determine the extremals of the functional $J_\lambda[\psi]$. Changing its argument from $|\psi\rangle$ to the state $|\psi\rangle + |\varepsilon\rangle$, where $|\varepsilon\rangle = \varepsilon|e\rangle$, with a normalized state $|e\rangle \in \mathcal{H}$ and a real parameter $\varepsilon \ll 1$, leads to

$$J_\lambda[\psi + \varepsilon] = J_\lambda[\psi] + \varepsilon J_\lambda^{(1)}[\psi] + O(\varepsilon^2). \tag{17}$$

The first-order variation $J_\lambda^{(1)}[\psi]$ only vanishes if $|\psi\rangle$ is an extremum of the functional $J_\lambda[\psi]$ or, equivalently, if

$$\frac{1}{3}\left(\frac{(\hat{p} - \langle\hat{p}\rangle)^2}{\Delta_p^2} + \frac{(\hat{q} - \langle\hat{q}\rangle)^2}{\Delta_q^2} + \frac{(\hat{r} - \langle\hat{r}\rangle)^2}{\Delta_r^2}\right)|\psi\rangle = |\psi\rangle \tag{18}$$

holds, which follows from generalizing a direct computation which had been carried out in [22] to determine the extremals of the product $\Delta p \Delta q$.

Equation (18) is nonlinear in the unknown state $|\psi\rangle$ due to the expectation values $\langle\hat{p}\rangle,\Delta_p^2$, etc. Its solutions can be found by initially treating these expectation values as constants to be determined only later in a self-consistent way. The unitary operator $\hat{U}_{\alpha,b,\gamma} = \hat{T}_\alpha \hat{G}_b \hat{S}_\gamma$ transforms the left-hand side of (18), which is quadratic in $\hat{p}$ and $\hat{q}$, into a standard harmonic-oscillator Hamiltonian,

$$\frac{1}{2}(\hat{p}^2 + \hat{q}^2)|\psi_{\alpha,b,\gamma}\rangle = \frac{3}{2c}|\psi_{\alpha,b,\gamma}\rangle, \tag{19}$$

where $|\psi_{\alpha,b,\gamma}\rangle \equiv \hat{U}_{\alpha,b,\gamma}^\dagger|\psi\rangle$, and c is a real constant. The unitary $\hat{U}_{\alpha,b,\gamma}$ consists of a *rigid phase-space translation* by $\alpha \equiv (q_0 + ip_0)/\sqrt{2\hbar} \in \mathbb{C}$,

$$\hat{T}_\alpha = \exp[i(p_0\hat{q} - q_0\hat{p})/\hbar], \tag{20}$$

followed by a *gauge transformation* in the momentum basis

$$\hat{G}_b = \exp(ib\hat{p}^2/2\hbar), \quad b \in \mathbb{R}, \tag{21}$$

and a *squeezing transformation*,

$$\hat{S}_\gamma \equiv \exp[i\gamma(\hat{q}\hat{p} + \hat{p}\hat{q})/2\hbar], \quad \gamma \in \mathbb{R}. \tag{22}$$

According to (19), the states $|\psi_{\alpha,b,\gamma}\rangle$ coincide with the eigenstates $|n\rangle, n \in \mathbb{N}_0$, of a harmonic oscillator with unit mass and frequency,

$$|n;\alpha,b,\gamma\rangle \equiv \hat{T}_\alpha \hat{G}_b \hat{S}_\gamma |n\rangle, \quad n \in \mathbb{N}_0, \tag{23}$$

where we have suppressed irrelevant constant phase factors; for consistency, the quantity $3/2c$ in (19) must only take the values $\hbar(n + 1/2)$ for $n \in \mathbb{N}_0$, as a direct but lengthy calculation confirms. The parameters b and γ must take specific values for (19) to hold, namely

$$b = \tfrac{1}{2} \quad \text{and} \quad \gamma = \tfrac{1}{2}\ln\tau. \tag{24}$$

We will denote the restricted set of states obtained from Eq. (23) by $|n;\alpha\rangle$. There are no constraints on the parameter α, which means that we are free to displace the states $|n\rangle$ in phase space without affecting the values of the variances. The variances of the observables $\hat{p}$, $\hat{q}$, and $\hat{r}$ are found to be equal, taking the value

$$\Delta_x^2[n;\alpha] = \tau\hbar\left(n + \tfrac{1}{2}\right), \quad x = p,q,r, \tag{25}$$

with the triple constant τ introduced in (6). Inserting these results into Eq. (18) we find that

$$\tfrac{1}{3}(\hat{p}^2 + \hat{q}^2 + \hat{r}^2)|n;\alpha\rangle = \tau\hbar\left(n + \tfrac{1}{2}\right)|n;\alpha\rangle, \tag{26}$$

where

$$|n;\alpha\rangle = \hat{T}_\alpha \hat{G}_{\frac{1}{2}} \hat{S}_{\frac{1}{2}\ln\tau}|n\rangle, \quad n \in \mathbb{N}_0, \alpha \in \mathbb{C}. \tag{27}$$

For each value of α, the *extremals* of the uncertainty functional (16) form a complete set of orthonormal states,

$$\sum_{n=0}^{\infty}|n;\alpha\rangle\langle n;\alpha| = \mathbb{I}, \tag{28}$$

since the set of states $\{|n\rangle\}$ has this property.

At its extremals the uncertainty functional (16) takes the values

$$J_\lambda[n;\alpha] = \left[\tau\hbar\left(n+\tfrac{1}{2}\right)\right]^{3/2}, \quad n \in \mathbb{N}_0, \tag{29}$$

according to Eq. (25), with the minimum occurring for $n = 0$. Thus the two-parameter family of states $|0;\alpha\rangle, \alpha \in \mathbb{C}$, which we will denote by

$$|\Xi_\alpha\rangle = \hat{T}_\alpha\big(\hat{G}_{\frac{1}{2}}\hat{S}_{\frac{1}{2}\ln\tau}|0\rangle\big), \tag{30}$$

minimizes the triple uncertainty relation (5).

The states $|\Xi_\alpha\rangle$ are *displaced generalized squeezed* states, with a squeezing direction along a line different from the position or momentum axes. To show this, it is sufficient to consider the state $|\Xi_0\rangle$, which satisfies (26) with $n \equiv 0$ and $\alpha \equiv 0$. The product of unitaries in (30) acting on the vacuum $|0\rangle$ is easily understood if one rewrites it using the identity

$$\hat{G}_b\hat{S}_\gamma = \hat{S}_\xi\hat{R}_\varphi, \tag{31}$$

where the unitary $\hat{R}_\varphi = \exp(i\varphi a^\dagger a)$ is a counterclockwise rotation by φ in phase space, while the operator

$$\hat{S}_\xi = \exp[(\overline{\xi}a^2 - \xi a^{\dagger 2})/2], \quad \xi = \gamma e^{i\theta}, \gamma > 0, \tag{32}$$

generalizes $\hat{S}_\gamma$ in (22) by allowing for squeezing along a line with inclination $\theta/2$; the annihilation operator and its adjoint $a^\dagger$ are defined by $a = (\hat{q} + i\hat{p})/\sqrt{2\hbar}$. Another standard BCH calculation (using the result from Sec. 6 of [23]) reveals that the values $\xi = (i/4)\ln 3$ and $\varphi = -\pi/12$ turn Eq. (31) into an identity for the values of b and γ given in (24). This confirms that the state of minimal triple uncertainty is the generalized squeezed state given in (7).

VI. SUMMARY AND DISCUSSION

We have established a tight inequality (5) for the triple uncertainty associated with a Schrödinger triple $(\hat{p},\hat{q},\hat{r})$ of pairwise canonical observables. Ignoring rigid translations in phase space, there is only one state $|\Xi_0\rangle$ which minimizes the triple uncertainty, shown in Eq. (30). The state $|\Xi_0\rangle$ is an eigenstate of the operator $\hat{Z}$ in (14) which describes the fundamental threefold cyclic symmetry of the Schrödinger triple $(\hat{p},\hat{q},\hat{r})$. Conceptually, the triple uncertainty and the one derived by Schrödinger and Robertson are linked because both incorporate the correlation operator $(\hat{p}\hat{q} + \hat{q}\hat{p})/2$, be it explicitly or indirectly via the expression $\hat{r}^2$.

The smallest possible value of the product $\Delta p\Delta q\Delta r$ is noticeably *larger* than the unachievable value $(\hbar/2)^{3/2}$, which follows from inequality (2) applied to each of the Schrödinger pairs $(\hat{p},\hat{q})$, $(\hat{q},\hat{r})$, and $(\hat{r},\hat{p})$. At the same time, the true minimum *undercuts* the value of the triple uncertainty in the vacuum state $|0\rangle$ by more than 10% [cf. Eq. (11)]. The experimental verification of these results is within reach of current quantum optical technology.

The results obtained in this paper add another dimension to the problem of earlier attempts to obtain uncertainty relations for more than two observables. In 1934, Robertson studied constraints which follow from the positive semidefiniteness of the covariance matrix for N observables [6], but the resulting inequality trivializes for an odd number of observables. Shirokov obtained another inequality [24] which contains little information about the canonical triple considered here.

The result for a Schrödinger triple obtained here suggests conceptually important generalizations. A tight bound for an *additive* uncertainty relation associated with the operators $(\hat{p},\hat{q},\hat{r})$ is easily established by a similar approach: the inequality

$$(\Delta p)^2 + (\Delta q)^2 + (\Delta r)^2 \geqslant \tau\frac{3\hbar}{2} \tag{33}$$

is saturated only by the state $|\Xi_0\rangle$ in (30), ignoring irrelevant rigid phase-space translations. This observation clashes with the relation between the *additive* and the *multiplicative* uncertainty relations for Schrödinger *pairs* $(\hat{p},\hat{q})$. According to [15] the states saturating the inequality $(\Delta p)^2 + (\Delta q)^2 \geqslant \hbar$ are a proper subset of those minimizing Heisenberg's *product* inequality (2).

Finally, an uncertainty relation for pairs of canonical observables also exists for the Shannon entropies S_p and S_q of their probability distributions [25,26]. We conjecture that the relation $S_p + S_q + S_r \geqslant (3/2)\ln(\tau e\pi)$ holds for the Schrödinger triple $(\hat{p},\hat{q},\hat{r})$, the minimum being achieved by the state $|\Xi_0\rangle$. This bound is tighter than $(3/2)\ln(e\pi)$, the value which follows from applying the bound $\ln(e\pi)$ for pairwise entropies to the triple.

ACKNOWLEDGMENTS

S.K. is supported via the act "Scholarship Programme of SSF by the procedure of individual assessment, of 2011–12" by resource of the Operational Programme for Education and Lifelong Learning of the ESF and of the NSF, 2007–2013. S.W. would like to thank V. I. Man'ko for an instructive discussion on aspects of the experimental realization. The authors thank J. Biniok, T. Bullock, P. Busch, and R. Colbeck for comments on drafts of this manuscript.

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