Dissipative solitons of self-induced transparency

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A theory of dispersive soliton of the self-induced transparency in a medium consisting of atoms or semiconductor quantum dots of two types is considered. A two-component medium is modeled by a set of two-level atoms of two types embedded into a conductive host material. These types of atoms correspond to passive atoms (attenuator atoms) and active atoms (amplifier atoms) with inverse population of the energetic levels. The complete solution is given of the Maxwell–Bloch equations for ensembles of two-type atoms with different parameters and different initial conditions by inverse scattering transform. The solutions of the Maxwell–Bloch equations for many-component atomic systems by inverse scattering transform are also discussed. The influence of the difference between dipole moments of atoms, the longitudinal and transverse relaxation times, pumping, and conductivity on the soliton is taken into account by means of perturbation theory. The memory effects are described in terms of generalized non-Markovian optical Bloch equations. The condition of a balance between the energy supplied and lost is obtained.

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I. INTRODUCTION

Optical solitons can propagate both in conservative as well as in dissipative physical systems. In Hamiltonian systems, there are different mechanisms providing undamped propagation of solitons such as, for instance, by a balance between nonlinearity and dispersion. In contrast to solitons in conservative systems, solitons in dissipative systems which are far from equilibrium are dynamical objects which are strongly dependent on an energy supply from an external source. Nonlinear effects in combination with gains and losses form dissipative solitons. For dissipative solitons, an external source of energy is essential, and this means that its properties are defined not only by means of the parameters of the pulse and medium but also by the parameters of the gain. The variety of physical systems in which exist dissipative solitons is enormous (see, for instant Refs. [1,2] and references therein).

In resonant systems, such as in two-level atomic systems or semiconductor quantum dots (SQDs), conservative solitons as well as dissipative solitons can exhibit the phenomenon of self-induced transparency (SIT). The resonant optical nonlinear wave can be formed with the help of the McCall–Hahn resonant mechanism, i.e., a nonlinear coherent interaction of an optical pulse with resonant optical two-level atoms or SQDs, when the conditions of SIT, $\omega T_p \gg 1$ and $T_p \ll T_{1,2}$, are fulfilled, where T_p and ω are the width and frequency of the pulse, respectively, and T_1 and T_2 are the longitudinal and transverse relaxation times of the resonant atoms or SQDs, respectively. When the area of the pulse, Θ, is larger than π, solitons (2π pulses) are generated. When $\Theta \ll 1$ resonance optical breathers (0π pulse) can be formed [3,4]. In addition to the SIT solution in an attenuator, a somewhat similar steady-state result (π pulse) in an amplifier can be obtained [3]. Experimental investigations of SIT solitons have been carried out on various types of materials such as vapors, SQDs, atomic systems in solids, and others (see, for example, Refs. [3–6]).

The theoretical explanation of the SIT effect is based on the representation of the resonant optical atoms or SQDs by an ensemble of two-level atoms whose evolution is caused by induced processes due to the interaction with a coherent light pulse. In general the theory of the interaction of electromagnetic radiation with an ensemble of two-level atoms is based on the Maxwell–Bloch equations, the complete solution of which, by means of the inverse-scattering transform (IST), are given [7,8].

Following from this, we consider a steady-state pulse in an amplifier or attenuator medium wherein we have two different states of the resonant atoms, one attenuating and one amplifying. For the attenuator medium, the initial condition is taken to be all atoms in their ground state and, for the amplifier medium, the initial condition is taken to be where all atoms are in their upper excited state. Normally, SIT studies are such that all the atoms are either initially attenuating or amplifying. But for investigations involving dissipative solitons, it becomes essential to consider a two-component medium consisting of a mixture of simultaneously interacting active (amplifying) and passive (absorbing) two-level atoms or SQDs.

Subject to the pulse width there are two different situations which require two different ways of consideration. The first is when the pulse width is longer than a few periods of the wave. The description of the wave dynamics in this case can be done

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with the usual slowly varying envelope approximation and the reduced Maxwell–Bloch equations [3,7–9]. This approach allows one to consider solitons in very wide regions of the pulse (soliton) widths from several tens of nanoseconds in dielectrics [3] up to several tens of femtoseconds in SQDs [10,11], which is a very wide field of potential applications for solitons.

The second situation is when a wave pulse has only a few cycles within its width. The appropriate description of these few-cycle pulses cannot be done in the slowly varying approximation for envelopes. For example, a conservative few-cycle soliton of the video type, with zero carrier frequency, and modeled by the nonreduced Maxwell–Bloch equations, was earlier considered [12]. More recently, numerical simulations have been carried out in the frame of the nonreduced Maxwell–Drude–Bloch model, including relaxations and conductivity, have demonstrated that few-cycle dissipative solitons can indeed be created [2,13,14].

Two-component dissipative systems can be studied analytically in addition to by numerical simulations. Since one aspect of this system is that parts of it are integrable by IST, one could then use perturbation theory to account for some the nonintegrable components, some of which are nonresonant gain and damping. The integrable parts of this system consists of the individual sets of the two atomic systems; ground-state atoms and excited-state atoms, each of which will resonantly exchange energy with the other part of the integrable system: the resonant pulse.

The purpose of the present work is to construct a theory of SIT in a two-component conductive medium consisting of a nonlinear optically active (amplifying) and passive (absorbing) two-level atoms or SQDs, under the condition of pumping the active atoms. The effect of the phase modulation, the inhomogeneous broadening of spectral lines, and the longitudinal and transverse relaxation processes are included. For the case of Markovian transverse relaxation we use the "$T_1 - T_2$-model" of the optical Bloch equations. The memory effects can be described in terms of generalized non-Markovian optical Bloch equations. We consider the possibility of the existence of stable dissipative envelope solitons and determine explicit analytical expressions for its parameters.

At the theoretical level, we use soliton perturbation theory based on the IST to consider the first-order effects of longitudinal and transverse relaxation processes, effects connected with the difference between dipole moments of atoms, the results of pumping the active atoms and, lastly, the influence of conductivity of the host medium on the propagation of a dissipative soliton [7,8]. We assume that the pulse propagation has a duration which is rather long compared to the period of the frequency of atomic transitions, so that we may use the slowly varying envelope approximation. The solution of the equations of SIT when there are many-component atomic (SQDs) systems will also be discussed.

To carry out this project, we first establish the connection between the reduced optical Bloch equations of SIT and the Zakharov–Shabat eigenvalue equations (ZSEs).

II. OPTICAL MAXWELL–BLOCH EQUATIONS

For investigation of the Bloch equations of a two-component medium consisting of nonlinear optically active (amplifying) and passive (absorbing) two-level atoms or SQDs, we consider the propagation of a circularly polarized optical coherent pulse through an ensemble of two-level atoms (SQDs). We take the pulse to be propagating along the positive z axis, with a pulse width of $T_p \ll T_{1,2}$, a frequency of $\omega \gg T_p^{-1}$, a wave vector $\vec{k}$, and an electric field vector of [15,16]

$$\vec{E}(z,t) = \vec{e}_- E_+ + \vec{e}_+ E_-, \tag{1}$$

where

$$E_\pm(z,t) = \hat{E}^\pm e^{\pm i(\omega t - kz)}, \quad \vec{e}_\pm = \frac{1}{\sqrt{2}}(\vec{x} \pm i\vec{y}),$$

where $\vec{e}_\pm$ are the complex polarization vectors, and $\vec{x}$ and $\vec{y}$ are unit vectors in the directions of the x and y axes. For the optical pulse, we first take it to be localized in some manner, such that $|E(z \to \pm\infty,t)| \to o(1/z)$ is always true. In addition to this, we use the slowly varying envelope approximation and assume that the complex envelope $\hat{E}^\pm(z,t) = \hat{E}(z,t)e^{\pm i\phi}$ satisfies the conditions

$$\left|\frac{\partial \hat{E}^\pm}{\partial t}\right| \ll \omega|\hat{E}^\pm|, \quad \left|\frac{\partial \hat{E}^\pm}{\partial z}\right| \ll k|\hat{E}^\pm|, \tag{2}$$

where $\phi(z,t)$ is the phase function.

For the description of the interaction of a SQD (or two-level atom, active or passive) with a circularly polarized optical pulse, we use the optical selection rules for quantum dots and consider a two-level system consisting of the ground state $|1\rangle$ and one excited (one exciton) state $|2\rangle$ (in SQDs such transitions are labeled 0-X transitions) where the level energies are $\mathcal{E}_1 = 0$ and $\mathcal{E}_2 = \hbar\omega_{0,1}$, respectively [17,18]. It follows that the Hamiltonian and the wave function are

$$H = H_0 + \hat{V},$$
$$|\Psi\rangle = \sum_{m=1,2} c_m(t) \exp\left(-\frac{i}{\hbar}\mathcal{E}_m t\right)|m\rangle,$$

where $H_0 = \hbar\omega_{0,1}|2\rangle\langle 2|$ is the ground-state Hamiltonian of the two-level SQD (or atom), whose frequency of excitation is $\omega_{0,1}$, $\hbar$ is Planck's constant divided by 2π,

$$\hat{V} = -\hat{\mu}\cdot\vec{E} = -\mu_1(\hat{\sigma}_- E_+ + \hat{\sigma}_+ E_-)$$

is the light-SQD interaction Hamiltonian,

$$\hat{\mu} = \vec{\mu}_{12}\hat{\sigma}_+ + \vec{\mu}^*_{12}\hat{\sigma}_-$$

is the SQD's dipole moment operator,

$$\vec{\mu}_{12} = \frac{\mu_1}{\sqrt{2}}(\vec{x} - i\vec{y})$$

is the electric dipole matrix element for the corresponding transition,

$$\mu_1 = |\vec{\mu}_{12}|, \quad \vec{\mu}_{21} = \vec{\mu}^*_{12},$$
$$\hat{\sigma}_\pm = \frac{1}{2}(\hat{\sigma}_x \pm i\hat{\sigma}_y),$$

and the $\{\hat{\sigma}_i\}$s are the Pauli matrices, which satisfy the commutation relation $[\hat{\sigma}_x,\hat{\sigma}_y] = 2i\hat{\sigma}_z$ and the cyclic permutations of the subscripts [19].

For the two-level SQDs, the general form of the wave function is $|\Psi\rangle = c_1|1\rangle + c_2|2\rangle$, where c_1 and c_2 are the

probability amplitudes. From the Schrödinger equation, their evolution is found to be

$$i\hbar\frac{\partial c_1(t)}{\partial t} = -\mu_1 E_+ c_2(t)e^{-i\omega_{0,1}t},$$
$$i\hbar\frac{\partial c_2(t)}{\partial t} = -\mu_1 E_- c_1(t)e^{i\omega_{0,1}t}.$$

The above equations can now be transformed into the ZSE [20–22], which is the eigenvalue problem of the Lax pair of the SIT equations. The transformation that carries this out is to simply take the probability amplitudes to be of the form

$$c_1 = iv_1^* e^{-i\frac{\Delta_1}{2}t}, \quad c_2 = v_2^* e^{i[kz+\frac{\Delta_1}{2}t]}, \tag{3}$$

where $\Delta_1 = \omega_{0,1} - \omega$. Now, from this, we can obtain the ZSE, but with ζ replaced by the value of $\zeta = \Delta_1/2$. In other words, turning this around, if we would start with the ZSE

$$\frac{\partial v_1}{\partial t} + i\zeta v_1 = qv_2, \quad \frac{\partial v_2}{\partial t} - i\zeta v_2 = rv_1, \tag{4}$$

we then obtain the evolution of c_1 and c_2 from Eqs. (3) and (4) upon taking

$$r = -\frac{\mu_1}{\hbar}\hat{E}^+, \quad q = \frac{\mu_1}{\hbar}\hat{E}^- = -r^*. \tag{5}$$

Important for us will be the average values of the Pauli operators, which are $s_1^i = Tr\langle\Psi|\hat{\sigma}^i|\Psi\rangle$, (where $i = x,y,z$) and are found to have the forms [19]

$$s_1^x = c_1^*(t)c_2(t)e^{-i\omega_{0,1}t} + c_1(t)c_2^*(t)e^{i\omega_{0,1}t},$$
$$s_1^y = ic_1^*(t)c_2(t)e^{-i\omega_{0,1}t} - ic_1(t)c_2^*(t)e^{i\omega_{0,1}t}, \tag{6}$$
$$s_1^z = c_2^*(t)c_2(t) - c_1^*(t)c_1(t).$$

Defining $s_1^\pm = \frac{1}{2}(s_1^x \pm is_1^y)$, then from Eqs. (6) and (3), we obtain

$$s_1^+ = iv_1^* v_2 e^{-i(kz-\omega t)}, \quad s_1^- = (s_1^+)^*,$$
$$s_1^z = |v_2(t)|^2 - |v_1(t)|^2. \tag{7}$$

Upon defining the quantities

$$\rho_1^+ = v_1^* v_2, \quad \rho_1^- = (\rho_1^+)^*, \tag{8}$$

then from Eqs. (4), (7), and (8), we obtain the undamped Bloch equations:

$$\frac{\partial\rho_1^+}{\partial t} = i\Delta_1\rho_1^+ - rs_1^z,$$
$$\frac{\partial s_1^z}{\partial t} = 2(r\rho_1^- - q\rho_1^+). \tag{9}$$

These equations above are given for the passive atoms ($j = 1$). The same equations will hold as well for the active atoms ($j = 2$) with one exception: when the active atoms are different from the passive atoms, generally the dipole moments will also be different. This is crucial since the dipole moments enter into the definition of q and r, as in Eq. (5), which are the ZSE potentials. We standardize this by letting the passive atoms define these potentials, as given in Eq. (5). There is a second point about Eq. (9), and that is that it is a linear homogeneous set of ordinary differential equations (ODEs). These equations will also describe the active atoms, but with different initial conditions. (And also upon replacing Δ_1 with the frequency shift for the active atoms, which will be Δ_2.)

We have to note that the connection between the reduced optical Bloch equations and ZSE for the linearly polarized pulse have been considered in Refs. [8,20].

Let us now define the macroscopic quantities. For either value of j, we take

$$\Lambda_j^\pm = \pm i\mu_j n_j \rho_j^\pm, \quad W_j = \mu_j n_j s_j^z, \Lambda_j^- = \Lambda_j^{+*}, \tag{10}$$

where the index $j = 1$ indicates the passive type of atoms and $j = 2$ indicates the active type of atoms. The quantities $\omega_{0,j}, \mu_j = |\vec{\mu}_{12}|_j$ and n_j are respectively the transition frequency between the levels, the dipole matrix element of the transition, and the concentrations of the jth kind of atom. Similarly, we use $\Delta_j = \omega_{0,j} - \omega$, which is the difference between the central frequency of the optical pulse and the atomic transition frequency for the jth type of atom.

With this notation, the evolution equations for the quantities in Eq. (10), for passive atoms, follow from Eq. (9) and are

$$\frac{\partial\Lambda_1^+}{\partial t} = i\Delta_1\Lambda_1^+ - irW_1,$$
$$\frac{\partial W_1}{\partial t} = 2i(r\Lambda_1^- + q\Lambda_1^+). \tag{11}$$

This is the integrable form for these quantities. The initial value of s^z for the excited atoms is $s^z = +1$ while the passive atoms will have $s^z = -1$.

We now introduce phenomenological decay constants into these equations. Doing so for Eq. (11) gives

$$\frac{\partial\Lambda_1^+}{\partial t} = i\Delta_1\Lambda_1^+ - irW_1 - \frac{\Lambda_1^+}{T_2^{(1)}} - \Lambda_{T,1}^+,$$
$$\frac{\partial W_1}{\partial t} = 2i(r\Lambda_1^- + q\Lambda_1^+) - \frac{W_1 - W_0^{(1)}}{T_1^{(1)}}. \tag{12}$$

Meanwhile, the expressions for the active atoms will be slightly different. First, their dipole moment μ_2 will generally be different from μ_1. Second, we have chosen to define q and r by Eq. (5), in which case the analogous equations for the active atoms will have corrections due to $\mu_2 - \mu_1$. These are

$$\frac{\partial\Lambda_2^+}{\partial t} = i\Delta_2\Lambda_2^+ - ir(1+\tilde{\epsilon})W_2 - \frac{\Lambda_2^+}{T_2^{(2)}} - \Lambda_{T,2}^+,$$
$$\frac{\partial W_2}{\partial t} = 2i(1+\tilde{\epsilon})(r\Lambda_2^- + q\Lambda_2^+) - \frac{W_2 - W_0^{(2)}}{T_1^{(2)}} + p, \tag{13}$$

where

$$\tilde{\epsilon} = \frac{\mu_2 - \mu_1}{\mu_1}.$$

In the above, $W_0^{(j)}$ is the equilibrium value, toward which the inversion W_j relaxes when $r = q = 0$. The quantities $T_1^{(j)}$ and $T_2^{(j)}$ are the longitudinal and transverse relaxation times. The quantity p is the rate of pumping needed to sustain the density of active atoms and affects the equilibrium value that W_2 decays toward, which is now $pT_1^{(2)} + W_0^{(2)}$.

We have also include terms in the above which would account for non-Markovian memory effects. These are given

by the quantities

$$\Lambda^{\pm}_{T,j} = M_2^{(j)} \int_{-\infty}^{\tau} \Lambda^{\pm}_j e^{\pm i\Delta_j(\tau-t')} e^{-\frac{N_2^{(j)}}{2}(\tau-t')^2} dt', \tag{14}$$

where the quantities $M_2^{(j)}$ and $N_2^{(j)}$ are the second moments of the correlation function and memory function of the two type of atoms, respectively [8].

Equations (12) and (13) are optical semiphenomenological Bloch equations for passive and active atoms in the rotating frame. As written, they are valid for Markovian as well as for non-Markovian transverse relaxations. If we set $\Lambda^{\pm}_{T,j} = 0$ in these equations, then they become the "$T_1 - T_2$ model" of the Bloch equations. This model is valid when the shape of the spectral line is Lorentzian. Such a situation occurs, for example, in gaseous media and in some solids where the transverse relaxations are described by the term $\Lambda^{\pm}_j/T_2^{(j)}$. However, this model is not valid in most solids, because quite often the free induction decay does not have an exponential character. In this case, it is more appropriate to include memory effects into the Bloch equations in order to obtain an accurate description of transverse relaxation processes. Then one would remove the term $\Lambda^{\pm}_j/T_2^{(j)}$ and leave the term $\Lambda^{\pm}_{T,j}$ to describe the transverse relaxations. This non-Markovian form of the optical Bloch equations has been studied in Ref. [8] for a single kind of passive atom. Due to the complexity of the non-Markovian calculations for multiple types of atoms, that problem will be studied later.

Finally, in addition to Eqs. (12) and (13), we require the description of the propagation of the optical pulse in the medium. The wave equation for the optical electric field $E(z,t)$ is given by

$$\frac{\partial^2 E}{\partial z^2} - \frac{\varepsilon}{c^2}\frac{\partial^2 E}{\partial t^2} - \frac{4\pi\sigma}{c^2}\frac{\partial E}{\partial t} = \frac{4\pi}{c^2}\frac{\partial^2 P}{\partial t^2}, \tag{15}$$

where c is the speed of light in vacuum, σ is the effective conductivity [σ could also be taken as the coefficient of nonresonance gain ($\sigma > 0$) or nonresonance absorption ($\sigma < 0$) [14]], $\varepsilon = \hat{n}^2$ is the coefficient of permittivity and $\hat{n}$ is the (frequency-dependent) index of refraction [23].

Upon taking into account the inhomogeneous broadening of the spectral lines, the polarization of the two-component system is given by

$$\vec{P} = \sum_{j=1}^{2} n_j \mu_j \int g_j(\Delta_j)(\vec{e}_- s_j^+ + \vec{e}_+ s_j^-) d\Delta_j, \tag{16}$$

where $g_j(\Delta_j)$ is the inhomogeneous broadening function for the jth type of atoms or SQDs, which is normalized to $\int_{-\infty}^{+\infty} g_j(\gamma)d\gamma = 1$.

Substituting the polarization (16) into Eq. (15), using the expansions (1) and then taking into account Eq. (2), we obtain the following nonlinear wave equation for the optical envelope:

$$\left(\frac{\partial}{\partial z} + \frac{\hat{n}}{c}\frac{\partial}{\partial t}\right) r = i\text{æ}[\langle\Lambda_1^+\rangle + \langle\Lambda_2^+\rangle] - \tilde{\sigma} r. \tag{17}$$

The averages in Eq. (17) are defined by

$$\langle\Lambda_l^{\pm}\rangle = \int g_l(\Delta_l)\Lambda_l^{\pm} d\Delta_l, \quad \text{æ} = \frac{2\pi\omega\mu_1}{\hat{n}\hbar c}, \quad \tilde{\sigma} = \frac{2\pi\sigma}{c\hat{n}}. \tag{18}$$

We take the retarded time $\tau = t - \frac{\hat{n}}{c}z$ to be a new timelike coordinate and $\chi = z$ to be a new independent spatial variable. In this coordinate system, the nonlinear wave equation for the optical envelope becomes

$$\frac{\partial r}{\partial \chi} = i\text{æ}\sum_j \langle\Lambda_j^+\rangle - \tilde{\sigma} r, \tag{19}$$

where we have generalized the system to have any number of types of atoms. Meanwhile, the material equations (12) and (13) are the same except for τ replacing t in the differentials, whence

$$\begin{aligned}
\frac{\partial \Lambda_j^+}{\partial \tau} &= i\Delta_j\Lambda_j^+ - i(1+\tilde{\epsilon}\delta_{j,2}) r W_j - \frac{\Lambda_j^+}{T_2^{(j)}} - \Lambda_{T,j}^+, \\
\frac{\partial W_j}{\partial \tau} &= 2i(1+\tilde{\epsilon}\delta_{j,2})(r\Lambda_j^- + q\Lambda_j^+) - \frac{W_j - W_0^{(j)}}{T_1^{(j)}} + p\delta_{j,2}.
\end{aligned} \tag{20}$$

Note that the functional dependencies of these quantities are $\Lambda_j^{\pm}(\tau,\chi;\Delta_j)$ and $W_j(\tau,\chi;\Delta_j)$.

In preparation for perturbation studies, we shall separate these equations into parts which are exactly integrable and collect all other terms into a set of perturbations. We take Eqs. (20) and (19) to be in the following form:

$$\frac{\partial \Lambda_j^+}{\partial \tau} = i\Delta_j\Lambda_j^+ - i r W_j + \delta(\Lambda_{j,\tau}^+), \tag{21}$$

$$\frac{\partial W_j}{\partial \tau} = 2i(r\Lambda_j^- + q\Lambda_j^+) + \delta(W_{j,\tau}), \tag{22}$$

$$\frac{\partial r}{\partial \chi} = i\text{æ}\sum_j \langle\Lambda_j^+\rangle + \delta(r_\chi), \tag{23}$$

where all perturbations of these equations are collected into the δ terms. For the special cases given above, these δ terms are

$$\delta(\Lambda_{j,\tau}^+) = -i\tilde{\epsilon}\delta_{j,2} r W_j - \frac{\Lambda_j^+}{T_2^{(j)}} - \Lambda_{T,j}^+, \tag{24}$$

$$\delta(W_{j,\tau}) = 2i\tilde{\epsilon}\delta_{j,2}(r\Lambda_j^- + q\Lambda_j^+) - \frac{W_j - W_0^{(j)}}{T_1^{(j)}} + p\delta_{j,2}, \tag{25}$$

$$\delta(r_\chi) = -\tilde{\sigma} r. \tag{26}$$

As has been noted before, for the Markovian case, one deletes the term $\Lambda_{T,j}^+$ in Eq. (24), while for the non-Markovian case, one deletes instead the term $\Lambda_j^+/T_2^{(j)}$.

The above is the completion of the governing equations for the system that we wish to study. Each atomic system (SQD) is defined by two key parameters: the atomic dipole moments μ_j and the frequencies of excitation $\omega_{0,j}$ of the transition, each of which could be different for each type of atom. We have only one optical pulse q which has to interact with both types of atoms.

As per the definition of Δ_j, following Eq. (3), the inhomogeneous broadening function for the jth atom has a zero argument when its resonant frequency is equal to the central frequency of the electric field. In principle, the resonant

frequency of each kind of atom will have a distribution given by $g_j(\Delta_j)$. The only real restriction is that the envelope spectrum of the optical field should have nonzero values at the various resonant frequencies, if any coupling between the kinds of atoms is to occur.

Equations (17) and (20) are the system of equations of SIT for the ensemble of the two-type atoms (SQDs) in a conductive medium including relaxations and pumping. In the next sections, we investigate the evolution of the single-soliton solutions of these equations by IST. Contributions to the evolution of the soliton's parameters due to the small terms of these equations, specifically $\Lambda_j^{\pm}/T_2^{(j)}$, $(W_j - W_0^{(j)})/T_1^{(j)}$, p, $\tilde{\sigma}$, and $\tilde{\epsilon}$ will be obtained by means of the standard IST perturbation theory, which is derived in Appendix A.

III. INVERSE SCATTERING TRANSFORM

The IST is usually given by a Lax pair where one member of the Lax pair is an eigenvalue problem and the other member is an evolution operator. These integrable problems are typically solved by using the eigenvalue problem to decompose the initial data into scattering data, and then using the evolution operator to obtain the evolution of the scattering data. For the integrable form of the above equations, specifically Eqs. (21)–(23) without the δ terms, the Lax pair can be obtained by generalizing the results given in Ref. [22].

However, a Lax pair is not the only way to obtain the evolution of scattering data. As shown in Ref. [24], one may obtain the evolution of the four scattering coefficients by evaluating the integral of $G(u,v;\zeta)$ given in the Appendix A, Eq. (A1). The same can be done similarly for the bound-state parameters ζ_k and D_k. This approach can be used for any integrable system. Thus, given the correct eigenvalue problem, one may bypass the evolution operator of the Lax pair and directly construct the evolution of the scattering data. This approach is also useful for perturbations of any integrable system, in that it will give the first-order shift due to a perturbation in the evolution of the scattering data.

For a mixture of two different kinds of atoms, each kind of atom could have different values for their physical constants as well as its own functional form of W_j, $\Lambda_j^{\pm}$, and $g_j(\Delta_j)$. The polarization of the two-level atoms (SQDs) will be a sum over j, as on the right-hand side of Eq. (16). It is understood that, for interactions to occur, the two resonant frequencies should be sufficiently close to each other so that they are both inside the spectral window of the optical field.

Taking the eigenvalue problem to be Eq. (4), upon inserting $q(\tau,\chi=0)$ and $r(\tau,\chi=0)$ for q and r, we obtain the initial scattering data [7,21,22]. As pointed out above, one can obtain the evolution in χ of the scattering data for the integrable part and, also, to first order for the perturbing parts. From the inverse scattering equations [22], one may then reconstruct q and r at any other value of χ. Once q and r are obtained, then integration of the linear differential equations (20) will give $\Lambda_j^{\pm}$ and W_j.

IV. ONE-SOLITON SOLUTIONS AND STATE VARIABLES

Given Eqs. (19) and (20) (when $\Lambda_j^{\pm}/T_2^{(j)} = 0 = \Lambda_{T,j}^{\pm} = (W_j - W_0^{(j)})/T_1^{(j)} = p = \tilde{\sigma}$) and the appropriate initial conditions, it follows that we can solve this system of equations by using the IST. To carry out this process, certain details need to be covered concerning the ZSE eigenvalue problem and to establish the notation.

There are two pairs of linearly independent solutions (Jost functions) of the ZSE: the first pair is denoted by

$$\Phi, \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$$

and

$$\bar{\Phi}, \begin{pmatrix}\bar{\phi}_1\\ \bar{\phi}_2\end{pmatrix},$$

and the second pair is

$$\Psi, \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

and

$$\bar{\Psi}, \begin{pmatrix}\bar{\psi}_1\\ \bar{\psi}_2\end{pmatrix}.$$

The first pair Φ and $\bar{\Phi}$ is defined by the asymptotic limit as $\tau \to -\infty$ to be

$$\Phi e^{i\zeta\tau} \to \begin{pmatrix}1\\ 0\end{pmatrix}, \quad \bar{\Phi} e^{-i\zeta\tau} \to \begin{pmatrix}0\\ -1\end{pmatrix},$$

and the second pair Ψ and $\bar{\Psi}$ is defined by the asymptotic limit as $\tau \to +\infty$ to be

$$\Psi e^{-i\zeta\tau} \to \begin{pmatrix}0\\ 1\end{pmatrix}, \quad \bar{\Psi} e^{i\zeta\tau} \to \begin{pmatrix}1\\ 0\end{pmatrix}.$$

For real ζ, the scattering coefficients a, b, $\bar{a}$, and $\bar{b}$ are defined from the asymptotic limits at $\tau \to +\infty$, where

$$\Phi \to \begin{pmatrix}a e^{-i\zeta\tau}\\ b e^{i\zeta\tau}\end{pmatrix}, \quad \bar{\Phi} \to \begin{pmatrix}\bar{b} e^{-i\zeta\tau}\\ -\bar{a} e^{i\zeta\tau}\end{pmatrix}.$$

On the real axis, one finds that $a\bar{a} + b\bar{b} = 1$. From the above definitions, one observes that, in general, the two pairs of solutions can be related as

$$\Phi = a\bar{\Psi} + b\Psi, \quad \bar{\Phi} = -\bar{a}\Psi + \bar{b}\bar{\Psi}, \tag{27}$$

and from the other direction

$$\Psi = -a\bar{\Phi} + \bar{b}\Phi, \quad \bar{\Psi} = \bar{a}\Phi + b\bar{\Phi}. \tag{28}$$

From the relation $r = -q^*$, it follows that $\bar{\Phi}(\zeta)$ and $\bar{\Psi}(\zeta)$ can be given in terms of $\Phi(\zeta)$ and $\Psi(\zeta)$:

$$\bar{\Phi}(\zeta) = \begin{pmatrix}\phi_2^*(\zeta^*)\\ -\phi_1^*(\zeta^*)\end{pmatrix}, \quad \bar{\Psi}(\zeta) = \begin{pmatrix}\psi_2^*(\zeta^*)\\ -\psi_1^*(\zeta^*)\end{pmatrix},$$

and $\bar{a}(\zeta) = a^*(\zeta^*)$, $\bar{b}(\zeta) = b^*(\zeta^*)$.

In addition to the continuous spectra, the ZSE (20) can also possess bound states. These occur whenever $a(\zeta)$ has a zero in the upper-half complex ζ plane, and they are designated by $\zeta_k = \xi_k + i\eta_k$ with both ξ and η real, but with $\eta > 0$, where k can be any integer up to N with N being the total number of such bound states. The one-soliton solution of

the ZSE corresponds to the case where a has only one such zero and the scattering coefficient $b(\zeta) = 0$ for ζ real. When $a(\zeta_k) = 0$, from (27), it follows that $\Phi(\zeta_k) = b_k \Psi(\zeta_k)$, where b_k is some constant value. Similarly, from (28), it follows that $\Psi(\zeta_k) = \bar{b}_k \Phi(\zeta_k)$ from whence $\bar{b}_k b_k = 1$. If the potentials are on compact support, then it also follows that these two coefficients are related to the values of $b(\zeta)$ and $\bar{b}(\zeta)$ at $\zeta = \zeta_k$, as $b_k = b(\zeta_k)$ and $\bar{b}_k = \bar{b}(\zeta_k)$. Since b_k will generally have real and imaginary parts, it can then be represented by $b_k = \exp(i\beta_k + 2\eta x_{0,k})$, where β_k is a phase and $x_{0,k}$ is the position of the kth soliton.

The one-soliton solution is a solution where there is only one bound state and no continuous spectrum. This solution has a closed analytical solution. For ζ real, these Jost functions are

$$\begin{aligned} \phi_1(\zeta)e^{i\zeta\tau} &= \frac{1 + e^{4\eta(\tau-\tau_0)}a(\zeta)}{1 + e^{4\eta(\tau-\tau_0)}}, \\ \phi_2(\zeta)e^{i\zeta\tau} &= \frac{i\eta e^{i(\beta+2\tau\xi)}}{(\zeta - \zeta_1^*)\cosh[2\eta(\tau-\tau_0)]}, \end{aligned} \tag{29}$$

$$\begin{aligned} \psi_1(\zeta)e^{-i\zeta\tau} &= \frac{i\eta e^{-i(\beta+2\tau\xi)}}{(\zeta - \zeta_1^*)\cosh[2\eta(\tau-\tau_0)]}, \\ \psi_2(\zeta)e^{-i\zeta\tau} &= \frac{a(\zeta) + e^{4\eta(\tau-\tau_0)}}{1 + e^{4\eta(\tau-\tau_0)}}, \end{aligned} \tag{30}$$

where the k subscripts on ξ, η, β, and τ_0 is dropped since there is only one zero. For a one-soliton solution, the scattering data for the continuous spectra is $a(\zeta) = \frac{\zeta-\zeta_1}{\zeta-\zeta_1^*}$, $b(\zeta) = 0 = \bar{b}(\zeta)$, where the eigenvalue is $\zeta_1 = \xi + i\eta$ with ξ real and η real and positive. The one-soliton solution for q is

$$q(\tau) = \frac{-2\eta e^{-i(\beta+2\tau\xi)}}{\cosh[2\eta(\tau-\tau_0)]}. \tag{31}$$

As can be seen from the definitions for q and r, below Eq. (4), $2\eta\hbar/\mu_1$ is the pulse height of the optical field, $\omega - 2\xi$ is the instantaneous frequency, $\beta + 2\eta\chi/c$ is the phase at fixed χ, and τ_0 is the central position of the pulse, which determines the delay time of the soliton in the medium.

At the bound-state eigenvalue in the upper-half ζ plane (UHP) ($\zeta = \zeta_1$), from Eqs. (29) and (30), the Jost solutions are

$$\begin{aligned} \phi_1(\zeta_1)e^{i\zeta_1\tau} &= \frac{1}{1 + e^{4\eta(\tau-\tau_0)}}, \\ \phi_2(\zeta_1)e^{i\zeta_1\tau} &= \frac{e^{i(\beta+2\xi\tau)}}{2\cosh[2\eta(\tau-\tau_0)]}, \end{aligned} \tag{32}$$

$$\begin{aligned} \psi_1(\zeta_1)e^{-i\zeta_1\tau} &= \frac{e^{-i(\beta+2\xi\tau)}}{2\cosh[2\eta(\tau-\tau_0)]}, \\ \psi_2(\zeta_1)e^{-i\zeta_1\tau} &= \frac{1}{1 + e^{-4\eta(\tau-\tau_0)}}. \end{aligned} \tag{33}$$

At the bound-state eigenvalue in the lower-half ζ plane (LHP) ($\zeta = \zeta_1^*$), the Jost functions are related to the complex conjugates of the appropriate Jost functions in the UHP.

As has been noted above, Eqs. (7) and (8), there is a direct relationship between the quantum mechanics Schrödinger equation for a two-level atom and the ZSE, which is given in Eq. (4). By using this relationship, all atomic expectation values may be given as quadratic combinations of the Jost functions of the ZSE upon replacing ζ by $\frac{1}{2}\Delta_j$, provided that $\mu_j = \mu_1$ (when $\mu_j \neq \mu_1$, then one can use a perturbation theory). For initial values, we assume at $\tau = \tau_0$ that all passive atoms are in their ground states and all active atoms are in their excited states. From Eqs. (7) and (8), we have the solutions for $\rho^{\pm}$ and s^z in terms of v. To determine v, one takes it to be some linear combination of Φ and $\bar{\Phi}$ since v must satisfy Eq. (4). Taking $q = 0$ at $\chi = 0$ when $\tau < \tau_0$ is equivalent to taking $q = 0$ at $z = 0$ and $t < 0$. Then if one specifies the initial values of s^z and $\rho^{\pm}$ at $\tau = \tau_0$, one can determine the correct linear combination of Φ and $\bar{\Phi}$, up to an overall sign which is arbitrary. From the initial values, one finds that $v = \Phi$ for the passive atoms and $v = \bar{\Phi}$ for the active atoms. This gives

$$\begin{aligned} \rho_j^+(\Delta_j) &= \pm\phi_1^*(\zeta)\phi_2(\zeta)|_{\zeta=\frac{1}{2}\Delta_j}, \\ s_j^z(\Delta_j) &= \pm[|\phi_2(\zeta)|^2 - |\phi_1(\zeta)|^2]|_{\zeta=\frac{1}{2}\Delta_j}, \end{aligned} \tag{34}$$

where the upper (lower) sign is for passive (active) atoms.

The above is the general solution for ρ_j^+ and s_j^z. However, if r and q are taken to be a soliton solution, then there are closed-form expressions for the components of Φ, given by Eq. (29). Substituting these into Eq. (34), for the passive atoms, one obtains

$$\rho_j^+(\tau,\Delta_j) = -2i\eta e^{i(\beta+2\xi\tau)}\frac{(\zeta_1 - \frac{1}{2}\Delta_j)e^{2\eta(\tau-\tau_0)} + (\zeta_1^* - \frac{1}{2}\Delta_j)e^{6\eta(\tau-\tau_0)}}{|\zeta_1 - \frac{1}{2}\Delta_j|^2(e^{4\eta(\tau-\tau_0)} + 1)^2}, \tag{35}$$

$$s_j^z(\tau,\Delta_j) = \frac{8\eta^2 e^{4\eta(\tau-\tau_0)}}{|\zeta_1 - \frac{1}{2}\Delta_j|^2(e^{4\eta(\tau-\tau_0)} + 1)^2} - 1. \tag{36}$$

From Eq. (10), one may now construct $\Lambda_j^{\pm}$ and W_j from the above. The expressions for the same quantities, but for the active atoms, are the negatives of the above, with Δ_1 replaced by Δ_2 if $\mu_2 = \mu_1$. When this latter is not the case, then one can obtain the first-order corrections due to the differences in the dipole moments by means of perturbation theory, as we obtain next.

V. INVERSE SCATTERING TRANSFORM PERTURBATIONS

The IST perturbation theory allows one to calculate the first-order shifts in the solution of the SIT nonlinear equations when small perturbation terms are present. Examples of which, in this system, are Markovian or non-Markovian transverse and longitudinal relaxations, different dipole moments, pumping of active atoms, and conductivity of the host medium. A general method for deriving the evolution of the SIT scattering data in the presence of first-order perturbations is given in Appendix A. In Appendix B, we briefly give the evolution of the SIT scattering data in the absence of any perturbation.

In the following, the various quantities $\Lambda_j^{\pm}/T_2^{(j)}$, $\Lambda_{T,j}^{\pm}$, $(W_j - W_0^{(j)})/T_1^{(j)}$, $\tilde{\sigma}$, p in the perturbations (24)–(26) are assumed to be small. We then proceed to calculate their first-order influence on the evolution of a one-soliton solution, Eq. (4).

The unperturbed evolution only requires the initial values of the scattering data. But when perturbations are present, then in order to carry out the associated perturbation calculations, one must first construct $h(u,v;\zeta)$ as given by Eq. (A5). To do this, one must have the values of the Jost functions, u and v, for all values of τ. In general, these can be constructed in a closed form only in the case of pure soliton solutions. Here we shall study the case of the perturbation of a single soliton propagating in a medium.

As shown in Refs. [7,8], the evolution of these perturbations can be given in terms of one function, $I(u,v;\zeta)$, which can be separated into two parts:

$$I(\psi,\psi,\zeta) = i\int_{-\infty}^{+\infty}\left(\frac{\partial q}{\partial\chi}\psi_2^2 - \frac{\partial r}{\partial\chi}\psi_1^2\right)d\tau = f(\tau;\zeta)|_{\tau\to-\infty}^{\tau\to+\infty} + \int_{-\infty}^{\infty} h(\tau;\zeta)d\tau, \quad (37)$$

where f contains the integrable part and h contains the perturbing parts. The general form of f and h for general perturbations of the SIT system are obtained in Appendix A.

Scattering data for this system consists of the reflection coefficient $\bar{b}(\zeta)/a(\zeta)$, which determines the continuous spectra and, if bound states are present, each bound state has its own bound-state parameters, which consist of an eigenvalue, ζ_k, and a normalization coefficient D_k, where the subscript k indicates the kth bound state. We assume in general that the number of bound states is finite. However, here, we shall only treat the one-soliton case.

The rules for determining the evolution of the scattering data are the following. From Refs. [7,8], given the function $I(u,v;\zeta)$, then the continuous spectra evolves as

$$\frac{\partial}{\partial\chi}\left(\frac{\bar{b}}{a}\right) = \frac{i}{a^2}I(\psi,\psi;\zeta) = \frac{i}{a^2}\int_{-\infty}^{\infty} d\tau G(\psi,\psi;\zeta), \quad (38)$$

where $\bar{b}(\zeta)/a(\zeta)$ is the reflection coefficient for inversion about $-\infty$ and $G(u,v;\zeta)$ is the integrand of $I(u,v;\zeta)$. For the SIT system, $G(u,v;\zeta)$ is given by Eqs. (A3)–(A5).

The kth bound-state eigenvalue will evolve as

$$\frac{\partial\zeta_k}{\partial\chi} = i\frac{\bar{b}(\zeta_k)}{a_k'}\int_{-\infty}^{\infty} d\tau G(\phi,\phi;\zeta)|_{\zeta=\zeta_k}, \quad (39)$$

where the evaluation must be done before the integration.

Meanwhile the kth normalization coefficient evolves as [25]

$$\frac{\partial D_k}{\partial\chi} = \frac{-i}{a_k'}\int_{-\infty}^{\infty} d\tau\left\{\frac{\partial}{\partial\zeta}\left[\frac{1}{a'(\zeta)}G(\psi,\psi;\zeta)\right]\right\}\Big|_{\zeta=\zeta_k}, \quad (40)$$

where

$$D_k = \frac{-\bar{b}(\zeta_k)}{a_k'}, \quad a_k' = \left(\frac{\partial a}{\partial\zeta}\right)\Big|_{\zeta=\zeta_k}. \quad (41)$$

As pointed out earlier, $\bar{b}_k = 1/b_k$ so one could also write $D_k = -1/(b_k a_k')$.

Next we shall address the solution of the perturbed equations (19) and (20) when r is initially a pure one-soliton solution ($k = 1$) and we have a collection of two kinds of atoms, one active and one passive.

VI. PARAMETERS OF DISSIPATIVE SOLITONS

Let us now apply the perturbation expansion for the ZSE developed in Refs. [7,8] to determine the evolution of the optical soliton's parameters ξ, η, β, and τ_0 when we have transverse relaxations, regular longitudinal relaxations, conductivity of the medium, and pumping. Appendix A details how to obtain the key quantity, $G(u,v;\zeta)$, which is the integrand of $I(u,v;\zeta)$, for any perturbation. Appendix B details the integrable evolutions of the parameters of the scattering data. Here we want to find how the evolution of these parameters vary from the integrable case when the perturbations are present.

The IST perturbation theory allows one to calculate the first-order shifts in the soliton solution (31) of the SIT nonlinear equations due to the presence of a small perturbation terms: (Markovian or non-Markovian) transverse and longitudinal relaxations, effects connected with the difference between dipole moments of atoms, pump of the active atoms, and conductivity of the host medium.

The initial scattering data consists only of the bound-state parameters: the eigenvalue ζ_1 and the normalization coefficient D_1. Perturbations will not only perturb these two parameters, but they also could cause a continuous spectrum to be generated. However, here we shall only be concerned with how the soliton parameters become perturbed.

To obtain these perturbations, one proceeds as follows: First, one constructs the nonintegrable part of $I(\psi,\psi;\zeta)$, which is $I_1(\psi,\psi;\zeta) = \int_{-\infty}^{+\infty} d\tau h(\tau;\zeta)$. Upon including all perturbations found in Eqs. (24)–(26), from Eq. (A5), for the Markovian "$T_1 - T_2$-model" of the Bloch equations, one

obtains

$$h(\tau,\chi;\zeta) = i\tilde{\sigma}(ru_1v_1 - qu_2v_2) + \frac{1}{2}æp(u_1v_2 + u_2v_1)\left\langle\frac{1}{2\zeta-\Delta}\right\rangle + \sum_{j=1}^{2} æ\tilde{\epsilon}\delta_{j,2}\left[-\left\langle\frac{W_j}{2\zeta-\Delta}\right\rangle(ru_1v_1 + qu_2v_2) + i\sum_{j=1}^{2}\left\langle\frac{r\Lambda_j^- + q\Lambda_j^+}{2\zeta-\Delta}\right\rangle(u_1v_2 + u_2v_1)\right] - \sum_{j=1}^{2}\frac{æ}{2T_1^{(j)}}\left\langle\frac{W_j - W_0^{(j)}}{2\zeta-\Delta}\right\rangle(u_1v_2 + u_2v_1) + i\sum_{j=1}^{2}\frac{æ}{T_2^{(j)}}\left[\left\langle\frac{\Lambda_j^-}{2\zeta-\Delta}\right\rangle u_2v_2 + \left\langle\frac{\Lambda_j^+}{2\zeta-\Delta}\right\rangle u_1v_1\right]. \quad (42)$$

In the above, $u = v = \Psi(\tau;\zeta)$ which, for the one-soliton solution, is given above in Eq. (30). Similarly, q and $r\,(=-q^*)$ for the one-soliton solution are given by Eq. (31). The macroscopic quantities required in Eq. (42) can be constructed from Eqs. (10), (35), and (36). When this is done, one has explicit τ and ζ dependencies for $h(\tau,\chi;\zeta)$. One should note that, according to Eq. (34), the $j=2$ active macroscopic quantities $\Lambda_2^{\pm}$ and W_2 are exactly the negative of the $j=1$ passive macroscopic quantities. This provides a simplification in later summations.

We assume that the experimental setup is such that the medium, which contains the passive and active atoms, is located in the region $z \geqslant 0$. The soliton enters the medium at $\tau = \tau_0$ and continues to propagate in the positive z direction. Thus there is nothing occurring between $\tau = -\infty$ and $\tau = \tau_0$. Consequently, the above integrals which have been given as an integral over τ, from $-\infty$ to $+\infty$, need to be replaced by an integral over τ, from τ_0 to $+\infty$.

Carrying out the above actions, one obtains for the evolution of the bound state eigenvalue, $\zeta_1 = \xi + i\eta$,

$$\partial_\chi\xi = -\frac{p}{8\eta}B_1^{(2)} - \frac{1}{2}\tilde{\epsilon}\eta\mu_2n_2A_1^{(2)} - \frac{1}{24\eta T_1^{(1)}} \times\left[4\mu_1n_1B_2^{(1)} - 3\left(W_0^{(1)} + \mu_1n_1\right)B_1^{(1)}\right] + \frac{1}{24\eta T_1^{(2)}} \times\left[4\mu_2n_2B_2^{(2)} + 3\left(W_0^{(2)} - \mu_2n_2\right)B_1^{(2)}\right] + [48\ln(2) - 29] \times\left[\frac{1}{6\eta T_2^{(1)}}\mu_1n_1B_2^{(1)} - \frac{1}{6\eta T_2^{(2)}}\mu_2n_2B_2^{(2)}\right], \quad (43)$$

$$\partial_\chi\eta = \frac{p}{8\eta}A_1^{(2)} + \frac{1}{2}\tilde{\epsilon}\eta\mu_2n_2B_1^{(2)} - \tilde{\sigma}\eta - \frac{1}{24\eta T_1^{(1)}}\left[4\mu_1n_1A_2^{(1)} - 3\left(W_0^{(1)} + \mu_1n_1\right)A_1^{(1)}\right] + \frac{1}{24\eta T_1^{(2)}}\left[4\mu_2n_2A_2^{(2)} + 3\left(W_0^{(2)} - \mu_2n_2\right)A_1^{(2)}\right] + \frac{\mu_1n_1}{12\eta T_2^{(1)}}\left\{[96\ln(2) - 58]A_2^{(1)} - 3A_1^{(1)}\right\} - \frac{\mu_2n_2}{12\eta T_2^{(2)}}\left\{[96\ln(2) - 58]A_2^{(2)} - 3A_1^{(2)}\right\}. \quad (44)$$

The new quantities in the above are defined as

$$A_n^{(j)} = \int_{-\infty}^{\infty}\frac{\alpha_j(\Delta_j)}{\left[\left(\frac{\Delta_j-2\xi}{2\eta}\right)^2+1\right]^n}d\Delta_j,$$

$$B_n^{(j)} = \int_{-\infty}^{\infty}\frac{\alpha_j(\Delta_j)\frac{\Delta_j-2\xi}{2\eta}}{\left[\left(\frac{\Delta_j-2\xi}{2\eta}\right)^2+1\right]^n}d\Delta_j, \quad n = 1,2,3,$$

and where the constant α_j is given by

$$\alpha_j(\Delta) = æg_j(\Delta_j).$$

Note that $A_n^{(j)}$ and $B_n^{(j)}$ are functions of ξ and η. Thus what we have in Eqs. (43) and (44) is a second-order set of nonlinear ordinary differential equations for the evolution of ξ and η, which are valid for sufficiently small values of $p, \tilde{\epsilon}, \tilde{\sigma}, 1/T_1^{(j)}$, and $1/T_2^{(j)}$. Integrating these equations for the evolutions of ξ and η will give a means of calculating the first-order influence of these small parameters on the soliton solution (31). Thus, one now has a means for constructing damped SIT solutions.

In addition, because longitudinal and transverse relaxation processes and conductivity are generally present in the above, the energy of the soliton

$$\mathcal{E} = \frac{c\hat{n}\hbar^2}{\mu_1^2\pi}\eta \quad (45)$$

will experience dissipation as it propagates through the medium. Since we have also included an external pump, energy losses could possibly be compensated, giving then a situation wherein a balance between the energy supplied and energy lost could arise. Then the soliton could reach an final state, wherein it would propagate at a constant speed and maintain a constant energy $\mathcal{E} = \text{const}$. In this case, a final state would be found from Eqs. (44) and (45), upon setting $d\xi/d\chi = 0 = d\eta/d\chi$, and then solving Eqs. (44) and (45) for possible final-state values of ξ and η as functions of the pump amplitude. Once one could have values of ξ and η which were constant, then it follows from the structure of the IST equations that the other soliton parameters, such as ξ, β, and τ_0, would became linear functions of distance χ. Thus one could then have traveling dissipative SIT solitons.

VII. CONCLUSION

We theoretically consider SIT in a two-component conductive medium consisting of nonlinear optical active (amplifying)

and passive (absorbing) two-level atoms or SQDs, under the condition of pumping the active atoms. If we neglect all damping effects, a soliton would have a fixed waveform (envelope) and energy, with its shape and energy given by Eqs. (31) and (45). But in physical systems, there always exist some damping effects, such as relaxations and conductivity. When this system is perturbed by relaxations, conductivity, and effects connected with different dipole moments, the shape of the waveform, its amplitude (width) and other parameters, will change slowly. A perturbation expansion based on the IST has been used to obtain the first-order effects of various perturbations on the propagation of such a 2π pulse. Explicit analytical but approximate expressions for the evolution of the amplitude (width), the instantaneous frequency (the frequency shifts), the phase modulation, and the decay rate of the pulse (velocity) can now be estimated by means of expressions (43) and (44).

Here we have treated the Markovian case. From these equations it is clear that the difference between dipole moments of the atoms, in lowest order, does influence the amplitude η and the instantaneous frequency ξ. The conductivity of the medium influences only the amplitude (width of the soliton) η and energy $\mathcal{E}$ and does not influence ξ. In the case when the dipole moments of the active and passive atoms are equal, $\tilde{\varepsilon} = 0$, the corresponding term in Eqs. (43) and (44) vanishes.

On the other hand, when there is a continuous supply of energy from an external source, which can be analytically expressed by means of a pumping term p then, with a balance between the energy supplied and energy lost, dissipative solitons can be formed. Their shape, amplitude, width, and energy could be conserved. In future work, we will consider the conditions for the existence of stable dissipative envelope solitons and will determine explicit analytical expression of its parameters.

From the expressions (43)–(45), it is clear that the addition of pumping, except for changing the value of η, does not change the soliton's shape (31) but does influence and shift all other soliton parameters η, ξ, β, τ_0, as well as $\mathcal{E}$ and they are linear functions of the propagation distance χ.

As an example for the coexistence of an ensemble of absorbing and amplifying transitions within the same quantum dot material, we refer to Ref. [26]. The coexistence results from a balanced electrical pumping mechanism in a p-n junction which prefers to invert energetically low-lying quantum transitions.

Summarizing the above results, there are differences between the usual conservative envelope soliton and possible dissipative envelope solitons. Unlike the dissipative envelope soliton of SIT considered above, the amplitude and energy of the conservative envelope soliton of SIT is constant only in the ideal case, which is when one could neglect any damping. But in conservative soliton systems in the presence of damping, η in lowest order tends to be a linear function of distance χ, which for higher orders of damping, relaxation or/and conductivity tend to take on an exponential decay. On the other hand, for the dispersive envelope soliton of SIT, η could actually become constant if the external pumping could compensate for atomic dissipation.

In the present work we adapted the theory of the SIT into a form convenient for investigation of dissipative envelope solitons of SIT in conducting two-component media. By the IST perturbation expansion, one can consider first-order effects of transverse and longitudinal relaxation processes and other small perturbations on the propagation of an optical dissipative 2π pulse. The approach which we have used here is of a rather general character and can be used also for investigations of dissipative solitons in many-component atomic or SQDs media.

By using IST, we consider the situation of a many-cycle SIT pulse in a two-component medium when the conductivity of the medium, $\tilde{\sigma}$, is constant. Another case of the few-cycle pulse propagation under the condition of the SIT and the frequency-dependent conductivity supplied by the Drude equation which is valid for semiconductors [23], numerically investigated earlier by Ref. [2], also can be investigated analytically by IST.

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APPENDIX A

The time evolution for simple integrable systems such as the Korteweg deVries equation or the nonlinear Schrödinger equation is done by using the second part of the Lax pair of equations to obtain the time evolution of the Jost functions, from which one directly obtains the time evolution of the scattering coefficients. When such systems are perturbed, the integrability is usually broken. In this case, both components of the Lax pair are no longer simultaneously valid. However, either one of the pair can be taken to be valid. For example, the eigenvalue problem of a Lax pair simply transforms a potential into scattering data. For an integrable system which is perturbed, the evolution of the scattering data is then correspondingly perturbed. Given any evolution of a potential, one can always determine the scattering data, which will also evolve. In particular, a method for determining the perturbations in the evolution of scattering data, directly the evolution of the potentials, was given in Ref. [24]. All that is required is the evaluation of an evolution function $I(u,v;\zeta)$ where u and v are Jost functions of the ZSE. In systems like SIT, there is an additional complexity. In addition to perturbations in the evolution of the electric field, there can arise additional perturbations from the macroscopic equations of the atomic variables. These perturbations have to be treated differently since they are not perturbations of the equations of motion for the potentials. A method for handling this class of perturbations was first presented in Ref. [7], albeit without a derivation. Here we shall provide a brief derivation as to how to include such perturbations of auxiliary equations into the IST perturbation theory.

To clarify this process, let us first consider the fully integrable case. Here, there are no perturbations, in which case $I(u,v,\zeta)$ can be expressed as an exact differential of some function, $f(\tau,\chi,\zeta)$ [24]. Then $I(u,v,\zeta)$ becomes simply the change in this function, f, evaluated between the two

endpoints, and therefore a function *only* of the scattering data. An example of how this comes about is given in Ref. [24].

When there are nonintegrable perturbations of the integrable equations, then $I(u,v,\zeta)$ will be a sum of an integrable part and a nonintegrable part [24]. The integrable part will be the differential of that same function $f(\tau,\chi,\zeta)$. The nonintegrable part will generally not be an exact differential. Instead, it would generally have to be evaluated by some other means, analytical and/or numerical.

In the perturbation theory for SIT, the auxiliary equations are the equations for the macroscopic functions $\Lambda^{\pm}$ and W. As the optical field itself evolves, these macroscopic functions are driven by their boundary values and the strength and shape of the optical field. They are determined by a set of ordinary differential equations, each of which could also contain various additional perturbations. The problem is to include these perturbations into the IST perturbation scheme.

Let us now turn our attention to the evaluation of $I(u,v;\zeta)$, starting from Eqs. (21)–(23). The general forms for $I(u,v;\zeta)$ was given by Eqs. (10.4)– (10.6) in Ref. [7]. Here we shall briefly provide a derivation. We shall only consider one specie of atoms since the general case follows upon including a sum over all species.

From Ref. [7], the integrand of $I(u,v;\zeta)$ is

$$G(u,v) = iq_{\chi}u_2v_2 - ir_{\chi}u_1v_1, \tag{A1}$$

where u and v are the appropriate Jost solutions of Eq. (4) and $I(u,v;\zeta) = \int_{-\infty}^{\infty} G(u,v)d\tau$. From Eq. (23), it follows that

$$\begin{aligned} G(u,v) = {} & u_1v_1\langle\Lambda^{+}\rangle - u_2v_2\langle\Lambda^{-}\rangle \\ & + iu_2v_2\delta(q_{\chi}) - iu_1v_1\delta(r_{\chi}). \end{aligned}$$

In order to compose an exact differential for the integrable parts, from Eqs. (4), (21), and (22), we start with the two identities

$$\begin{aligned} \partial_{\tau}(\Lambda^{-}u_2v_2) = {} & i(2\zeta-\Delta)\Lambda^{-}u_2v_2 - iqu_2v_2W \\ & + \Lambda^{-}r(u_1v_2+u_2v_1) + u_2v_2\delta(\Lambda^{-}_{\tau}), \\ \partial_{\tau}(\Lambda^{+}u_1v_1) = {} & -i(2\zeta-\Delta)\Lambda^{+}u_1v_1 - iru_1v_1W \\ & + \Lambda^{+}q(u_1v_2+u_2v_1) + u_1v_1\delta(\Lambda^{+}_{\tau}). \end{aligned}$$

Observe that, if one divides each of these expressions by $2\zeta-\Delta$ and then averages, as in Eq. (18), one can use these expressions to replace the $u_1v_1\langle\Lambda^{+}\rangle$ and $u_2v_2\langle\Lambda^{-}\rangle$ terms in $G(u,v)$, obtaining

$$\begin{aligned} G(u,v;\zeta) = {} & i\text{æ}\partial_{\tau}\left[\left\langle\frac{u_1v_1\Lambda^{+}}{2\zeta-\Delta}\right\rangle + \left\langle\frac{u_2v_2\Lambda^{-}}{2\zeta-\Delta}\right\rangle\right] \\ & - i\text{æ}(u_1v_2+u_2v_1)\left[\left\langle\frac{r\Lambda^{-}}{2\zeta-\Delta}\right\rangle + \left\langle\frac{q\Lambda^{+}}{2\zeta-\Delta}\right\rangle\right] \\ & - \text{æ}(qu_2v_2+ru_1v_1)\left\langle\frac{W}{2\zeta-\Delta}\right\rangle \\ & - i\text{æ}u_1v_1\left\langle\frac{\delta(\Lambda^{+}_{,\tau})}{2\zeta-\Delta}\right\rangle - i\text{æ}u_2v_2\left\langle\frac{\delta(\Lambda^{-}_{,\tau})}{2\zeta-\Delta}\right\rangle \\ & + iu_2v_2\delta(q_{\chi}) - iu_1v_1\delta(r_{\chi}). \end{aligned}$$

where now $G(u,v;\zeta)$ also picks up an additional dependence on ζ through the macroscopic averages. Next, as above, from Eq. (22) we can obtain the identity

$$\begin{aligned} & \partial_{\tau}[W(u_1v_2+u_2v_1)] \\ & \quad = 2W(qu_2v_2+ru_1v_1) + 2i(r\Lambda^{-}+q\Lambda^{+}) \\ & \qquad \times(u_1v_2+u_2v_1) + (u_1v_2+u_2v_1)\delta(W_{,\tau}), \end{aligned} \tag{A2}$$

which allows us to eliminate the averaged W term in Eq. (A2). Doing so then gives the following for the integrand $G(u,v;\zeta)$:

$$G(u,v;\zeta) = \partial_{\tau}f(\zeta,\tau) + h(\zeta,\tau), \tag{A3}$$

where

$$\begin{aligned} f(\zeta,\tau) = {} & \text{æ}\left\{iu_1v_1\left\langle\frac{\Lambda^{+}}{2\zeta-\Delta}\right\rangle + iu_2v_2\left\langle\frac{\Lambda^{-}}{2\zeta-\Delta}\right\rangle\right. \\ & \left. - \frac{1}{2}(u_1v_2+u_2v_1)\left\langle\frac{W}{2\zeta-\Delta}\right\rangle\right\}, \end{aligned} \tag{A4}$$

$$\begin{aligned} h(\zeta,\tau) = {} & -\text{æ}\left[iu_1v_1\left\langle\frac{\delta(\Lambda^{+}_{,\tau})}{2\zeta-\Delta}\right\rangle + iu_2v_2\left\langle\frac{\delta(\Lambda^{-}_{,\tau})}{2\zeta-\Delta}\right\rangle\right. \\ & \left. - \frac{1}{2}(u_1v_2+u_2v_1)\left\langle\frac{\delta(W_{,\tau})}{2\zeta-\Delta}\right\rangle\right] \\ & + iu_2v_2\delta(q_{,\chi}) - iu_1v_1\delta(r_{,\chi}) \end{aligned} \tag{A5}$$

which gives

$$\begin{aligned} I(u,v,\zeta) = {} & \int_{-\infty}^{\infty} G(u,v;\zeta)d\tau = f(\zeta,\tau)\Big|_{\tau\to-\infty}^{\tau\to+\infty} \\ & + \int_{-\infty}^{\infty} h(\zeta,\tau)d\tau. \end{aligned}$$

Thus we have been able to separate $I(u,v;\zeta)$ into two parts—the first part is an exact differential which results because the unperturbed system is an integrable system. The second part contains the contributions from all perturbations of the five fundamental equations of the SIT system.

For the multiple-atom case, since Eq. (23) requires a sum over the various populations, the expression for $G(u,v;\zeta)$ will also be a sum over $\Lambda^{\pm}$ and W for all populations. For the term $h(\zeta,\tau)$, this sum is also extended to be a sum over each individual type of perturbation, as in Eqs. (24)–(26).

APPENDIX B

The unperturbed evolutions of the scattering data will be briefly given here. First, one must obtain the unperturbed state variables and use them to construct $f(u,v;\zeta)$ as given by Eq. (A4). The unperturbed-state variables are defined in Eqs. (7) and (8) and can be given in terms of the Jost solutions. To match the initial values of the passive atoms, one must take $v=\Phi$, which gives

$$\begin{aligned} & \rho^{+}(\Delta_1) = \phi_1^{*}\phi_2, \quad \rho^{-}(\Delta_1) = \phi_1\phi_2^{*}, \\ & s^{z}(\Delta_1) = |\phi_2|^2 - |\phi_1|^2. \end{aligned} \tag{B1}$$

The evaluation of $f(u,v;\zeta)$ only requires the values as $\tau\to-\infty$ and as $\tau\to+\infty$. As $\tau\to-\infty$,

$$\rho^{\pm}(\Delta_1)\to 0, \quad s^{z}(\Delta_1)\to -1. \tag{B2}$$

For the active atoms, as per Eqs. (7) and (8), one only needs to reverse the signs of $\rho^{\pm}$ and s^z and then replace Δ_1 by Δ_2. In the limit of $\tau \to +\infty$, for the passive atoms, we use Eq. (27) to express Φ in terms of Ψ. Then, from the asymptotics of Ψ, one obtains

$$\begin{aligned} \rho^{+}(\Delta_1) &\to [e^{2i\zeta\tau}\bar{a}b]|_{\zeta=\frac{1}{2}\Delta_1}, \\ \rho^{-}(\Delta_1) &\to [e^{-2i\zeta\tau}a\bar{b}]|_{\zeta=\frac{1}{2}\Delta_1}, \\ s^{z}(\Delta_1) &\to -[a\bar{a}-\bar{b}b]|_{\zeta=\frac{1}{2}\Delta_1}. \end{aligned} \tag{B3}$$

where a, $\bar{a}$, b, and $\bar{b}$ are the ZSE scattering coefficients, in which ζ is replaced by $\frac{1}{2}\Delta_1$. For the active atoms, again one simply reverses the signs of $\rho^{\pm}$ and s^z with Δ_1 replaced by Δ_2.

Let $I_0(u,v;\zeta)$ designate the unperturbed part of $I(u,v;\zeta)$, which is $\sum_j . f_j(\tau;\zeta)|_{\tau\to-\infty}^{\tau\to+\infty}$. For our case here, j is only summed from 1 to 2. From Eqs. (27), (28), and (B1)–(B3), as first given in Ref. [22], one finds that

$$\begin{aligned} I_0(\psi,\psi;\zeta) = {} & æ a(\zeta)\bar{b}(\zeta)\sum_{j=1}^{2} n_j\mu_j(-)^{j-1} \\ & \times\left[\int_P \frac{g_j(\Delta)d\Delta}{\Delta-2\zeta} + i\pi g_j(2\zeta)\right], \end{aligned} \tag{B4}$$

where the subscript P on the integral symbol indicates a principle value integral. From Eq. (38), the evolution of the unperturbed continuous spectra [22] is given by

$$\begin{aligned} \partial_\chi\left(\frac{\bar{b}}{a}\right) = {} & \left(\frac{\bar{b}}{a}\right) æ \sum_{j=1}^{2} n_j\mu_j(-)^{j-1} \\ & \times\left[i\int_P \frac{g_j(\Delta)d\Delta}{\Delta-2\zeta} - \pi g_j(2\zeta)\right]. \end{aligned} \tag{B5}$$

The integrable part of the evolution of the bound-state eigenvalues, from Eq. (39), can be quickly evaluated since $a(\zeta)$ vanishes at any eigenvalue in the UHP. From Eq. (27), we see that $\Phi = b_k\Psi$ at an eigenvalue, in which case, $I(\phi,\phi;\zeta_k)$ becomes $b_k^2 I(\psi,\psi;\zeta_k)$ which, due to Eq. (B4), vanishes since it is proportional to $a(\zeta_k)$, which is zero. Thus, as is well known, the soliton eigenvalues are stationary when no perturbations are present.

For the integrable evolution of the kth normalization coefficient, from Eq. (40), one obtains the well-known result for unperturbed motion [22],

$$\partial_\chi D_k = i æ D_k \sum_{j=1}^{2} n_j\mu_j(-)^{j-1}\int_R \frac{g_j(\Delta)d\Delta}{\Delta-2\zeta_k}, \tag{B6}$$

where D_k is given by Eq. (41) and the subscript R on the integral symbol indicates that the integral is to be taken along the real axis.

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