

**Superradiance and collective gain in multimode optomechanics**T. Kipf<sup>1</sup> and G. S. Agarwal<sup>2</sup><sup>1</sup>*Institut für Optik, Information und Photonik, Universität Erlangen-Nürnberg, D-91058 Erlangen, Germany*<sup>2</sup>*Department of Physics, Oklahoma State University, Stillwater, Oklahoma 74078, USA*

(Received 22 September 2014; published 5 November 2014)

We present a description of a strongly driven multimode optomechanical system that shows the emergence of cooperative effects usually known from systems of atom-light interaction. Our calculations show that under application of a coherent pump field the system's response can be switched from a superradiant regime to a collective gain regime by varying the frequency detuning of the pump. In the superradiant regime, enhanced optical cooling of a single vibrational mode is possible, whereas the collective gain regime would potentially enable one to achieve almost thresholdless phonon laser action. The threshold pumping power scales as  $1/N$ .

DOI: [10.1103/PhysRevA.90.053808](https://doi.org/10.1103/PhysRevA.90.053808)

PACS number(s): 42.50.Wk, 42.50.Gy

**I. INTRODUCTION**

Collective scattering effects based on coherent interaction of resonant systems have been of interest since the seminal work by Dicke in 1954 [1]. Systems of atom-light interaction showing superradiance have since been studied both theoretically [2–4] and experimentally [5–7] for many years, but have gained increasing attention only in the last decade owing to progress in laser cooling of atomic clouds [8,9]. These systems allow the direct observation of cooperative scattering, but are limited in their range of experimentally accessible parameters. Furthermore, the coherent control of atoms within small distances, that are required for this task, is generally difficult to achieve. In recent years, interest has shifted to a new class of systems of artificial atoms, such as quantum dots [10] and Cooper pair boxes [11], that were found to show analogous collective effects such as in ensembles of atoms. With the rapid advances in the field of optomechanics, both in the optical [12,13] and the microwave regime [14,15], new candidates have emerged for the study of collective behavior on the quantum level. The on-circuit implementation of the optomechanical interaction at microwave wavelengths [14,15] hereby introduces the possibility of coupling multiple nanomechanical oscillators to a common cavity, thus offering a versatile approach to studying cooperative dynamics over a wide range of parameters.

In this paper, we present a theoretical description of a multimode optomechanical system with regard to the emergence of collective behavior. The flexibility of this system lies in the possibility of bringing it from a superradiative state to a state, where collective gain can be observed, by varying the detuning of the driving pump. We derive the collective equations governing the dynamics of the system starting from an optomechanical Hamiltonian description and discuss the dependence of the system on its parameters. We specialize in the case of  $N = 2$  mechanical oscillators to provide a physical explanation for the emergence of superradiance and collective gain and then generalize to an arbitrary number of oscillators.

We mention that collective effects in optomechanical systems have attracted some attention in recent years. Shkarin *et al.* [16] and Buchmann *et al.* [17] have noticed how collective effects can lead to the coupling of two mechanical oscillators which can be used to transfer energy from one

mechanical oscillator to the other. Mumford *et al.* [18] have studied the possibility of a Dicke-type phase transition in a system involving two cavity modes and one phonon mode. In extensive studies Xuereb *et al.* [19–21] consider the possibility of long-range interactions in optomechanical arrays. They show extreme sensitivity of the optomechanical interactions to the net reflection coefficient of the dielectric array [19,20]. They report [21] collective behavior of the array in the bad cavity limit, i.e., when optomechanical coupling  $\ll$  mechanical frequency  $\ll$  cavity damping. The collective behavior that we report is in a different regime of parameters, which is directly relevant to superconducting electromechanical systems [14,15] and graphene-based systems in superconducting resonators [22].

Whereas we present our analysis and results with regard to the set of systems as in [14,15,22], the extensive work of Xuereb *et al.* [19–21] brings out many new possibilities which depend on the overall reflectivity of the mechanical array, making the optomechanical coupling vary significantly from one element to the other.

This paper is structured as follows. In Sec. II we introduce our model, which is based on recent experimental progresses in on-circuit implementations of optomechanics and derive a linearized Hamiltonian description the system. In Sec. III we analytically solve the system's equations for the response function under resonant driving on the anti-Stokes sideband. We briefly discuss the emergence of superradiant collective behavior resulting from the coupling of the array of similar mechanical oscillators to a common reservoir. This superradiance behavior in such systems is free from the complications arising from the dipole-dipole interactions which can destroy superradiance in atomic ensembles [23]. In Sec. IV we extend this analysis for the case of driving on the Stokes sideband. After deriving a criterion for stable operation of the optomechanical system, we analyze the characteristics of the response function. Here, our calculations suggest that the coupling mediated by the cavity field leads to collective gain in the output field of the cavity. We conclude in Sec. V.

**II. MODEL**

Let us consider a system of  $N$  independent mechanical oscillators coupled to a common photonic cavity strongly

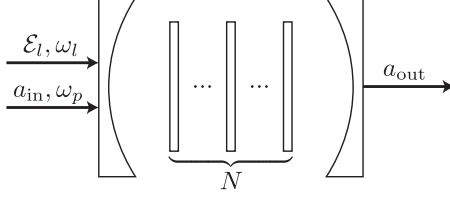


FIG. 1. Schematic of the (theoretically) studied optomechanical system in multimirror assembly, consisting of  $N$  mechanical oscillators coupled to a common cavity. The system is strongly driven with a pump of frequency  $\omega_l$  and electric field strength  $\mathcal{E}_l$ . The output field  $a_{out}$  is studied for an input probe field  $a_{in}$  of frequency  $\omega_p$ .

driven with a pump of frequency  $\omega_l$  as schematically shown in Fig. 1. Experimentally, such a system can be realized in an on-circuit implementation using electromechanical capacitors connected to a superconducting microwave resonator, as recently demonstrated in [14,15]. Via capacitive coupling, the mechanical oscillators modulate the resonance frequency of the common microwave cavity. As opposed to optical systems, the on-circuit implementation is coherently driven with microwave photons and can in principle hold an arbitrary number of mechanical oscillators with individual resonance frequencies. With that, these systems are a promising candidate for the investigation of collective effects in ensembles of harmonic oscillators, as theoretically described in [24].

The optomechanical Hamiltonian of our proposed system is given by

$$H = \hbar\omega_c c^\dagger c + \sum_{j=1}^N [\hbar\omega_j b_j^\dagger b_j - \hbar c^\dagger c g_j (b_j^\dagger + b_j)] + H_l, \quad (1)$$

where  $c$  and  $b_j$  are the bosonic annihilation operators for the cavity mode and the mechanical modes, respectively. The cavity resonance frequency is given by  $\omega_c$ ,  $\omega_j$  denotes the resonance frequency of the  $j$ th mechanical oscillator, and  $g_j$  is the optomechanical coupling rate.

The system is strongly driven with a pump of frequency  $\omega_l$  and power  $P_l$ . The corresponding Hamiltonian reads

$$H_l = i\hbar\mathcal{E}_l (e^{-i\omega_l t} c^\dagger - e^{i\omega_l t} c), \quad (2)$$

with an amplitude  $\mathcal{E}_l = \sqrt{2\kappa_E P_l / \hbar\omega_l}$ , where  $2\kappa_E$  is the cavity linewidth associated with external coupling. The total cavity linewidth is given by  $2\kappa = 2\kappa_E + 2\kappa_I$ , whereas  $\kappa_I$  accounts for all internal losses.

In a frame rotating with the pump frequency  $\omega_l$ , linearized about a steady state, the Hamiltonian reads

$$H \approx \hbar\Delta a^\dagger a + \hbar \sum_{j=1}^N [\omega_j b_j^\dagger b_j - (G_j a^\dagger + G_j^* a)(b_j^\dagger + b_j)], \quad (3)$$

where we have defined the enhanced coupling rate as  $G_j = \alpha g_j$  and  $c \equiv \alpha + a$ , with the system's steady-state amplitude  $\alpha = \mathcal{E}_l / (\kappa + i\Delta)$ .  $\Delta = \omega_c - \omega_l$  is the detuning from the cavity resonance frequency. Note that we have dropped purely classical and small terms. Note also that  $a$  is slowly varying as we are in the rotating frame.

### III. SUPERRADIANCE

In the following, we work in the resolved sideband regime  $\omega_m \gg \kappa$  and assume similar mechanical resonators with  $\omega_j \approx \omega_m$ . For cooperative effects to occur, the coupling needs to become resonant. This can be achieved by choosing the pump frequency  $\omega_l$  such that  $\Delta \approx \pm\omega_m$ . With this choice, the slowly varying intracavity field oscillates at the mean frequency of the mechanical oscillators.

Let us first consider driving on the anti-Stokes sideband, i.e.,  $\Delta \approx \omega_m$ . The physical process that we consider in this section corresponds to the generation of a phonon  $-\omega_l + \omega = \omega_m$  which then can combine with a pump photon  $\omega_l$  to produce an anti-Stokes photon  $\omega_l + \omega_m$ . Here, the interaction terms  $G_j a^\dagger b_j^\dagger$  and  $G_j^* a b_j$  become off-resonant and can be neglected in the rotating wave approximation (RWA).

We introduce dissipative dynamics, accounting for leakage of photons and phonons, in the form of the quantum Langevin equations for the Heisenberg operators  $a$  and  $b_j$ ,

$$\dot{a} = -(\kappa + i\Delta)a + i \sum_{j=1}^N G_j b_j + f_a(t), \quad (4a)$$

$$\dot{b}_j = -(\Gamma_j + i\omega_j)b_j + iG_j^* a + f_{b_j}(t), \quad (4b)$$

where  $f_i(t)$  are the quantum Langevin forces, which account for vacuum noise and any thermal noise entering the system. The correlation functions associated with the quantum and thermal fluctuations are given by [25]

$$\langle f_a^\dagger(t) f_a(t') \rangle = 0, \quad (5a)$$

$$\langle f_a(t) f_a^\dagger(t') \rangle = 2\kappa \delta(t - t'), \quad (5b)$$

$$\langle f_{b_j}^\dagger(t) f_{b_j}(t') \rangle = 2\Gamma_j \bar{n}_j \delta(t - t'), \quad (5c)$$

$$\langle f_{b_j}(t) f_{b_j}^\dagger(t') \rangle = 2\Gamma_j (\bar{n}_j + 1) \delta(t - t'), \quad (5d)$$

$$\langle f_{b_j}^\dagger(t) f_{b_k}(t') \rangle = 0 \quad (\forall j \neq k). \quad (5e)$$

Here,  $\bar{n}_j$  denotes the thermal occupation of the heat bath associated with the mechanical mode  $b_j$ . Note that we have adopted the standard Markov approximation for the correlation functions given in Eq. (5), i.e., we have assumed delta-correlated noise without memory.

For the case of  $N = 2$  degenerate mechanical oscillators with  $\omega_1 = \omega_2$ , the form of the quantum Langevin equations (4) has been studied in [26]. Here, we focus on the more general case of *near-degenerate* mechanical oscillators and study the emergence of collective behavior depending on the detuning of the mechanical oscillator frequencies. To proceed, we set  $f_a(t) = \sqrt{2\kappa_E} a_{in}(t)$ , hence assuming that a probe field  $a_{in}(t)$ , strong enough compared to single photons but yet much weaker than the pump, is applied. For simplicity, we also neglect the force terms  $f_{b_j}(t)$ .

We solve Eq. (4) in frequency space, transforming functions and operators as  $f(\omega) = \int_{-\infty}^{+\infty} dt e^{i\omega t} f(t)$  of  $f(t)$ . Using the input-output relation [27],

$$a_{out}(\omega) = \sqrt{2\kappa_E} a(\omega) - a_{in}(\omega) \equiv R(\omega) a_{in}(\omega), \quad (6)$$

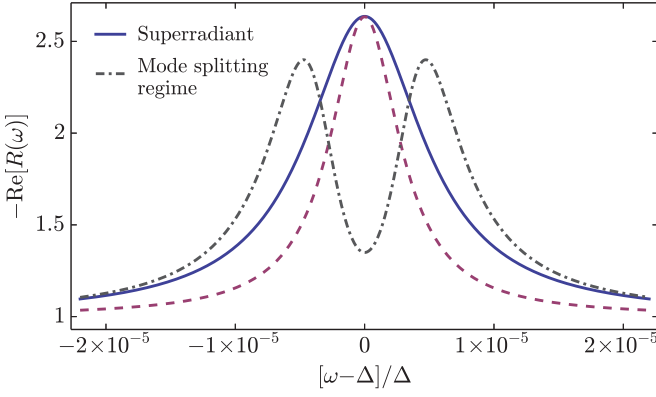


FIG. 2. (Color online) Response function for  $\Delta\omega = 0$  and  $N = 2$  (solid curve) showing an increased linewidth compared to a system with  $N = 1$  (dashed red curve, scaled and translated to facilitate comparison). The black dash-dotted curve corresponds to  $\Delta\omega = 4.5\Gamma$  and shows clearly separated modes. Parameters were chosen as follows:  $\kappa/2\pi = 1.0$  MHz,  $\omega_m = 10\kappa$ ,  $\Gamma = 10^{-4}\kappa$ ,  $G_1 = G_2 = 1.5\sqrt{\kappa\Gamma}$ , close to typical experimental values [14,15].

we find the following solution for the cavity field in terms of the response function of the cavity,

$$R(\omega) = \frac{2\kappa_E}{\chi_c^{-1}(\omega) + \sum_{j=1}^N |G_j|^2 \chi_j(\omega)} - 1, \quad (7)$$

with the cavity response  $\chi_c(\omega)$  and the mechanical response  $\chi_j(\omega)$  functions, given by

$$\chi_c(\omega) = \frac{1}{\kappa + i(\Delta - \omega)}, \quad \chi_j(\omega) = \frac{1}{\Gamma_j + i(\omega_j - \omega)}. \quad (8)$$

First, let us briefly discuss the results for driving on the anti-Stokes sideband, while limiting our discussion to  $N = 2$  mechanical modes. Without loss of generality we assume that  $G_j$  are real valued. We furthermore require that  $\Gamma_j \ll \kappa$ , which is usually the case for typical realizations of the proposed system [14,15]. In addition, let us denote the frequency difference of the mechanical oscillators by  $\Delta\omega \equiv \omega_1 - \omega_m = -(\omega_2 - \omega_m)$ . Figure 2 shows the real part of the response function in the vicinity of the resonant anti-Stokes sideband. For the degenerate case of identical mechanical elements [26], the system divides into a bright superradiant mode (solid curve) with linewidth  $\Gamma_+ = \Gamma + 2\Gamma_{\text{opt}}$  and a dark mode  $\Gamma_- = \Gamma - 2\Gamma_{\text{opt}}$ , which is effectively decoupled from the cavity. The broadening is proportional to the optomechanical damping rate  $\Gamma_{\text{opt}} = G_j^2 \chi_c$ . An analysis of the roots of the denominator of  $R(\omega)$  shows that the formation of a bright and a dark mode occur only for  $\Delta\omega < G_1 G_2 / \kappa$  and the dark mode asymptotically decouples from the cavity for  $\Delta\omega \rightarrow 0$ . We provide a detailed derivation of this result for the similar case of driving on the Stokes sideband.

We finally note the connection between the superradiance of the mechanical oscillators and the superradiance of the atomic system in a cavity [28]. In both cases the system interacts with the cavity photons of a single common cavity mode which in turn interacts with the vacuum modes of the outside world—the bath is made up of all the outside vacuum modes. It is well known that the single atom decay rate in the cavity is given

by the Purcell formula  $g^2/\kappa$  where  $g$  is the coupling of the atom to the cavity. For the mechanical elements, the coupling with the cavity gives an additional decay rate  $G^2/\kappa = \Gamma_{\text{opt}}$ . In the atomic case, the decay rate of a single atom in the presence of the other atoms is given by  $Ng^2/\kappa$ , whereas in our optomechanical system the decay rate of a single mechanical oscillator is  $NG^2/\kappa$ . Thus the similarity between the atomic and the mechanical case is striking.

As a matter of fact, the whole process can be thought of as a scattering of phonons of different mechanical oscillators into a common cavity mode of indistinguishable photons. In this case, we observe superradiance and each mechanical oscillator is more rapidly damped, i.e., emits phonons more rapidly which are converted into anti-Stokes cavity photons.

#### IV. COLLECTIVE GAIN

We will now study the system under resonant driving in the Stokes sideband with  $\Delta \approx -\omega_m$ . The physical process here is different—it leads to the spontaneous generation of a phonon and a Stokes photon. Thus the phonon field can grow and one can have phonon laser action.

In what follows, we describe the characteristics and the origin of the collective behavior resulting in this regime. Using the same approach as before, we start by dropping nonresonant terms (RWA) in the linearized optomechanical Hamiltonian (3). This yields the quantum Langevin equations,

$$\dot{a} = -(\kappa + i\Delta)a + i \sum_{j=1}^N G_j b_j^\dagger + f_a(t), \quad (9a)$$

$$\dot{b}_j^\dagger = -(\Gamma_j - i\omega_j)b_j^\dagger - iG_j^* a + f_{b_j}^\dagger(t), \quad (9b)$$

with quantum Langevin forces defined by Eq. (5). A short calculation yields the system's response function,

$$R(\omega) = \frac{2\kappa_E}{\chi_c^{-1}(\omega) - \sum_{j=1}^N |G_j|^2 \chi_j(\omega)} - 1, \quad (10)$$

with  $\chi_c(\omega)$  as above and  $\chi_j(\omega) = 1/[\Gamma_j - i(\omega_j + \omega)]$ .

For a critical driving power in the Stokes sideband, the system exhibits self-sustained oscillations [29] and becomes unstable. We thus limit our investigation to the stable driving regime, which can be found by evaluating the Routh-Hurwitz stability criterion (see, e.g., [30]) for the quantum Langevin equations (9). For similar mechanical oscillators with  $\Gamma \approx \Gamma_j$  and  $\Delta\omega \approx 0$  the stability condition evaluates to

$$\sum_{j=1}^N |G_j|^2 < \Gamma\kappa. \quad (11)$$

Under this condition, the system will also remain stable with finite frequency detuning  $\Delta\omega \neq 0$ . For typical structures [14,15] we have  $\Gamma \ll \kappa$  and are thus limited to the weak coupling regime, i.e.,  $G_j^2 \ll (\kappa/2)^2$ , for stable operation. When the condition (11) does not hold, phonon lasing occurs and Eq. (9) have to be generalized to include nonlinearities to reach stable operation [31]. With that, the cavity response  $\chi_c(\omega)$  is approximately independent of frequency near the mechanical

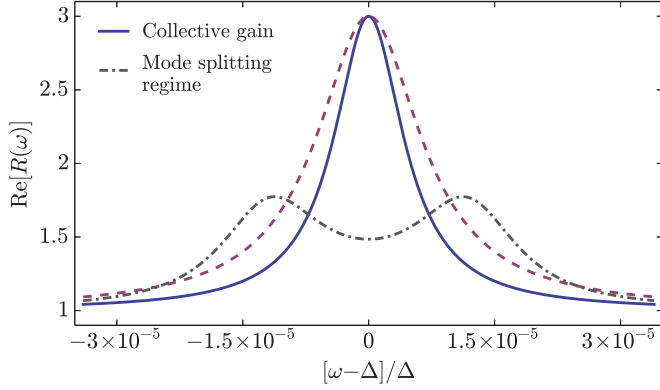


FIG. 3. (Color online) Same as in Fig. 2 except for driving on the Stokes sideband with  $G = 0.5\sqrt{\kappa}\Gamma$ . The response function of the degenerate multimode system (solid curve) shows a decreased linewidth. Here, the dash-dotted curve corresponds to  $\Delta\omega = 1.25\Gamma$ .

resonance frequencies  $\omega_j$ :

$$\chi_c(\omega) \approx \chi_c(-\omega_j) = \frac{1}{\kappa + i(\Delta + \omega_j)} \approx \frac{1}{\kappa}. \quad (12)$$

The second approximation in Eq. (12) holds for near-degenerate mechanical frequencies, i.e.,  $\Delta + \omega_j \ll \kappa$ .

In the following, we put special emphasis on the case of  $N = 2$  mechanical oscillators and generalize our discussion to more than two modes afterwards. Figure 3 shows the numerical evaluation of  $R(\omega)$  for this case in the vicinity of the resonance. For the multimode system, the linewidth of the resonant feature is decreased in comparison to the system of a single mechanical oscillator, whereas the amplitude is strongly increased (collective gain). It should be borne in mind that the collective gain, that we discuss, is in terms of the phonon variables, as we are investigating phonon laser action. This should be differentiated from the narrowing of the cavity linewidth which was found in [19].

To get an understanding of the cooperative effects leading to this signature, we analyze the roots of the denominator of  $R(\omega)$ . In the approximation of Eq. (12) the roots are given by

$$\omega_{\pm}^c = \frac{1}{2}[\omega_1^e + \omega_2^e \pm \sqrt{(\omega_1^e - \omega_2^e)^2 - 4\chi_c^2 G_1^2 G_2^2}], \quad (13)$$

with the effective complex frequencies of the mechanical oscillators  $\omega_j^e = -\omega_j - i(\Gamma_j - \Gamma_{\text{opt}})$  and the optomechanical damping rate  $\Gamma_{\text{opt}} = G_j^2 \chi_c$ . We can identify  $\chi_c G_1 G_2$  as the effective coupling between the mechanical modes, mediated by the photonic field. The real and imaginary parts of Eq. (13) are shown in Fig. 4. The square root term gives rise to two regimes separated by a bifurcation point at  $\Delta\omega = \chi_c G_1 G_2$ . For a frequency detuning  $\Delta\omega$  smaller than the effective coupling, the effective frequencies of the mechanical modes become degenerate and form two collective normal modes. This applies to resonant driving both on the Stokes and on the anti-Stokes sideband. For off-resonant driving the bifurcation point vanishes; the formation of collective modes, however, persists.

A similar case of normal mode splitting is known in the context of strong coupling between a single mechanical oscillator and the cavity mode [32]. In our case, however, we are required to work in the weak coupling regime, where

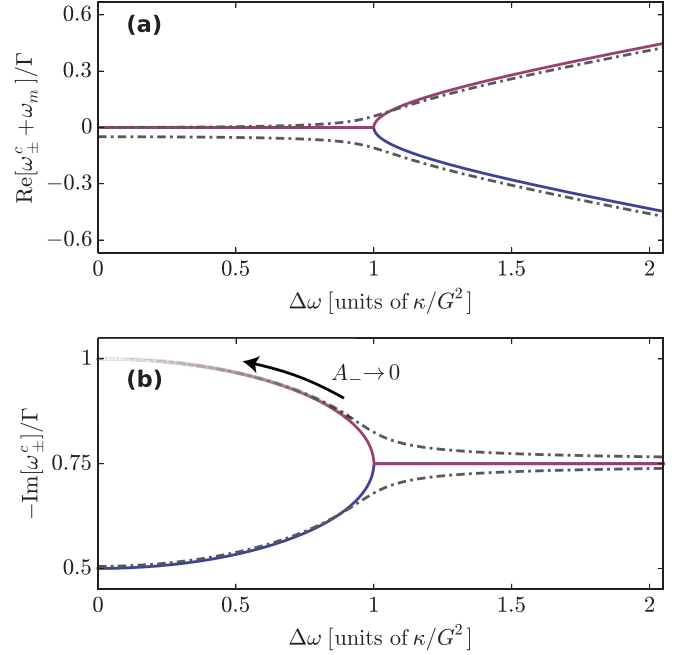


FIG. 4. (Color online) Real (a) and imaginary (b) part of the denominator roots of  $R(\omega)$  for on-resonance driving on the Stokes sideband. Upper red (lower blue) curve corresponds to the negative (positive) sign in (13). Dashed curves were calculated for off-resonant driving with  $\delta \equiv \omega_c - \omega_j + \omega_m = 0.1\kappa$ . The bifurcation point vanishes for  $\delta \neq 0$ . In (b) the upper curve with amplitude  $A_-$  decouples from the cavity for  $\Delta\omega \rightarrow 0$  (see main text for details). Parameters were chosen as in Fig. 3.

the cavity response function is approximately independent of frequency in the vicinity of the mechanical resonances. In this system, the collective modes form due to the effective coupling between the mechanical oscillators, mediated by the photonic field.

The mode showing collective gain with  $\Gamma_+ = \Gamma - 2\Gamma_{\text{opt}}$  (lower curve in Fig. 4) defines the response of the optomechanical system, as other mode (upper curve) asymptotically decouples from the cavity for  $\Delta\omega \rightarrow 0$ . Here, the underlying physical process can be seen as a scattering from indistinguishable photons in the cavity field into a single *collective* phonon mode, whereas (in the case of  $N$  mechanical oscillators) the other  $N - 1$  phononic modes do not couple to the light field at all. This can also be understood by analyzing response function  $R(\omega)$  in terms of partial fractions,

$$R(\omega) = 2\kappa_E \left[ \frac{A_+}{\omega - \omega_+^c} - \frac{A_-}{\omega - \omega_-^c} - 1 \right] - 1. \quad (14)$$

This analysis shows that  $A_- \rightarrow 0$  for  $\Delta\omega \rightarrow 0$ . This also applies to the anti-Stokes sideband, where only the superradiant mode with  $\Gamma_+ = \Gamma + 2\Gamma_{\text{opt}}$  contributes to the response function.

In a somewhat different picture this effect can be understood in terms of the structure of the response function.  $R(\omega)$  of the fully degenerate system with identical mechanical oscillators is equal to the response function of a system with a *single* oscillator in a cavity with an effective coupling rate of  $G' = \sqrt{2}G$ . This applies both to the Stokes and the anti-Stokes sideband

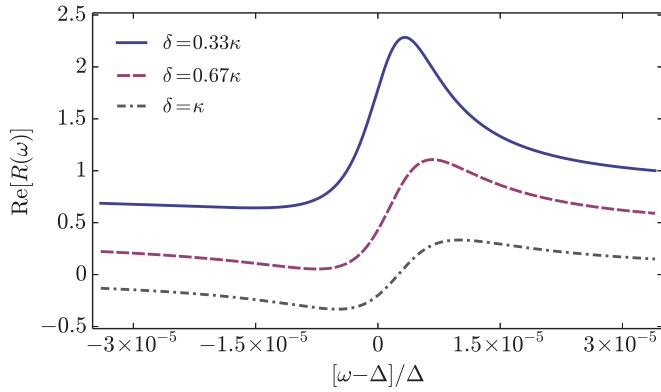


FIG. 5. (Color online) Real part of response function for off-resonant driving with detuning  $\delta \equiv \omega_c - \omega_l + \omega_m$  in the vicinity of the Stokes sideband. Parameters were chosen as in Fig. 3.

and can readily be generalized for a system of  $N$  oscillators. Here, the effective coupling rate reads  $G' = \sqrt{NG}$ . In the same way, generalization of the collective linewidth yields  $\Gamma_{\pm} = \Gamma \pm N\Gamma_{\text{opt}}$ , whereas the plus (minus) sign holds for the anti-Stokes (Stokes) sideband. This implies that the threshold power for phonon laser action would drop by a factor of  $N$  [33].

For off-resonant driving the collective behavior persists; the response function, however, gradually changes from an absorptive to a dispersive structure, as shown in Fig. 5. The resonant feature considerably broadens and decreases in

amplitude. Qualitatively, this behavior occurs for any number of oscillators and is not distinctive for the multimode system. In the here discussed case of frequency degeneracy, the system of  $N$  mechanical oscillators with coupling rates  $G$  is equal to a system of a single oscillator with a coupling rate of  $\sqrt{NG}$ .

## V. CONCLUSIONS

In summary, we have derived and analyzed the emergence of superradiance and collective gain in optomechanical multimode systems. We have discussed the necessary conditions for these collective effects to occur and outlined the experimental feasibility. In light of recent progress in on-circuit implementations of optomechanics [14,15], the experimental realization of the proposed system should be within reach. We showed that the system can easily be switched from a superradiant state, allowing enhanced cooling of a single vibrational mode, to a collective gain regime where phonon laser action [34] with a single collective phononic mode is potentially possible.

## ACKNOWLEDGMENTS

We would like to thank Junho Suh for providing us with information about their multimode optomechanical device. T.K. gratefully acknowledges support and advice from Joachim von Zanthier, the German National Academic Foundation (Studienstiftung des Deutschen Volkes) and the hospitality of the Oklahoma State University, while this work was done.

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