Increase of degeneracy improves the performance of the quantum adiabatic algorithm

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We propose a strategy to improve the performance of the quantum adiabatic algorithm (QAA) on an NP-hard (nondeterministic-polynomial-time-hard) problem exact cover, by increasing the ground-state degeneracy of the problem Hamiltonian. Our strategy is based on the empirical finding that for the QAA the difficulty of random instances decreases with the degeneracy of the ground state. We increase the degeneracy by adding extra qubits to form additional clauses. Our numerical results show that on average our strategy can provide an increase in the minimum gap size along the linear interpolation path of Hamiltonian for both easy and difficult instances. The success probability at fixed total evolution time is thus increased.

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I. INTRODUCTION

Since the first appearance of the quantum adiabatic algorithm (QAA) [1,2], there has been debate over its ability to solve the classical NP-hard (nondeterministic-polynomial-time-hard) problem in polynomial time (satisfiability (SAT) [1,3–6], max independent set [7,8]). While there is no evidence of exponential speedup, some strategies have been proposed to improve the performance of the QAA [9–13]. In order to have a well-defined gap through the evolution, people have focused on SAT problems with either unique satisfying assignment (USA) [1,3,5,10] or USA up to total spin flip degeneracy [6]. The influence of the degeneracy of the space of satisfying assignments on the performance of the QAA is not well understood.

In this paper, we focus on the random instances of exact cover (i.e., 1-in-3 SAT) to investigate the influence of the ground-state degeneracy on the performance of the QAA. We measure the performance of the QAA by a modified minimum gap size along the adiabatic path. Using this measure, we find some evidence suggesting a "phase transition" of the difficulty with the change of "order parameter" clause density. Similar to the classical result [14], the critical point is at the SAT-UNSAT (unsatisfiable) threshold. For both SAT and UNSAT cases, the difficulty decreases as the ground-state degeneracy increases.

Based on this finding, we propose a strategy of improving the performance of the QAA by adding additional clauses to increase the ground-state degeneracy of the problem Hamiltonian. Numerical simulation of relatively small systems (10 qubits) shows that with significant probability our strategy will increase the minimum gap size, thus increasing the success probability. Due to limited numerical power, the scaling of the improvement with the system size is unknown. Note that a recent paper [15] also considers the strategy of increasing the degeneracy; however, their result requires a penalty which scales polynomially in the system size and only perturbative crossings are shown to disappear. Here we fix the energy penalty to be constant and our numerical evidence is nonperturbative. Reference [11] also considers changing the problem Hamiltonian to increase the minimum gap size; however, their approach is to change the scaling of parameters, and the degeneracy of the eigenvalues is unchanged.

This paper is organized as follows: In Sec. II we give a brief introduction to the QAA; in Sec. III we describe the basic QAA scheme for the problem of exact cover; in Sec. IV we propose our strategy to increase the degeneracy; in Sec. V we define the minimum gap size as the quantitative measure of the performance of a QAA scheme on a specific instance of problem, i.e., the difficulty of the instance for the QAA; in Sec. VI we investigate the "phase transition" of the QAA difficulty; in Sec. VII we numerically demonstrate that our strategy can increase the gap size; in Sec. VIII we consider how the strategy will increase the success probability; and in Sec. IX we summarize our results.

II. QUANTUM ADIABATIC ALGORITHM (QAA) SCHEME

A QAA scheme is described by three components: (i) a problem Hamiltonian H_P that we want to minimize, with D degenerate ground states $|\phi_d^P\rangle, d \in [1,D]$; (ii) a beginning Hamiltonian H_B and an isolated physical system initially prepared in its ground state $|\psi(0)\rangle$; and (iii) the path of Hamiltonian at time t described by $H(s = t/T)$ with fixed boundary $H(0) = H_B, H(1) = H_P$. Since the total running time T can be easily changed in experimental realizations of the QAA schemes, we view T as a tunable parameter rather than a component.

The dynamics of the isolated quantum system are governed by the Schrödinger equation parameterized by the scaled time $s = t/T \in [0,1]$ (we set $\hbar = 1$)

$$i\frac{\partial}{\partial s}|\psi(s)\rangle = T H(s)|\psi(s)\rangle . \tag{1}$$

The success of a QAA scheme is measured by the success probability, defined as

$$P_{sc}(T) = \sum_{d=1}^{D} \left|\langle\phi_d^P|\psi(s=1)\rangle\right|^2, \tag{2}$$

Note that $P_{sc}(T)$ depends on the parameter T nonmonotonically [13]. Its behavior can be highly instance dependent at small T, while at large enough T, it approaches unity, according to the adiabatic theorem.

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The goal for a QAA scheme can be set as follows: Given certain limited resources, maximize the success probability P_{sc}. Assuming zero temperature, the resources of the QAA are typically (i) time, which constraints the total running time parameter, T; (ii) the power and ability to engineer a nondiagonal Hamiltonian that controls the quantum tunneling, thus constraining the path of Hamiltonian $H(s)$; and (iii) the precision of parameter control and measurement, which limits the number of different energy scales available. For example, the Hamiltonian for solving the maximum independent set in Ref. [7]'s Eq. (1) has one coefficient scale in the system size and the other constant. This means the precision of energy limits the maximum calculable system size.

III. A SIMPLE QUANTUM ADIABATIC SCHEME FOR EXACT COVER

In order to investigate whether a strategy can improve the performance of the QAA solving exact cover, in principle, one needs to compare the performance of the QAA before and after applying the strategy for all possible choices of the QAA scheme. However, this is impossible, since there are infinite possible schemes. Instead, we demonstrate the improvement by giving a nontrivial example. The QAA scheme before applying any strategy is chosen to be as simple as possible, as follows:

(1) Problem Hamiltonian H_P

Given m clauses $c_a(x_{a1},x_{a2},x_{a3}),a \in [1,m]$ formed by n Boolean variables $x_i,i \in [1,n]$, each clause $c_a(x_{a1},x_{a2},x_{a3})$ is true only when there are 2 false and 1 true in the variables x_{a1},x_{a2},x_{a3} it contains. The problem of exact cover asks whether all the m clauses can be satisfied at the same time; i.e., whether the Boolean formula

$$F(\vec{x}) \equiv \wedge_{a=1}^{m} c_a(x_{a1},x_{a2},x_{a3}) \tag{3}$$

is satisfiable (SAT) or not (UNSAT), where the notation "$\wedge$" means Boolean "and." For further use, define $v_i^k = 1$ when clause c_k contains x_i and $v_i^k = 0$ otherwise. Note that to be more precise, we are considering the positive 1-in-3 SAT problem, which is equivalent to exact cover 3. There has been ambiguity among papers on the usage of these terms, so we do not distinguish between them anymore. For simplicity, we call the problem we are considering "exact cover."

To formulate the quantum adiabatic algorithm, the variables are translated into qubits (true as $S^z = -1$, false as $S^z = 1$). Rather than using a 3-local Hamiltonian [1,3,6], we adopt the 2-local Hamiltonian in Ref. [4]:

$$H_P = \sum_{a=1}^{m} H_a, \quad H_a \equiv \frac{1}{4}\left(S_{a1}^z + S_{a2}^z + S_{a3}^z - 1\right)^2 \tag{4}$$

In the spin glass formulism, after a constant shift and rescaling, the Hamiltonian can be written as an SK model $H_P = \sum_i h_i S_i^z + \frac{1}{2}\sum_{ij} J_{ij} S_i^z S_j^z$, where $h_i = -\sum_{k=1}^{m} v_i^k$ is the negative of the number of times variable x_i appears in Boolean formula F and $J_{ij} = \sum_{k=1}^{m} v_i^k v_j^k$ is the number of times variable x_i and x_j appear together in one clause. Note that here for simplicity the unit of energy is set to unity since only the relative value matters for the discussion in this paper.

(2) Beginning Hamiltonian H_B

For simplicity, we adopt the same transverse field as Farhi *et al.* [1] as the beginning Hamiltonian:

$$H_B = \sum_{j=1}^{n} \Gamma_j S_j^x, \tag{5}$$

where Γ_j equals the number of clauses containing the variable x_j and is equal in amplitude to the field h_j. The ground state of $H(0) = H_B$ is a uniform superposition of all possible assignments $|\psi(0)\rangle = \frac{1}{\sqrt{2^n}} \sum_{\{S_k^z\}} \bigotimes_{k=1}^{n} |S_k^z\rangle$.

(3) Adiabatic path

We adopt the linear interpolation path of Hamiltonian along the QAA evolution

$$H(s) = sH_P + (1-s)H_B. \tag{6}$$

IV. STRATEGY OF INCREASING THE DEGENERACY

For each component of the QAA, strategies can be designed to improve the performance: changing the problem Hamiltonian by reduction [11], choosing a different path of Hamiltonian [13], etc. In this paper, we consider altering the ground-state degeneracy of the problem Hamiltonian H_P by introducing extra bits.

Given Eq. (3), consider the ground state $|\phi_i^P\rangle$ with energy E_g of the problem Hamiltonian, Eq. (4). As long as there is certain variable $x_k = 0$ in the corresponding assignment, we can add an extra clause containing variable x_k and two additional variables x_k^{e1},x_k^{e2} to increase the degeneracy of energy level E_g. The new Boolean formula with $m+1$ clauses and $n+2$ variables

$$G\left\{F(\vec{x}),x_k^{e1},x_k^{e2}\right\} = F(\vec{x}) \wedge c_{m+1}\left(x_k,x_k^{e1},x_k^{e2}\right) \tag{7}$$

has the Hamiltonian $H'_P = H_P + H_{m+1}$ in our QAA scheme, where H_{m+1} is the penalty of the additional clause. The eigenstates of H'_P are in a larger Hilbert space. Every original eigenstate of H_P corresponds to four new states due to the combination of extra qubits $x_k^{e1} = 0/1,x_k^{e2} = 0/1$. Depending on whether the new clause is true or false, there is zero or constant energy penalty H_{m+1} on this clause for the four states as in Table I.

In this way, any state with $x_k = 0$ and energy E_0, including the ground state $|\phi_i^P\rangle$, becomes two new states with the same energy E_0 and two new states with energy $E_0 + 1/4$. So, the original energy level E_0 gets one extra degeneracy from the extra bits' assignments $(x_k = 0,x_k^{e1} = 1,x_k^{e2} = 0)$ and $(x_k = 0,x_k^{e1} = 0,x_k^{e2} = 1)$. On the contrary, any state with $x^k = 1$ and energy E_1 becomes one new state with the same energy E_1, two new states with energy $E_1 + 1/4$, and one new state with energy $E_1 + 1$. The original energy level E_1 has the same degeneracy.

TABLE I. Energy penalty H_{m+1}.

(x_k^{e1},x_k^{e2})	(0,0)	(0,1)	(1,0)	(1,1)
$x_k = 0$	1/4	0	0	1/4
$x_k = 1$	0	1/4	1/4	1

The problem is that we do not know what the ground state is yet. Luckily, for exact cover, every clause is TRUE only when the variables in the clause have two 0's and one 1. Thus for the ground state, statistically there are more 0's than 1's. So by picking a random bit and adding an additional clause, we will have an increase of degeneracy in the ground state with probability larger than $1/2$.

V. PERFORMANCE MEASURE OF QUANTUM ADIABATIC SCHEMES

In order to determine whether our strategy improves the performance of the simple QAA scheme for exact cover or not, we need to first introduce some quantitative measure for the performance of a QAA scheme.

The performance of a QAA scheme for a fixed instance is directly evaluated by the success probability $P_{sc}(T)$ as a function of total running time T. Since obtaining such a function is computationally costly and comparing them is not simple, a different measure needs to be defined. A common measure is to fix a certain T and compare the $P_{sc}(T)$ of different strategies, but this measure is not well defined, since the choice of T is ambiguous. When T is large enough, all cases will have success probability close to unity. To avoid this problem, we adopt a measure based on the structure of eigenvalues along the path $H(s)$ in Eq. (6) that is independent of T. A simple measure can be the minimum gap size Δ, which appears in the estimation of adiabatic condition.

However, the traditional minimum gap size is only well defined for the nondegenerate ground state, since for the degenerate case, the gap will approach zero at the end of the evolution as $s \to 1$. But if only a single diabatic transition is considered, transitions to states that become the degenerate ground states at the end of evolution will not decease the final success probability (we call them "good states"). This allows us to redefine the gap size as the minimum energy difference between the instantaneous ground state and any state to which a single diabatic transition will decrease the final success probability (we call them "bad states"). For clarification, in this paper all instances of "minimum gap" refer to the above redefined minimum gap.

Numerically we track from $s = 1$ to $s = 0$ which states are "good states" and which states are "bad states." The procedure is as follows: (i) At $s = 1$, mark all ground states as "good" and others as "bad." (ii) Decrease s slowly and based on continuation of the spectrum evolution as well as the separation between different levels determine the unavoided crossings between the lowest "bad state" and the first "good state" lying below it. When there is no unavoided crossing, continue the same ordering of labels. Whenever an unavoided crossing happens between them, exchange the "good" and "bad" labels between the two energy levels. (iii) Repeat until $s = 0$. Then the minimum gap can be determined by considering the ground state and the lowest "bad state" at different $0 < s < 1$. Note that multiple diabatic transitions between different levels that lead the ground state to the "bad states" is ignored when we use the minimum gap determined by the above numerical scheme as a measure of performance.

An example of the definitions of gap, good states and bad states are given in Fig. 1. In Fig. 1(a), there is an avoided crossing at $s \sim 0.78$, and at the avoided crossing, diabatic transition to the good state marked by the green (gray) line will not change the success probability at the end if only one transition is considered; thus the minimum gap is defined as the energy difference between the instantaneous ground state (blue; bottom line) and bad state (red; top line). Note that more complicated situations can happen; e.g., Fig. 1(b) shows one good state crosses the bad state and it is not between the instantaneous ground state and bad state at the minimum gap point.

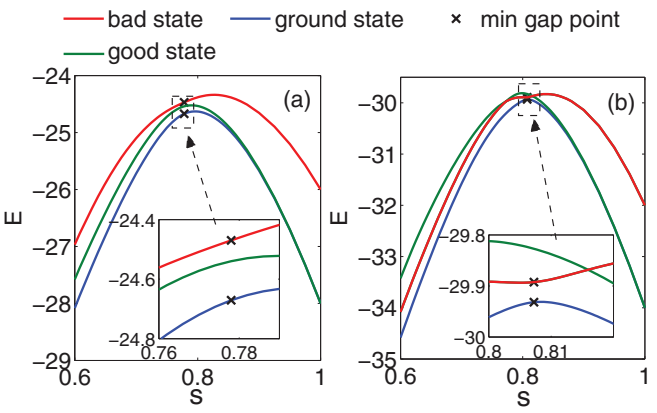

FIG. 1. (Color online) Example of definition of minimum gap. In both cases, the problem Hamiltonian has 2-fold degenerate ground state ($D = 2$) and we plot the three lowest eigenstates. The subpanels are the zoom in of the minimum gap point.

VI. PHASE TRANSITION, EMPIRICAL DIFFICULTY, AND DEGENERACY

Now that the minimal gap size is defined, we generate random instances of Eq. (3) and evaluate the gap size of each instance. Since it is hard to analytically solve the minimum gap size, we use numerical diagonalization of relatively small size systems of $n = 10$ qubits. Then, we can calculate how modifying the problem Hamiltonian by adding extra qubits will change the minimum gap size. Before this, we want to show that $n = 10$ is large enough as a demonstration and that the minimum gap size is a valid measure of the QAA scheme's performance on certain instances, i.e., the difficulty of an instance. We also want to show some more intuition for the strategy.

In the SAT problem, the classical difficulty of an instance has been well studied [14,16–18]. The difficulty for an instance of a problem can be defined with respect to an algorithm by the time it takes to solve it. Moreover, one can define certain order parameters to characterize this difficulty, and difficult problems occur at "phase transitions" of such order parameters. This definition of difficulty and order parameter is called "empirical hardness models" in Ref. [17]; we thus add the word "empirical" before "difficulty" in accordance. For the SAT problem, the order parameter is the clause density m/n. There exist two critical values for clause density c_{sat} and c_{cluster}. For random instances, when $m/n > c_{\text{sat}}$, the probability of SAT goes to zero and $m/n < c_{\text{sat}}$ to one as the size $n \to \infty$. DPLL backtracking–based classical algorithms [19,20] have

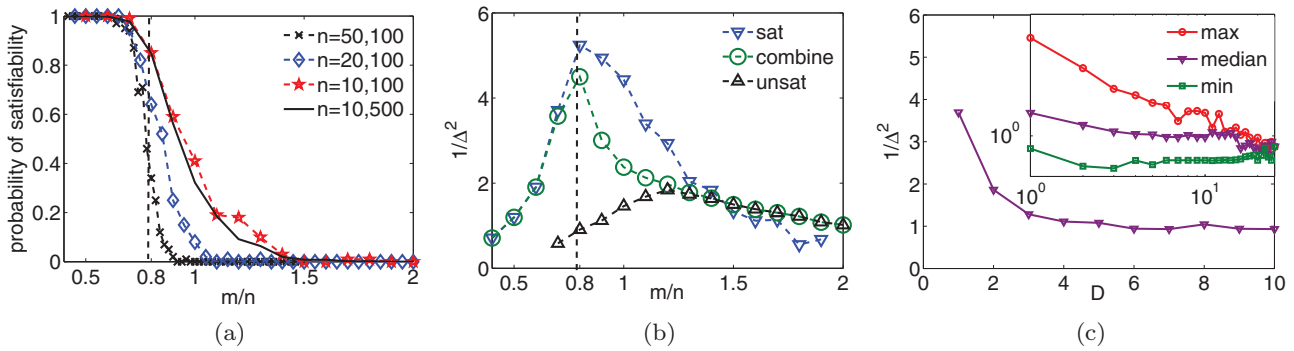

FIG. 2. (Color online) (a) The SAT probability with clause density for $n = 10/20/50$ variables. Each data point of probability is estimated over 100/500 random samples, as indicated in the legend. (b) Median $1/\Delta^2$ with the clause density. The UNSAT does not have large average empirical difficulty. We sample 1000 instances of $n = 10$ fixed for each clause density. The dashed line in subfigures (a) and (b) indicates the position of phase transition at $m/n \sim 0.79$. (c) Correlation of $1/\Delta^2$ and degeneracy of ground state of H_P ($n = 10$ fixed); each data point is the median of 1000 random instances. The subpanel is the corresponding log-log plot, which also includes the minimum and maximum $1/\Delta^2$.

the most difficult instances at c_{sat}. When $c_{\text{cluster}} < m/n < c_{\text{sat}}$, there is an intermediate phase with many metastable local minimums of the problem Hamiltonian. Classical local search algorithms have the most difficult instances at c_{cluster}. The relevant critical clause density for our interest turns out to be c_{sat}. For exact cover, $c_{\text{sat}} \sim 0.79$.

In Fig. 2(a), we show that for $n = 10$, the phase transition of SAT-UNSAT is already distinct, though as n increases, the phase transition does become sharper. This means that $n = 10$ is large enough to show some statistical characteristics. Note that the way we generate random instances and what we define as clause density is different from Refs. [17,21], but the same as Ref. [14], and we have verified that the position of the "phase transition" point is consistent with Ref. [21]'s result. For details of our methods of generating random instances, see the Appendix. In Fig. 2(a), the classical exact-cover solver [22] is used, which enables us to solve problems of $n = 50$ bits.

To investigate the quantum version of the order parameter for empirical difficulty, we generate 1000 random instances of Eq. (3) at each fixed clause density (17 000 samples in total). For each random instance, we numerically diagonalize Hamiltonian Eq. (6) at different s and search for the minimum gap size Δ defined in Sec. V. In analogy to the classical result of Fig. 3 in Ref. [14], in Fig. 2(b) we plot median $1/\Delta^2$ with the clause density for the SAT, UNSAT, and the combined cases. $1/\Delta^2$ is used here so a larger value means a more difficult instance. We also see a phase transition in the combined median empirical difficulty (green [gray]). The difference between our result and the classical result in Ref. [14] is that the UNSAT case is not more difficult than the SAT case on average.

The reason for the difference is that QAA does not know whether the ground state is SAT or UNSAT; it only solves the ground states gapped by Δ. On the other hand the classical algorithm in Ref. [14] can stop after finding a SAT assignment and often needs to eliminate more possible assignments for UNSAT case. However, here the classical algorithm gives a determinstic proof while QAA is probabilistic, so no conclusion on quantum classical distinction can be made.

Due to this intuition, we suspect that the degeneracy of the ground states may be correlated with the empirical difficulty measure $1/\Delta^2$. And we find in Fig. 2(c) it is indeed the case. For instances with different ground-state degeneracy D, we see the median, maximum, and minimum empirical difficulty indeed decrease as the degeneracy increases. However, this does not establish a causal relation, so in the following sections, we evaluate the success of our strategy.

VII. INCREASED GAP SIZE

Since cases with nondegenerate ground states are the most difficult, we focus on these cases in evaluating our strategy. We generate random instances of Eq. (3) with $n = 10$ variables and keep those with unique ground states. Then we numerically diagonalize Eq. (6) at different s and find the minimum gaps Δ_1. The distribution among 200 000 random instances of the minimum gap Δ_1 and the corresponding s is given in Figs. 3(a) and 3(b). As a reminder, the minimum gap in our paper is redefined in terms of "good states" and "bad states." We see the avoided crossings all happen around $s \sim 0.68$ and the cases with a small gap are rare.

In order to reduce the time required for numerical analysis after the extra qubits are introduced in our strategy, we randomly choose instances with Δ_1 as uniformly as possible distributed between [0,0.5] from the 200 000 instances to form a smaller representative sample. Then, we apply our strategy on the sample instance and calculate the new minimum gap size Δ_2. For the purpose of demonstration, we choose qubits assigned as zero in the unique ground state and add an additional clause containing this qubit. In practice, this is not possible since the ground state is unknown, but as mentioned before, random variables are more likely to be zero than one in the ground state. As shown in Fig. 3(c), on average there is a near constant increase on the minimum gap size for instances with different original gap size. The significance of this average constant increase is that for the most difficult cases with small Δ_1, the ratio of increase in gap size $(\Delta_2 - \Delta_1)/\Delta_1$ can be very large, leading to a significant improvement of the QAA performance for difficult cases.

Note that since small gaps are rare, the subpanel in in Fig. 3(c) shows at small Δ_1 the sample number is smaller, so the average of gap change can be less accurate. Also, despite

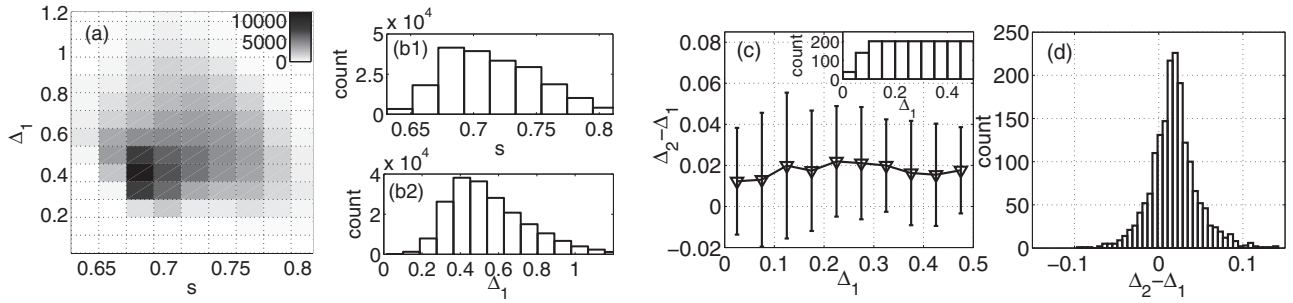

FIG. 3. (a) Statistics of minimum gap for 200,000 instances with nondegenerate ground states. The joint distribution over s, Δ_1. (b1) and (b2) Marginal distribution of each variable. We sample from the 20,0000 instances with gap size in [0,0.5] uniformly to study how adding degeneracy will change the gap size. (c) Average gap change $\Delta_2 - \Delta_1$ and original gap Δ_1. Every data point is the average of an interval of ± 0.025, and the error bar is $\pm$ standard deviation in each interval. The subpanel shows the number of samples averaged over; it is fixed as 150 except the rare small gap cases. (d) The distribution of gap change $\Delta_2 - \Delta_1$; the percentage above zero is around 0.78.

that on average there is a constant increase, the gap change varies for each instances. The distribution of the gap change is shown in Fig. 3(d). We see that around 78% of the time the gap increases, while around 22% of the time the gap decreases.

VIII. INCREASED SUCCESS PROBABILITY AT FIXED T

Increasing the success probability is the ultimate goal. To see how the improvement in performance measured by the minimum gap size reflects on the actual success probability, we fix the total running time T and numerically integrate the dynamical evolution of Eq. (1) on the same samples to calculate the success probability p_1 before applying the strategy and p_2 afterward. The increase of success probability depends on the total running time T; we choose two cases $T = 80$ and $T = 10$ representing two different regions.

When T is small, diabatic transition can occur with high probability. Recent results [13] show that for difficult cases, the success probability can be higher when T is small. Because of the complicated avoided crossings between states, smaller T can allow multiple diabatic transitions that go back to ground state. Indeed, we see in Figs. 4(a) and 4(c) that when $T = 10$ [Fig. 4(c)] the success probability p_1 is centered at around 0.25 and poor cases ($p_1 < 0.1$) as well as good cases ($p_1 > 0.9$) are rare; when $T = 80$ [Fig. 4(a)], the success probability p_1 centers around 0.9 but there are more poor cases ($p_1 < 0.1$). We call the region of running time represented by $T = 80$ the adiabatic region and $T = 10$ diabatic region. In the adiabatic region, the minimum gap has a good correlation with the success probability, while in the diabatic region, the correlation is weaker and the success probability is determined mainly by the structure of the avoided and unavoided crossings.

Due to the different conditions limiting the success probability, the change $p_2 - p_1$ in success probability after applying the strategy is also different in the two regions. For $T = 80$ [Fig. 4(b)], we see that around 86% of the time the success probability increases. For $T = 10$ [Fig. 4(d)], we see that only around 62% of the time the success probability increases. In both cases, the average $p_2 - p_1$ decreases with the original success probability p_1 [Figs. 4(a) and 4(c)]. This is because when probability is close to unity, there is no room for improvement. And at smaller p_1 there is significant increase in success probability on average.

To further confirm that the change of the success probability is indeed mainly due to the increase of the gap in the $T = 80$ case, we plot the ratio of gap square Δ_2^2/Δ_1^2 with the ratio of success probability p_2/p_1 for the $p_1 < 0.5$ instances. The reason we do not include $p_1 > 0.5$ instances is that as we can see in Fig. 4(a), for $p_1 > 0.5$ instances, the average $p_2 - p_1$ starts to decrease. The amount of increase is limited by the upper bound of unity; consequently excluding those data can

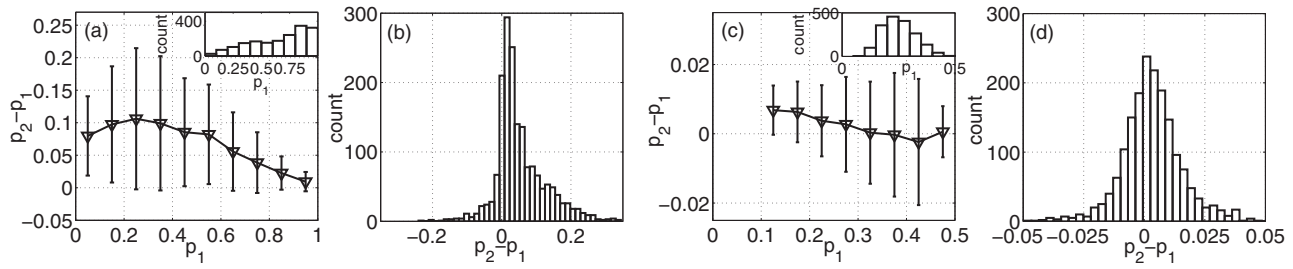

FIG. 4. (a) For total running time $T = 80$, average success probability change $p_2 - p_1$ and original success probability p_1. Every data point is the average of an interval of ± 0.05. The error bar is $\pm$ standard deviation. The subpanel shows the number of samples averaged over; at small p_1 the number of sample is smaller. (b) The distribution of success probability change $p_2 - p_1$. The percentage above zero is around 0.86. (c) For total running time $T = 10$, average success probability change $p_2 - p_1$ and original success probability p_1. Every data point is the average of an interval of ± 0.025. The error bar is $\pm$ standard deviation. The subpanel shows the number of samples averaged over. At small p_1 the number of samples is smaller. (d) The distribution of success probability change $p_2 - p_1$. The percentage above zero is around 0.62.

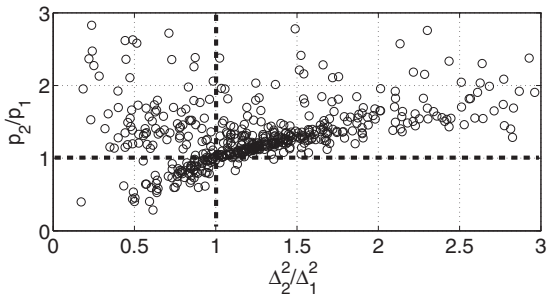

FIG. 5. Correlation between the change of gap Δ_2^2/Δ_1^2 and the change of success probability p_2/p_1 for cases with $p_1 < 0.5$. Total running time $T = 80$.

demonstrate better how the change of success probability is correlated with gap change.

The result is shown in Fig. 5, where we can see that nearly all cases with an increased gap have success probability increased proportional to the relative gap change. Surprisingly, when the gap is decreased, there are also some instances in which the success probability increases substantially. This increase may be caused by a change of the multiple avoided crossings between eigenstates.

IX. SUMMARY

In this paper, we propose a strategy to improve the performance of the QAA on random instances of exact cover by increasing the ground-state degeneracy. We define a modified minimum gap size as a measure of the performance of the QAA on a specific instance. This measure is a QAA analog for classical empirical difficulty. A "phase transition" in the quantum case with the clause density as the "order parameter" is numerically observed at the SAT-UNSAT threshold. Furthermore, this empirical difficulty for the QAA decreases with the degeneracy of ground state. We numerically observe that with significant probability, our strategy can increase the minimum gap size along the path of Hamiltonian and thus increase the success probability at fixed total running time. Our finding indicates that when you have idle extra qubits in your QAA machine, you can make use of them to improve the overall performance.

More detailed analyses of the mechanism of the increase in success probability for the cases where the gap decreased need to be done in future works. Note that how the probability of improving the performance scales with the system size is still unknown. There are mechanisms that may impair the success when the number of qubits becomes very large, e.g., local minima with more variables assigned zero than the global minimum [23]. Further numerical or analytical results on the scaling with system size is required to settle this question.

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APPENDIX: METHODS OF GENERATING RANDOM EXACT COVER INSTANCE

Since we want to fix the number of qubits so that the minimum gap size along the adiabatic path is not influenced by changes in the number of qubits, we generate random instances of Eq. (3) of $n = 10$ variables throughout the paper, except in Fig. 2(a), where different numbers of variables are considered for comparison. The number of clauses m varies for different purposes.

To generate an instance of Eq. (3) with n fixed variables and m fixed clauses, we first consider $N > n$ variables (N is somewhat larger than n to make the process more efficient). Then, we generate each of the m clauses by randomly choosing three different variables from the N variables. We count the number of variables n' appeared in the m clauses and repeat generating the m clauses until $n' = n$.

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