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Exact localized and oscillatory solutions of the nonlinear spin and pseudospin symmetric Dirac equations

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We derive exact analytic solutions of nonlinear Dirac equations with Kerr-like cubic nonlinearity. The equations model, among many physical systems, a Bose-Einstein condensate confined by a two-dimensional honeycomb optical lattice as well as relativistic solitons in optical waveguide arrays. The nonrelativistic limit of the solutions is derived, and the role of the nonlinearity on localization is discussed. The possibility of realizing some of these solutions in waveguide arrays and Bose-Einstein condensates is pointed out.

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I. INTRODUCTION

The recent increasing interest in the nonlinear Dirac equation (NLDE) was stimulated by the realization that it models many physical systems, such as electron transport in graphene [1], Bose-Einstein condensates (BECs) confined by a honeycomb optical lattice [2], and optical pulses in waveguide arrays [3,4]. The possibility of observing relativistic effects in these systems at very low velocities is particularly appealing. For instance, the effective speed of light in graphene equals the Fermi velocity, which is two orders of magnitude smaller than the speed of light in vacuum [1]. In Bose-Einstein condensates it is even more dramatic, where the effective speed of light equals the speed of sound, which is ten orders of magnitude smaller than the speed of light in vacuum [2]. Relativistic effects have indeed been investigated in some of these systems [5–8].

Many generalizations of the NLDE have been considered and studied in the literature, including extensions to higher dimensions [9–18] and different nonlinearities [19–22] such as the well-known Thirring model [19] and the Gross-Neveu model [21]. The fact that solutions of the NLDE and its generalizations correspond to extended particles of the associated physical systems [20,23] has stimulated extensive interest in the existence and stability of its solutions [9–22,24–26].

Recently, Haddad and collaborators have extended their earlier work on the NLDE of Bose-Einstein condensates confined by a two-dimensional (2D) honeycomb lattice [2] by presenting a thorough analysis of the soliton solutions of the 1D NLDE corresponding to an isolated zigzag or armchair line in the 2D lattice [24]. Using conformal degree arguments, Haidari has presented a generalized NLDE with various forms of the cubic nonlinear coupling from which these two models, in addition to many other well-known models such as the Thirring and Gross-Neveu models, arise as special cases [27].

In Ref. [24], near exact (graphical) solutions were obtained for the massless NLDE. In Ref. [26], one single soliton solution was derived for the massive version of the same equation within the context of waveguide arrays. In both of these works, the so-called symmetric coupling was considered, i.e., linear coupling. Here, we consider the same 1D NLDE for the massless and massive cases with symmetric (linear) and nonsymmetric (nonlinear) coupling. We derive exact localized and oscillatory analytic solutions for the relativistic case as well as the nonrelativistic limit. We show that by properly taking the nonrelativistic limit, the relativistic solutions indeed tend to their nonrelativistic counterparts. We calculate physical quantities associated with these solutions, such as the total energy, momentum, and spin, and we point out the possibility of realizing these solution waveguide arrays and Bose-Einstein condensates.

In the next section, we present the NLDEs to be considered. Section III is devoted to presenting our solution method and the resulting solutions for the various cases. We discuss in Sec. IV the possible extensions of the solutions found here to NLDEs in a moving frame and with external potentials, and we end with final remarks and an outlook for future work.

II. THE NONLINEAR DIRAC EQUATIONS

Requiring the different components of the spinor Lagrangian to have the same conformal degree, Haidary [27] has constructed a general Lagrangian that includes nonlinear coupling. Generalized nonlinear Dirac equations were then obtained by variation of the spinor action. It turns out that many of the well-known NLDE models, in addition to the ones considered here, emerge as special cases of this generalized model, as will be detailed below. The general model for the one-dimensional coupled nonlinear Dirac equations reads

$$i\partial_t\psi_+ = m\psi_+ + \partial_x\psi_- + (\alpha_+|\psi_+|^2 + \alpha_-|\psi_-|^2)\psi_+ + \alpha_W(\psi_+\psi_-^* + \psi_+^*\psi_-)\psi_-, \quad (1)$$

$$i\partial_t\psi_- = -m\psi_- - \partial_x\psi_+ + (\alpha_-|\psi_+|^2 + \alpha_+|\psi_-|^2)\psi_- + \alpha_W(\psi_+\psi_-^* + \psi_+^*\psi_-)\psi_+, \quad (2)$$

where $\psi_\pm(x,t)$ are the spinor field components of a fictitious spin-1/2 system. Here, m corresponds to the rest mass, and $\alpha_\pm = \alpha_V \pm \alpha_S$ and α_W are real dimensionless parameters corresponding to coupling strengths of the nonlinear self-interaction. The massive Thirring model [19] is obtained as the special case of a pure vector coupling mode, where $\alpha_S = \alpha_W = 0$, and the massive Gross-Neveu model [21] is obtained as the special case of a pure scalar coupling mode, with $\alpha_V = \alpha_W = 0$. Furthermore, the above-mentioned physical models of graphene, BEC, and waveguide arrays with cubic nonlinearity are obtained with the special case of $\alpha_W = \alpha_- = 0$. This case is denoted below as the *spin symmetric* case, in contrast with another special case with $\alpha_W = \alpha_+ = 0$, which we denote as the *pseudospin symmetric*

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case [27]. For the spin symmetric special case, the NLDE takes the form

$$i\partial_t\psi_+ = m\psi_+ + \partial_x\psi_- + \alpha_+|\psi_+|^2\psi_+, \tag{3}$$

$$i\partial_t\psi_- = -m\psi_- - \partial_x\psi_+ + \alpha_+|\psi_-|^2\psi_-, \tag{4}$$

and the pseudospin symmetric case corresponds to

$$i\partial_t\psi_+ = m\psi_+ + \partial_x\psi_- + \alpha_-|\psi_-|^2\psi_+, \tag{5}$$

$$i\partial_t\psi_- = -m\psi_- - \partial_x\psi_+ + \alpha_-|\psi_+|^2\psi_-. \tag{6}$$

Equations (3) and (4) are the massive NLDE with cubic (Kerr-type) nonlinearities. The rest of this paper is devoted to finding exact analytic stationary solutions to Eqs. (3)–(6). Solutions will be obtained in both the relativistic and nonrelativistic limits.

III. SOLUTION METHOD AND RESULTS

The simpler nonrelativistic case is started within the following subsection. We derive the relativistic solutions in Sec. III B, from which the proper nonrelativistic results are then extracted by a limiting procedure in Sec. III C. Oscillatory solutions are derived for the pseudospin symmetric case in Sec. III D.

A. Nonrelativistic limit

1. Spin symmetric case

We look for the stationary solutions of Eqs. (3) and (4), which can be written as

$$\psi_+(x,t) = \phi_+(x)\, e^{i\lambda_+ t}, \tag{7}$$

$$\psi_-(x,t) = \phi_-(x)\, e^{i\lambda_- t}, \tag{8}$$

where $\phi_\pm(x)$ are real functions and $\lambda_\pm$ are real constants corresponding to the energy of the excitations. Substituting in Eqs. (3) and (4), we get

$$\phi'_-\, e^{-i(\lambda_+-\lambda_-)t} + \alpha_+\phi_+^3 + (m+\lambda_+)\phi_+ = 0, \tag{9}$$

$$\phi'_+\, e^{i(\lambda_+-\lambda_-)t} - \alpha_+\phi_-^3 + (m-\lambda_-)\phi_- = 0, \tag{10}$$

where a prime denotes a derivative with respect to x. The nonrelativistic case is attained when the energy equals the rest mass, namely $\lambda_- = \lambda_+ = m$. In this case, the last equations simplify to

$$\phi'_- + \alpha_+\phi_+^3 + 2m\phi_+ = 0, \tag{11}$$

$$\phi'_+ - \alpha_+\phi_-^3 = 0. \tag{12}$$

Solving these coupled equations is detailed in Appendix A, where we obtain the analytic solutions

$$\phi_+(x) = \pm 2\sqrt{\frac{-m}{\alpha_+}}\frac{1}{\sqrt{1+16m^4(x-x_0)^4}}, \tag{13}$$

and Eq. (A1) gives

$$\phi_-(x) = \pm 4m\sqrt{\frac{-m}{\alpha_+}}\frac{x-x_0}{\sqrt{1+16m^4(x-x_0)^4}}. \tag{14}$$

Clearly, for real solutions, α_+ and m should have opposite signs. In Fig. 1(a), we plot both components for some specific values of the parameters.

2. Pseudospin symmetric case

Following a similar procedure, the solutions of the pseudospin case, Eqs. (5) and (6), can be derived and take the form

$$\phi_+(x) = \frac{1}{\frac{1}{4}e^{-2c_1+e^{c_1}(x-x_0)} + 2m\alpha_- e^{-e^{c_1}(x-x_0)}}, \tag{15}$$

$$\phi_-(x) = -\frac{1}{\alpha_-}\left(\frac{1}{4}e^{-c_1+e^{c_1}(x-x_0)} - 2m\alpha_- e^{c_1-e^{c_1}(x-x_0)}\right), \tag{16}$$

where c_1 and x_0 are arbitrary real constants. The solution is plotted in Fig. 1(b).

B. Relativistic regime

1. Spin symmetric case

Here, we solve Eqs. (3) and (4) for the relativistic case, namely with $m \neq \lambda_\pm$. It should be noted that throughout this section, it is assumed that $\lambda_\pm > m$. While localized solutions are obtained with this assumption, the case $\lambda_\pm < m$ apparently corresponds to oscillatory solutions. Substituting the stationary solutions Eqs. (7) and (8) in Eqs. (3) and (4), we obtain for $\lambda_- = \lambda_+ = \lambda$

$$\phi'_- + \alpha_+\phi_+^3 + (m+\lambda)\phi_+ = 0, \tag{17}$$

$$\phi'_+ - \alpha_+\phi_-^3 + (m-\lambda)\phi_- = 0. \tag{18}$$

Attempting to solve this system in a similar manner as in the previous section, namely solving Eq. (18) for ϕ_- and then substituting in Eq. (17), will not be possible here since ϕ_- turns out to be a complicated function of ϕ'_+ and thus Eq. (17) will not be separable. Alternatively, we follow the following approach. Defining the auxiliary functions

$$F_+(x) = m + \lambda + \alpha_+\phi_+^2(x), \tag{19}$$

$$F_-(x) = m - \lambda - \alpha_+\phi_-^2(x), \tag{20}$$

Eqs. (17) and (18) take the form

$$F'_- - 2F_+\sqrt{F_+-m-\lambda}\,\sqrt{m-\lambda-F_-} = 0, \tag{21}$$

$$F'_+ + 2F_-\sqrt{F_+-m-\lambda}\,\sqrt{m-\lambda-F_-} = 0. \tag{22}$$

Multiplying Eq. (21) by F_-, Eq. (22) by F_+, and then adding, we find the following conservation relation for F_- and F_+:

$$F_-^2 + F_+^2 = c_1^2, \tag{23}$$

where c_1 is an arbitrary real constant. This result can be used to decouple Eqs. (21) and (22) and derive one equation for, say, F_+,

$$F'_+ + 2\sqrt{c_1^2-F_+^2}\,\sqrt{F_+-m-\lambda}\,\sqrt{m-\lambda-\sqrt{c_1^2-F_+^2}} = 0. \tag{24}$$

This equation can be solved by transforming to polar coordinates, as detailed in Appendix B. Then using Eq. (19), the

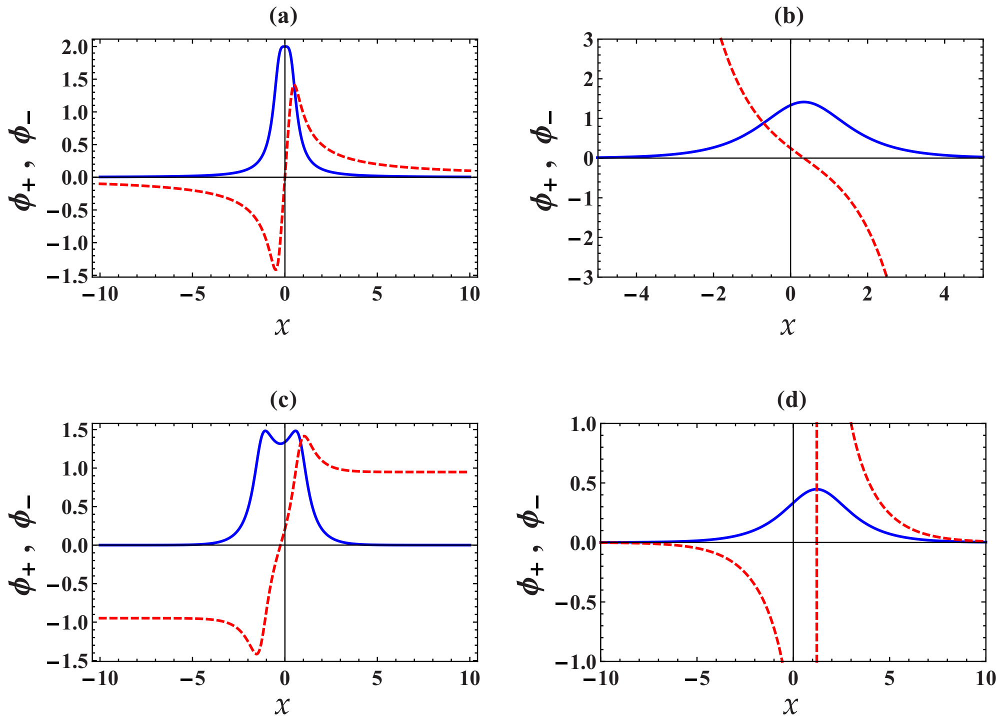

FIG. 1. (Color online) The exact stationary solutions (13)–(16), (26), (25), (32), and (33) of the NLD Eqs. (3) and (4). (a) Nonrelativistic spin symmetric case with $m = -\alpha_+ = 1$ and $x_0 = 0$. (b) Nonrelativistic pseudospin symmetric case with $m = \alpha_-/4 = 1/4$ and $c_1 = x_0 = 0$. (c) Relativistic spin symmetric case with $\lambda = 10m = -\alpha_+ = 1$ and $x_0 = 0$. (d) Relativistic pseudospin symmetric case with $\lambda = 1$, $\alpha_- = 0.8$, $m = 1.2$, and $x_0 = 0$. The solid blue line corresponds to ϕ_+ and the dashed red line corresponds to ϕ_-.

solution ϕ_+ is obtained,

$$\phi_+(x) = -\frac{8\sqrt{\lambda_1}}{\sqrt{\alpha}}\frac{y}{\sqrt{\frac{1-c}{c}[16(c^2-1)^2 + 32c(c^2-1)y + 8(3c^2+1)y^2 + 8cy^3 + y^4]}}, \tag{25}$$

where $y(x) = e^{\sqrt{2}\sqrt{a}\sqrt{b}\,x}$ and we have set the arbitrary peak position to zero. The expression for $\phi_-(x)$ can be derived using Eqs. (20) and (23), which reads

$$\phi_-(x) = \frac{\sqrt{2\lambda}}{\sqrt{\alpha}(1-c)^2}\frac{4(1-c^2)+y^2}{\sqrt{\frac{-1}{(1-c)^3}[8(3c^2+1)y^2 + 32c(c^2-1)y + 16(c^2-1)^2 + 8cy^3 + y^4]}}. \tag{26}$$

In Figs. 1(c) and 1(d), we plot these solutions, where it is observed that they are off-centered. The shift from the center is given by the root of the odd component, ϕ_-, namely

$$x_r = -\frac{\log[(m-\lambda)^2/16m\lambda]}{\sqrt{8}\sqrt{a}\sqrt{b}}, \tag{27}$$

such that when m approaches λ the two peaks in ϕ_+ merge and shift to the left and the saturation values of ϕ_- approach zero.

2. *Pseudospin symmetric case*

A different approach will be used here since there will be no need to define the auxiliary functions $F_\pm$. In terms of the stationary profiles $\phi_\pm$, the pseudospin symmetric NLDEs, Eqs. (5) and (6), become

$$\phi_-' + \alpha_-\phi_-^2\,\phi_+ + (m+\lambda)\phi_+ = 0, \tag{28}$$

$$\phi_+' - \alpha_-\phi_+^2\,\phi_- + (m-\lambda)\phi_- = 0, \tag{29}$$

where we have assumed again $\lambda_- = \lambda_+ = \lambda \neq m$. We multiply Eq. (28) by ϕ_+', Eq. (29) by ϕ_-', subtract the two equations, and then solve the resulting differential equation for ϕ_- to obtain

$$\phi_- = \pm\frac{\sqrt{\alpha_-(m+\lambda)\phi_+^2 + \lambda^2 - m^2 - c_1}}{\sqrt{-\alpha_-(\alpha_-\phi_+^2 + \lambda - m)}}, \tag{30}$$

where c_1 is an arbitrary constant. Substituting back in Eq. (6), we obtain a differential equation for ϕ_+, which is separable, and, upon using the substitution $\phi_+ = \sqrt{(m-\lambda)/\alpha_-}\cos[\theta(x)]$, it can be integrated to give

$$(x-x_0)\,\alpha_-\sqrt{(m-\lambda)/\alpha_-}\sqrt{(m-\lambda)\,\alpha_-}\sqrt{m^2-\lambda^2} = \ln\left(\frac{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}\right), \tag{31}$$

where x_0 is another arbitrary constant corresponding to the center of the localized solution. Notice that the quantities under the square roots may be negative, hence we did not combine them under a single square root sign. Invoking that $\phi_+ = \sqrt{(m-\lambda)/\alpha_-}\,\cos[\theta(x)]$, we solve this equation for ϕ_+ to finally get

$$\phi_+(x) = \frac{2\,(m-\lambda)}{\alpha_-\,(m-\lambda)\,e^{\sqrt{m-\lambda}\,\sqrt{m+\lambda}\,(x-x_0)} + e^{-\sqrt{m-\lambda}\,\sqrt{m+\lambda}\,(x-x_0)}}, \tag{32}$$

and upon using this result in Eq. (30), we obtain

$$\phi_-(x) = \frac{2\sqrt{(m-\lambda)\,\alpha_-}\,\sqrt{(m+\lambda)\,\alpha_-}}{\alpha_-\,[\alpha_-\,(m-\lambda)\,e^{\sqrt{m-\lambda}\,\sqrt{m+\lambda}\,(x-x_0)} - e^{-\sqrt{m-\lambda}\,\sqrt{m+\lambda}\,(x-x_0)}]}. \tag{33}$$

The two solutions are plotted in Fig. 1(d). It is clear that ϕ_- diverges at the peak position of the well-behaved ϕ_+.

C. Extracting the nonrelativistic results from the relativistic ones

Here we outline the derivation of the nonrelativistic solutions, Eqs. (13) and (14), from the relativistic ones, Eqs. (26) and (32).

The nonrelativistic limit is obtained by taking the limit $\lambda \to m$. Before taking this limit, it should be noted that the center of the relativistic solutions is shifted by the value of x_r given by Eq. (27). Since the nonrelativistic solutions are centered at the origin, we start, without loss of generality, by shifting back the relativistic solutions to the origin with the transformation $x \to x + x_r$ applied to Eq. (26). This step turns out to be crucial since x_r itself diverges as $1/(\lambda - m)$. Then we expand in powers of the small quantity $\lambda - m$. In the limit $\lambda \to m$, only the zeroth-order term remains while all higher-order terms vanish. This leads to the relativistic result Eq. (13), namely

$$\lim_{\lambda\to m} \phi_+(x + x_r) = \pm 2\sqrt{\frac{-m}{\alpha_+}}\,\frac{1}{\sqrt{1+16m^4x^4}}. \tag{34}$$

In a similar manner, the nonrelativistic pseudospin solution Eq. (14) can be derived from the relativistic solution Eq. (32).

D. Oscillatory solutions

As noted earlier, the localized solutions found here for the spin symmetric case are valid only for $\lambda > m$. Attempting to obtain oscillatory solutions, apparently for $\lambda < m$, requires $\phi_\pm$ to be complex functions. Such solutions have indeed been found by Haddad *et al.* [24], though for the massless case. While the field components $\phi_\pm$ are oscillatory, the total field amplitude is a soliton over a finite background.

For the pseudospin symmetric case the situation is different, in that real oscillatory solutions can be found, as we show below. Here we solve the pseudospin Eqs. (28) and (29) following the same approach of Sec. III B, namely by defining the auxiliary functions $F_\pm$, as in Eqs. (19) and (20). It turns out that this procedure results in oscillatory solutions.

In terms of $F_\pm$, Eqs. (28) and (29) take the form

$$F_-' = -2\sqrt{F_+ - m - \lambda}\,\sqrt{m - \lambda - F_-}(F_- - 2m), \tag{35}$$

$$F_+' = 2\sqrt{F_+ - m - \lambda}\,\sqrt{m - \lambda - F_-}(F_+ - 2m). \tag{36}$$

Dividing the two equations and then integrating, the following conservation relation is obtained:

$$\frac{1}{F_-} + \frac{1}{F_+} = \frac{1}{2m} + \frac{c_1}{2m\,F_-\,F_+}, \tag{37}$$

where c_1 is an arbitrary constant. Invoking the dependence of $F_\pm$ on $\phi_\pm$, through Eqs. (19) and (20), we solve the last equation for ϕ_-,

$$\phi_- = \sqrt{\frac{-\alpha_-\,(m+\lambda)\,\phi_+^2 + \lambda^2 + 3m^2 - c_1}{\alpha_-\,(\alpha_-\phi_+^2 + \lambda - m)}}. \tag{38}$$

Substituting back in Eq. (28), we obtain the following differential equation for ϕ_+:

$$\phi_+' + (m - \lambda - \alpha_-\,\phi_+^2)\sqrt{\frac{-\alpha_-\,(m+\lambda)\,\phi_+^2 + \lambda^2 + 3m^2 - c_1}{\alpha_-\,(\alpha_-\phi_+^2 + \lambda - m)}} = 0, \tag{39}$$

with the solution

$$\phi_+(x) = \frac{\sqrt{m-\lambda}}{\sqrt{\alpha_-}}\,\mathrm{sn}\left(\sqrt{a}\,(x - x_0)\middle|\frac{b}{a}\right), \tag{40}$$

where $\mathrm{sn}(x|n)$ is the Jacobi elliptic function, $a = 3m^2 + \lambda^2 - c_1$, and $b = \lambda^2 - m^2$. Substituting back in Eq. (38), and then using the relations $\mathrm{sn}(x|n)^2 + \mathrm{cn}(x|n)^2 = 1$ and noticing that $a - b^2\,\mathrm{sn}(\sqrt{a}\,x|\frac{b^2}{a})^2 = a\,\mathrm{dn}(\sqrt{a}\,x|\frac{b^2}{a})^2$, the following

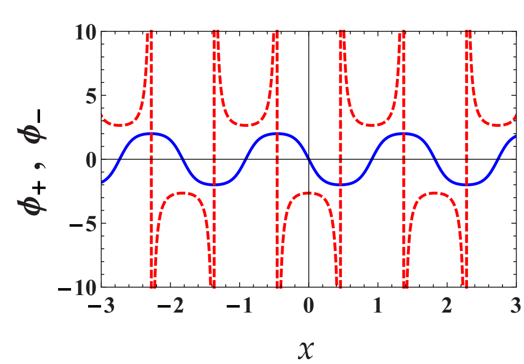

FIG. 2. (Color online) The exact oscillatory solutions (40) and (41) of the NLD Eqs. (28) and (29) for the relativistic pseudospin symmetric case with $m = -\alpha_- = 1$, $\lambda = 5$, and $c_1 = x_0 = 0$. The solid blue line corresponds to ϕ_+ and the dashed red line corresponds to ϕ_-.

simplified expression for ϕ_- is obtained:

$$\phi_-(x) = -\sqrt{\frac{a}{\alpha_-(m-\lambda)}\frac{\mathrm{dn}\left[\sqrt{a}\,(x-x_0)|\frac{b}{a}\right]}{\mathrm{cn}\left[\sqrt{a}\,(x-x_0)|\frac{b}{a}\right]}}, \tag{41}$$

where $\mathrm{cn}(x|n)$ and $\mathrm{dn}(x|n)$ are Jacobi elliptic functions. In Fig. 2, we plot these solutions for some values of the parameters.

IV. DISCUSSION AND OUTLOOK

A. Conserved quantities

It is instructive to use the exact solutions found here to calculate important physical quantities such as total energy, total linear momentum, and total spin (charge). Since the solutions we derive here of the pseudospin symmetric case diverge, we restrict the calculation to the spin symmetric case. We start by constructing the Lagrangian density,

$$L = \left(\frac{i}{2}\right)(\bar{\boldsymbol{\Psi}}\gamma^\mu\partial_\mu\boldsymbol{\Psi} - \partial_\mu\bar{\boldsymbol{\Psi}}\gamma^\mu\boldsymbol{\Psi}) - \alpha_+\,\bar{\boldsymbol{\Psi}}(\bar{\boldsymbol{\Psi}}\mathbf{A_1}\boldsymbol{\Psi}\mathbf{A_1} + \bar{\boldsymbol{\Psi}}\mathbf{A_2}\boldsymbol{\Psi}\mathbf{A_2})\boldsymbol{\Psi}, \tag{42}$$

from which Eqs. (3) and (4) are derived by $\partial L/\partial\bar{\boldsymbol{\Psi}}$. Here, $\bar{\boldsymbol{\Psi}} = \boldsymbol{\Psi}^\dagger\gamma^0$, where $\boldsymbol{\Psi}^\dagger$ is the transpose conjugate of the spinor,

$$\boldsymbol{\Psi} = \begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}, \tag{43}$$

and the matrices $\mathbf{A}_{1,2}$ are given by

$$\mathbf{A_1} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \quad \mathbf{A_2} = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}. \tag{44}$$

The Einstein summation rule is implied such that $\gamma^\mu\partial_\mu = \gamma^0\partial_t + \gamma^1\partial_x$, with $\gamma^{0,1}$ being related to the Pauli matrices as follows:

$$\gamma^0 = \sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad \gamma^1 = i\sigma_1 = i\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \tag{45}$$

The momentum energy tensor is then written as

$$T^{\mu\nu} = \left(\frac{i}{2}\right)(\bar{\boldsymbol{\Psi}}\gamma^\mu\partial^\mu\boldsymbol{\Psi} - \partial^\mu\bar{\boldsymbol{\Psi}}\gamma^\mu\boldsymbol{\Psi}) - g^{\mu\nu}L, \tag{46}$$

where the metric $g^{\mu\nu}$ is equivalent to γ^0. This gives the Hamiltonian density

$$\begin{aligned} H &= T^{00}\\ &= \left(\frac{i}{2}\right)(\bar{\boldsymbol{\Psi}}\gamma^1\partial_x\boldsymbol{\Psi} - \partial_x\bar{\boldsymbol{\Psi}}\gamma^1\boldsymbol{\Psi}) + m\bar{\boldsymbol{\Psi}}\boldsymbol{\Psi}\\ &\quad + \alpha_+\,\bar{\boldsymbol{\Psi}}(\bar{\boldsymbol{\Psi}}\mathbf{A_1}\boldsymbol{\Psi}\mathbf{A_1} + \bar{\boldsymbol{\Psi}}\mathbf{A_2}\boldsymbol{\Psi}\mathbf{A_2})\boldsymbol{\Psi} \end{aligned} \tag{47}$$

and current

$$J = T^{01} = \left(\frac{i}{2}\right)(\bar{\boldsymbol{\Psi}}\gamma^0\partial_x\boldsymbol{\Psi} - \partial_x\bar{\boldsymbol{\Psi}}\gamma^0\boldsymbol{\Psi}). \tag{48}$$

The spin density is defined as

$$s = \bar{\boldsymbol{\Psi}}\gamma^0\boldsymbol{\Psi}. \tag{49}$$

The total energy, momentum, and spin are thus given by

$$E = \int_{-\infty}^{\infty} H\,dx, \quad P = \int_{-\infty}^{\infty} J\,dx, \quad Q = \int_{-\infty}^{\infty} s\,dx. \tag{50}$$

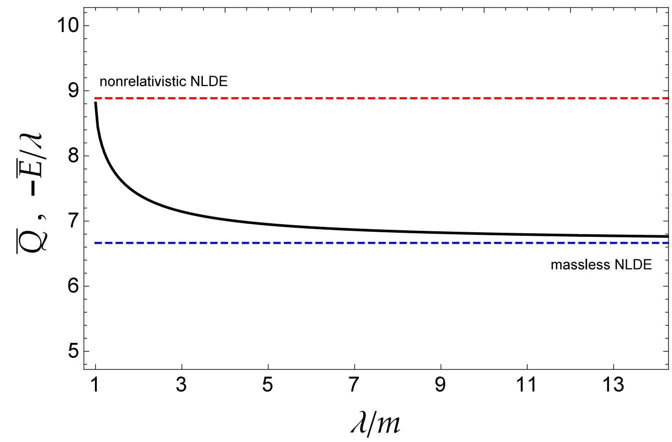

FIG. 3. (Color online) Total spin $\bar{Q}$ and total energy $\bar{E}$ vs λ/m. The upper and lower dashed horizontal lines indicate the nonrelativistic and massless values $[\sqrt{8}\pi/\alpha_+$ and $3\pi/(\sqrt{8}/\alpha_+)]$, respectively. The value $\alpha_+ = -1$ is used.

Clearly, for stationary solutions the linear momentum vanishes, $P = 0$. For the nonrelativistic spin symmetric case, Eqs. (13) and (14) are used to calculate the total energy and spin, where the following analytical results are obtained:

$$E = -\frac{Q}{m} = -\frac{\sqrt{8}\pi}{\alpha_+}. \tag{51}$$

It should be noted that $\alpha_+ < 0$ is a condition for the solutions $\phi_\pm$ of Eqs. (13) and (14) to be real. This leads to the fact that Q is positive and E is negative. The latter indicates that the solutions at hand correspond to a bound state, as expected for a solitonic solution. For the relativistic spin symmetric case, the energy and spin integrals diverge due to the fact that the solutions (32) and (33) have nonzero background. Nonetheless, one can calculate the renormalized energy and spin where the contribution of the background is canceled as follows:

$$\bar{E} = \int_{-\infty}^{\infty}(H - H_\infty)\,dx, \quad \bar{Q} = \int_{-\infty}^{\infty}(s - s_\infty)\,dx, \tag{52}$$

where

$$H_\infty = \lim_{x\to\pm\infty} H = \frac{\lambda - m}{\alpha_+}\lambda, \quad s_\infty = \lim_{x\to\pm\infty} s = \frac{m-\lambda}{\alpha_+}, \tag{53}$$

and Eqs. (32) and (33) have been used to explicitly calculate these limits. Scaling x with the rest mass as $m\,x \to x$ shows that the integrals in Eq. (52) become a function of only λ/m, $\bar{E} = \bar{E}(\lambda/m)$, and $\bar{Q} = \bar{Q}(\lambda/m)$.

In contrast with the nonrelativistic case, the results of the integrations for the relativistic case cannot be obtained analytically. Therefore, we compute the integrations numerically for a range of λ/m values and show the result in Fig. 3. Since our solutions are valid only for $m \leqslant \lambda$, the ratio λ/m ranges from 1 to ∞, where the lower boundary corresponds to the nonrelativistic limit and the upper boundary corresponds to the massless limit. As noted in Sec. III C, the peak of the solutions shifts away from the origin with an amount that is proportional to how deep the solution is in the relativistic regime, namely x_r as given by Eq. (27). For numerical integrations, this causes a problem when the boundaries of the integration are finite. Similar to the procedure of Sec. III C, we shift the solutions $\phi_+(x) \to \phi_+(x + x_r)$ before performing the integrations. This brings the nontrivial part of the solution back to the origin and it results in shorter convergence times. The curves of the quantities $\bar{Q}$ and $-\bar{E}/\lambda$ turn out to be identical, which agrees with the nonrelativistic result Eq. (51) for $m = \lambda$. The curve starts at $\lambda/m = 1$ from the nonrelativistic value $\sqrt{8}\,\pi/\alpha_+$ and saturates for large vales of λ/m at the value that corresponds to the massless NLDE. To calculate this limiting value, we set $m = 0$ in the solutions (32) and (33) and then calculate the integrals (52). Scaling x to λ as $x\lambda \to x$, the resulting integral becomes independent of λ and the integration gives $3\pi/(\sqrt{8}\alpha_+)$.

Another conserved quantity for the relativistic spin symmetric case was found in Sec. III B, namely $F_-^2 + F_+^2$. Using the solutions (32) and (33) gives $F_-^2 + F_+^2 = (m + \lambda)^2$, which is equivalent to the T^{11} element of the momentum energy tensor, as noticed by Ref. [28].

B. Movable solutions and solutions in the presence of external potentials

Here we investigate the possibility of finding transformations that lead to solutions in a moving frame and in the presence of external potentials. Such solutions would be useful to study the solitons dynamics and scattering by potentials.

In the moving frame, the solution $\mathbf{\Psi}'(x',t')$ should be related to the solution at the rest frame, $\mathbf{\Psi}(x,t)$, through the Lorentz transformation $\mathbf{\Psi}' = \mathbf{S}\,\mathbf{\Psi}$, where

$$\mathbf{S} = \begin{pmatrix} a_1(v) & a_2(v) \\ a_3(v) & a_4(v) \end{pmatrix}, \tag{54}$$

$x' = x - v\,t$, $t' = t - v\,x$, and $a_{1-4}(v)$ are constants that depend only on the speed of the moving frame v. For such movable solutions to exist, the NLDEs (3) and (4) should be covariant under the Lorentz transformation. Applying the Lorentz transformation on these equations, it turns out that the system (3) and (4) is not covariant under this transformation. This means that movable solutions cannot be obtained for the present case. This result is in fact expected due to the fact that the NLDE contains a nonlinear term that violates Lorentz invariance, as was shown on a fundamental level by Ref. [2].

It will also be interesting to find solitonic solutions of the NLDE in the presence of an external potential, $V(x)$, such as that of an electromagnetic field [29]. The NLDE that corresponds to the Lagrangian (42) [namely Eqs. (13) and (14)] will then be modified by adding the term $-\gamma^0 V(x)\mathbf{\Psi}$ [29]. To find solutions of this inhomogeneous NLDE in terms of the solutions of the homogeneous NLDE, one typically performs a similarity transformation for the fields and their dependent variables as follows: $\mathbf{\Psi}(x,t) \to e^{iA(x,t)}\mathbf{\Psi}(X(x,t),t)$. Requiring this field to be a solution of the inhomogeneous NLDE forces $A(x,t)$ and $X(x,t)$ to satisfy a system of two coupled differential equations. The solutions of these equations define the similarity transformation, which can then be used to map the solutions of the homogeneous NLDE to those of the inhomogeneous NLDE. For the NLDE considered in this paper, it turns out that only trivial solutions for $A(x,t)$ and $X(x,t)$ exist, namely $A(x,t) = \text{const}$ and $X(x,t) = x$, which indicates that the similarity transformation used here is insufficient to obtain solutions of the NLDE in the presence of external potentials. It should be noted that in Ref. [29] a different nonlinear term is used, which allows for the existence of alternative solutions of the inhomogeneous NLDE.

C. Applications to physical systems

There are two physical systems that are described by the NLDE with Kerr-like nonlinearity, namely light solitons in waveguide arrays and Bose-Einstein condensates in a honeycomb optical lattice. As suggested theoretically by Longhi [6], the first experimental realization of an optical analog for relativistic quantum mechanics was demonstrated by simulating the *Zitterbewegung* of a free Dirac electron in an optical superlattice [7]. In this setup, binary waveguide arrays represent a rather simple physical system that is described by the NLDEs (3) and (4). In terms of the electric field amplitude in the nth waveguide, a_n, the fields $\psi_\pm$ are given by $(\psi_+,\psi_-) = (-1)^n(a_n, i a_{2n-1})$, which means that ψ_+ and ψ_- correspond to the even and odd waveguide amplitudes, respectively. In the continuum limit, the equation of motion that describes the evolution of the discrete modes in the tight-binding approximation becomes identical with the NLDEs (3) and (4), namely [26]

$$i\partial_t \mathbf{\Psi} = -i\kappa\gamma^0\partial_x\mathbf{\Psi} + \sigma\gamma^1\mathbf{\Psi} - \alpha_+\mathbf{G}, \tag{55}$$

where $\mathbf{G} = (|\psi_+|^2\psi_+, |\psi_-|^2\psi_-)^T$. Here, κ is the coefficient of coupling between two adjacent waveguides, σ is the propagation mismatch, and α_+ is the nonlinear coefficient of waveguides. By comparing the last equation with NLDEs (3) and (4), we conclude that the rest mass m corresponds to the propagation mismatch σ, the nonlinear coupling coefficient α_+ is the nonlinear coefficient of waveguides, and the waveguide coupling coefficient $\kappa = 1$ in our case. The latter is of course equivalent to scaling κ with x, as $x/\kappa \to x$. It is thus expected that the solutions found here will be realizable by such an experimental setup.

Another physical system that is modeled by the NLDEs of the present paper is a Bose-Einstein condensate confined by a honeycomb optical lattice, as was first shown by Haddad and Carr [2]. They pointed out later that a one-dimensional version of the NLDE describes the condensate along a zigzag or armchair lines on the lattice [28], and then they proposed a detailed experimental procedure to excite relativistic vortices [30]. The NLDEs of the present paper, (3) and (4), describe in particular the condensate along the zigzag line. The NLDE of this system reads [2]

$$i\hbar\partial_t\mathbf{\Psi} = -ic_s\gamma^0\partial_x\mathbf{\Psi} - U\mathbf{G}. \tag{56}$$

Here, the speed of sound c_s is the effective speed of light, and U is the strength of the interatomic interaction that is proportional to the s-wave scattering length. The field components ψ_+ and ψ_- correspond to the condensate wave function in two degenerate sublattices [2]. In comparison with the solutions found by Haddad *et al.* [24] for the massless NLDE, our solutions correspond to the massive NLDE. In the context of BEC in a honeycomb lattice, the massive NLDE would model a break in the sublattice degeneracy [31]. In current experimental setups, the values of the two parameters can be controlled within a wide range, allowing for the possibility of exciting the localized solutions derived in this paper.

D. Final remarks and outlook

The solutions found here are stationary solitonic solutions since they preserve their shape over time. They are analogous to the sech solutions of the nonlinear Schrödinger equation (NLSE). However, there is an important difference between the two cases. The solitons of the NLSE have their width and amplitude dependent on the strength of the nonlinearity such that their norm is constant. In the NLDE solitons, say (13) and (14), the width is independent of the strength of nonlinearity and the amplitude is inversely proportional to it. Therefore, it will be interesting to study the interaction of two NLDE solitons, as it is expected to be fundamentally different from the case of NLSE solitons.

It is noticed that the solutions of the pseudospin symmetric case diverge. As a result, the associated physical quantities such as total energy and spin also diverge. For waveguide arrays and Bose-Einstein condensates this is not physical, hence the pseudospin symmetric solutions cannot be realized in these systems.

Oscillatory solutions for the spin symmetric case are yet to be sought. It should be noted that the solutions derived here are restricted to $\lambda \geqslant m$. For $\lambda < m$, the expressions for $\phi_\pm$ become complex and do not solve the designated NLDEs. Thus, the problem of finding oscillatory solutions needs to be treated separately and is left for future work.

For the pseudospin symmetric case, localized solutions are obtained for energies larger than the rest mass, $\lambda > m$, while oscillatory solutions are obtained for $\lambda < m$. This situation is opposite to that of the linear case. Solving Eqs. (28) and (29) without nonlinearity, $\alpha_- = 0$, we get

$$\phi_+(x) = c_1 \cosh(x\sqrt{m^2-\lambda^2}) - c_2 \frac{\sqrt{m^2-\lambda^2}}{m+\lambda}\sinh(x\sqrt{m^2-\lambda^2}), \tag{57}$$

$$\phi_-(x) = c_2 \cosh(x\sqrt{m^2-\lambda^2}) - c_1 \frac{m+\lambda}{\sqrt{m^2-\lambda^2}}\sinh(x\sqrt{m^2-\lambda^2}). \tag{58}$$

Clearly, oscillatory solutions are obtained for $|\lambda| > m$. This exhibits the role of nonlinearity in binding the extended particles to which the localized solutions correspond.

The NLDEs considered in Refs. [25,29] are similar to the one considered here but with a different cubic nonlinear term that is similar to that of the Thirring model [19] and the Gross-Neveu model [21]. Due to the fact that these cubic terms preserve the Poincaré covariance, movable solutions were possible to obtain. In our case, the cubic nonlinearity breaks the Poincaré covariance, and movable solutions were not possible to obtain with a Lorentz transformation.

The solution found by Tran *et al.* [26], in the context of discrete waveguide arrays, is a sech-like localized solitonic solution for the massive NLDE. Our solutions are fundamentally different in the sense that they are localized on a finite background. Finally, it should be mentioned that the stability of these solutions is yet to be investigated via the typical modulational stability analysis.

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APPENDIX A: DERIVING THE SOLUTION OF EQS. (11) AND (12)

Solving Eq. (12) for ϕ_-,

$$\phi_- = \left(\frac{\phi'_+}{\alpha_+}\right)^{1/3}, \tag{A1}$$

and then substituting in Eq. (11), we get

$$\frac{\phi''_+}{3\alpha_+^{1/3}\phi_+'^{\,2/3}} + \alpha_+\phi_+^3 + 2m\phi_+ = 0. \tag{A2}$$

Using $\phi''_+ = \phi'_+ \times d\phi'_+/d\phi_+$, this equation becomes separable and can be integrated to give

$$c_1 - 3m\phi_+^2 - \frac{3}{4}\alpha_+\phi_+^4 = \frac{3}{4\alpha_+^{1/3}}\phi_+'^{\,4/3}, \tag{A3}$$

where c_1 is a constant of integration. Integrating the last equation, we get

$$\left(\frac{4\alpha_+^{1/3}}{3}\right)^{3/4}(x-x_0) = \int \frac{d\phi_+}{[c_1 - 3m\phi_+^2 - (3/4)\phi_+^4]^{3/4}}, \tag{A4}$$

where x_0 is another constant of integration. Performing the integral, the last equation reads

$$\left(\frac{4\alpha_+^{1/3}}{3}\right)^{3/4}(x-x_0) = \frac{\sqrt{2}\,{}_2F_1\left(\frac{1}{2},\frac{3}{4};\frac{3}{2};y\right)\phi_+\left(\frac{-6m+2q-3\alpha_+\phi_+^2}{-3m+q}\right)^{3/4}\left(\frac{6m+2q+3\alpha_+\phi_+^2}{3m+q}\right)^{1/4}}{(4c_1 - 12m\phi_+^2 - 3\phi_+^4)^{3/4}}, \tag{A5}$$

where $q = \sqrt{9m^2 + 3c_1\alpha_+}$, $y = 2q\phi_+^2/[2c_1 + (-3m+q)\phi_+^2]$, and ${}_2F_1(\frac{1}{2},\frac{3}{4};\frac{3}{2};y)$ is the hypergeometric function. To find ϕ_+, the inverse of the functional on the right-hand side has to be found. For such a complicated form, this may not be feasible. Simple special cases are therefore sought. The special case of $c_1 = 0$ leads to an analytic solution. Setting $c_1 = 0$ in Eq. (A4) and performing the integration, we get Eqs. (13) and (14).

APPENDIX B: SOLVING EQ. (24)

Although Eq. (24) is separable, direct integration leads to an integral that cannot be performed. However, with the change of variables $F_+ = c_2 \cos\theta$, where c_2 is an arbitrary real constant, the last equation takes the form

$$-2\,dx = \frac{-c_2\,\sin\theta\,d\theta}{\sqrt{c_1^2 - c_2^2\cos^2\theta}\,\sqrt{c_2\,\cos\theta - m - \lambda}\,\sqrt{m - \lambda - \sqrt{c_1^2 - c_2^2\,\cos^2\theta}}}. \tag{B1}$$

With $c_2 = c_1 = -(m+\lambda)$, this equation simplifies to

$$-2\,dx = \frac{-d\theta}{\sqrt{a(1+\cos\theta)}\,\sqrt{b(1+c\sin\theta)}}, \tag{B2}$$

where $a = -(m+\lambda)$, $b = m - \lambda$, and $c = -a/b$. Integrating both sides of Eq. (B2), we obtain

$$2(x - x_0) = \frac{\sqrt{2}}{\sqrt{a}\,\sqrt{b}}\,\log\left[2(c + \sqrt{\sec^2(\theta/2)(1 + c\,\sin\theta)} + \tan(\theta/2))\right], \tag{B3}$$

which simplifies to

$$z = \sqrt{\frac{2}{1+f}}\,\sqrt{1 + c\sqrt{1-f^2}} + \sqrt{\frac{1-f}{1+f}}, \tag{B4}$$

where $z = e^{(\sqrt{2}\,\sqrt{a}\,\sqrt{b})\,x}/2 - c$ and $f = \cos\theta = F_+/c_1$. The last equation is quadratic in f with the solutions

$$f_1 = -\frac{1 - 4c^2 + 8c\,z - 6z^2 + z^4}{4c^2 - 8c\,z + (1+z^2)^2}, \tag{B5}$$

$$f_2 = \frac{-1 + 4c^2 + 8c\,z + 6z^2 - z^4}{4c^2 + 8c\,z + (1+z^2)^2}. \tag{B6}$$

Choosing the second solution, we substitute $F_+ = c_1\,f_2$ in Eq. (19) and then solve for ϕ_+ to finally obtain Eq. (25), from which ϕ_- of Eq. (26) can be derived.

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