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Entanglement classification of pure symmetric states via spin coherent states

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We give explicit expressions for canonical states labeling the vast majority of entanglement equivalent classes of symmetric states of qubits and efficient algorithms for reducing a given state to the representative of the class it belongs. This way, we achieve an almost complete classification under local unitary and local invertible transformations for symmetric states. The main tool is a technique introduced in this work, enabling to decompose in a unique way, spin symmetric states into a superposition of spin-1/2 coherent states. For the case of two qubits, the proposed decomposition reproduces the Schmidt decomposition and therefore, in the case of a higher number of qubits, can be considered as its generalization.

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Symmetric states under permutations have recently drawn a lot of attention in the field of quantum information. The essential reason is that the number of parameters needed for the description of a state in a symmetric subspace scales just linearly with the number of parties. This simplification makes symmetric states a good test ground for complex quantum information tasks such as the description of multipartite entanglement [1–8] and quantum tomography [9].

In this Rapid Communication we present our results and methods regarding the classification of entanglement for symmetric states. More precisely we provide explicit expressions for canonical states labeling the equivalence classes under the action of local operations, both unitary and invertible. We therefore identify the vast majority of entanglement classes for symmetric states for any number of qubits. To date, there are two well-known representations for symmetric states: the Dicke basis [10] and the Majorana representation [11]. In the present work, in order to prove our results we introduce a representation that permits one to decompose symmetric states into sums of coherent states. The introduced representation strongly resembles a generalized Schmidt decomposition, and it supports a geometric representation of symmetric states over the Bloch sphere. Furthermore, an immediate consequence of our methods is a straightforward estimation of a well-established measure of entanglement, the so-called Schmidt measure [12] for the symmetric states.

Every symmetric state of N qubits can be expressed in a unique way over the Dicke basis formed by the $N+1$ joined eigenstates $\{|N/2,m\rangle\}$ of the collective operators $\hat{S}_Z = \sum_{i=1}^N \hat{\sigma}_i^Z$ and $\hat{S}^2$, where $\vec{\hat{S}} = \sum_{i=1}^N \vec{\hat{\sigma}}_i$:

$$|N/2,m\rangle = d_{N,m}^{-1} \sum_{\text{perm}} \underbrace{|1\rangle|1\rangle\cdots|1\rangle}_{m+N/2}\underbrace{|0\rangle|0\cdots|0\rangle}_{N/2-m}, \tag{1}$$

where $d_{N,m} = \sqrt{\binom{N}{m+N/2}}$. More specifically $\hat{S}_Z|N/2,m\rangle = m|N/2,m\rangle$ and $\hat{S}^2|N/2,m\rangle = N(N/2+1)/2|N/2,m\rangle$ with $m = -N/2, -N/2+1, \ldots, N/2$. Even if the number of $\hat{S}_Z$ eigenstates is $N+1$, using the freedom of choice of the global phase and the normalization condition, one remains with N complex numbers expressing in a unique way every symmetric state over this basis.

A commonly used alternative to the Dicke basis is the Majorana representation [11] initially proposed to describe states of spin-j systems. This attributes to each state N points on the Bloch sphere in the following way: One projects the given symmetric state $|\Psi\rangle = \sum_{m=-N/2}^{N/2} c_m |N/2,m\rangle$ on a spin coherent state [13] of N qubits defined as

$$|\alpha\rangle = e^{\alpha^* \hat{S}_+}|N/2,-N/2\rangle \tag{2}$$

$$= \sum_{m=-s}^{s} (\alpha^*)^{N/2+m} d_{N,m} |N/2,m\rangle, \tag{3}$$

where $\hat{S}_+|N/2,m\rangle = \sqrt{(N/2+m)(N/2-m+1)}|N/2,m+1\rangle$. This projection leads to a polynomial of Nth order on the complex parameter α, the so-called Majorana polynomial:

$$\Psi(\alpha) = \langle\alpha|\Psi\rangle = \sum_{m=-N/2}^{N/2} \lambda_m \alpha^{N/2+m}, \tag{4}$$

with $\lambda_m = d_{N,m} c_m$. The N complex roots $\{\alpha_n\}$ (Majorana roots) of the polynomial $\Psi(\alpha)$,

$$\Psi(\alpha) \propto \prod_{n=1}^{N} (\alpha - \alpha_n), \tag{5}$$

fully characterize the state $|\Psi\rangle$. It is possible to introduce a geometric picture by attributing N Bloch vectors $\{\mathbf{v}_n\}$ to the roots $\{\alpha_n\}$ via the inverse stereographic mapping $\{\alpha_n\} \to \{e^{i\varphi_n}\tan(\theta_n/2)\} \equiv \{\mathbf{v}_n\}$. The edges of these vectors define a set of N points on the Bloch sphere, the so-called Majorana stars. Furthermore, using the states $|\chi_n\rangle = \sin(\theta_n/2)|0\rangle - e^{-i\varphi_n}\cos(\theta_n/2)|1\rangle$ which are orthogonal to the states associated to $\{\mathbf{v}_n\}$, one can write in a unique way every symmetric state as

$$|\Psi\rangle = A \sum_{\text{perm}} |\chi_1\rangle \otimes |\chi_2\rangle \otimes \cdots \otimes |\chi_N\rangle, \tag{6}$$

where A is a complex number that stands for the normalization factor and the global phase.

Majorana representation, among other applications [14–17], has proven very useful to the study and classification of entanglement in symmetric states [2–7,18]. There are two ways to classify entanglement, or in other words, to regroup states in classes according to their entanglement properties. In the first classification, each class contains states which can be transformed in each other by local unitary (LU) transformations. States belonging to the same, so-called, LU class of entanglement have identical entanglement properties. In the second classification, stochastic local operations and classical communication (SLOCC) are also allowed. In that case, it can be shown [19,20] that two states are in the same class if and only if one state can be converted to the other via the use of invertible local (IL) operations mathematically implemented by the SL(2,$\mathbb{C}$) group. States belonging to same so-called IL class of entanglement, are entangled in the same way.

Focusing on the case of symmetric states of qubits, the LU transformations which leave states in the symmetric subspace are equivalent to collective $\mathbf{SU(2)}$ rotation [$\mathbf{SU(2)} = \mathrm{SU}(2) \times \mathrm{SU}(2) \times \cdots \times \mathrm{SU}(2)$] [4] where all the SU(2) (single qubit) transformations are identical and are parametrized by three real numbers. A symmetric state of qubits is defined by $2N$ real parameters but identifying its invariant part under LU transformations requires only $2N-3$ real numbers, the so-called LU invariants. States with the same LU invariants belong to the same LU entanglement class. There are different ways of identifying a set of LU invariants for a given state [21]. One way is to calculate the values of a complete set of polynomial invariant quantities. Alternatively with the help of LU transformations one can reduce a given state to a properly chosen LU canonical form [22,23], described by $2N-3$ real numbers. For symmetric states, the Majorana representation offers an overcomplete set of LU invariants with a geometric aspect, the inner products among $\{\mathbf{v}_n\}$. This can be easily understood noting an essential aspect of Majorana representation: Majorana stars rotate uniformly under collective $\mathbf{SU(2)}$ rotation [see Eq. (6)].

In the case of IL transformations for symmetric states, it has been proven [4] that it is sufficient to search for interconvertibility via just collective $\mathbf{SL(2,\mathbb{C})}$ operations, i.e., $\mathbf{SL(2,\mathbb{C})} = \mathrm{SL}(2,\mathbb{C}) \times SL(2,\mathbb{C}) \times \cdots \times \mathrm{SL}(2,\mathbb{C})$. The SL(2,$\mathbb{C}$) group is the complexification of the SU(2) group and thus a collective IL transformation on a symmetric state is parametrized by six real numbers. Similarly to the case of LU transformations, the $2N-6$ invariants of a symmetric state under IL transformations (IL invariants), can be calculated in different ways. It has been recently proven that under SL operations Majorana roots follow Möbius transformations on the complex plane [2,18,24] and that a complete set of IL invariants [18] is given by combinations of the roots of the Majorana polynomials. Finally, it is worth mentioning that a third way for classifying entanglement relevant only for symmetric states, has been recently suggested [3]. This last classification is related to the fact that an IL transformation cannot change the classification of degeneracies of the Majorana polynomial, Eq. (4).

In what follows we only consider pure states and we call a *generic* symmetric state of N qubits, a pure state whose highest degeneracy degree (γ) in Majorana's roots satisfies the condition $\gamma < \frac{N+1}{2}$ or $\gamma = N$ for N odd, and $\gamma < \frac{N}{2}+1$ or $\gamma = N$ when N is even. In the class of generic states, the states with no degeneracies are included, which are the states covering the vast majority of space of symmetric states.

Let us start with the presentation of our principal results which consist of explicit expressions for canonical states labeling the equivalence classes under the action of the local operations in the space of generic states. We state that by the application of local $\mathbf{SU(2)}$ transformation, any generic symmetric N-qubits state can be reduced to the following canonical form:

$$\left|\Psi^{\mathrm{LU}}_{(\mathrm{odd})}\right\rangle = A\left(|\mathbf{0}\rangle + y_1|\mathbf{X}_1\rangle + \sum_{m=2}^{(N-1)/2} y_m e^{il_m}|\mathbf{X}_m\rangle\right), \tag{7}$$

for odd N and

$$\left|\Psi^{\mathrm{LU}}_{(\mathrm{even})}\right\rangle = A\left(|\mathbf{0}\rangle + y_1|\mathbf{1}\rangle + \sum_{m=2}^{N/2} y_m e^{il_m}|\mathbf{X}_m\rangle\right) \tag{8}$$

for N even, where $y_m \in [0,1]$ and $l_m \in [0,2\pi]$ are real parameters and $|\mathbf{X}_m\rangle = |X_m\rangle \otimes |X_m\rangle \otimes \cdots \otimes |X_m\rangle$, with the one-qubit states parametrized as $|X_m\rangle = \cos(\theta_m/2)|0\rangle + e^{i\phi_m}\sin(\theta_m/2)|1\rangle$. Therefore, the $2N-3$ real parameters [22] y_m, l_m, θ_m, and ϕ_m constitute a complete set of LU invariants, that is, they completely parametrize the equivalence classes corresponding to orbits under the action of the local $\mathbf{SU(2)}$ transformations group, on the generic symmetric states.

A similar result can be proven for local $\mathbf{SL(2,\mathbb{C})}$, and the corresponding canonical states are

$$\left|\Psi^{\mathrm{IL}}_{(\mathrm{odd})}\right\rangle = A\left(|\mathbf{0}\rangle + |\mathbf{1}\rangle + \sum_{m=2}^{(N-1)/2} \lambda_m e^{il_m}|\mathbf{X}_m\rangle\right), \tag{9}$$

for odd N and

$$\left|\Psi^{\mathrm{IL}}_{(\mathrm{even})}\right\rangle = \left|\Psi^{\mathrm{IL}}_{(\mathrm{even})}\right\rangle = A\left(|\mathbf{0}\rangle + |\mathbf{1}\rangle + \lambda|\mathbf{c}\rangle + \sum_{m=3}^{N/2}\lambda_m e^{il_m}|\mathbf{X}_m\rangle\right), \tag{10}$$

for N even, where λ_m and ξ_m are real parameters and $|c\rangle = (c|0\rangle + |1\rangle)/\sqrt{1+|c|^2}$. The complex numbers c and λ are not independent and they are related to each other via a parametric relation which is provided in [25]. As in the case of $\mathbf{SU(2)}$, the $2N-6$ real numbers λ_m, l_m, θ_m, and ϕ_m in Eq. (9), together with the real and imaginary parts of λ in Eq. (10), form a complete set of IL invariants.

By definition, states equivalent under LU or IL operations belong to the same LU or IL class of entanglement and these are entangled in the same way [19,20]. In consequence the canonical forms in Eqs. (7) and (8) and Eqs. (9) and (10) permit us to identify all LU or IL classes for generic symmetric states of qubits as well as representative states of these.

Let us now present the two main elements of the proof for the canonical forms, Eqs. (7) and (8) and Eqs. (9) and (10). In a first step, we show that each generic symmetric state can be decomposed in a unique way as a linear combination of spin coherent states. In a second step, we straightforwardly employ local operations on the obtained decomposition in order to reach the canonical forms. The proof is constructive since it provides us the algorithm for classifying a given generic symmetric state of N qubits.

Any generic symmetric state $|\Psi_{(\mathrm{odd})}\rangle$ of N qubits where N is *odd* can be decomposed in a unique way as a superposition of at most $(N-1)/2$ spin coherent states $|\Phi_m\rangle$:

$$|\Psi_{(\text{odd})}\rangle = \left(\sum_{m=0}^{(N-1)/2} c_m|\Phi_m\rangle\right),$$
$$|\Phi_m\rangle = |\phi_m\rangle \otimes |\phi_m\rangle \cdots |\phi_m\rangle. \tag{11}$$

As a convention we arrange the complex amplitudes c_m in decreasing sequence $|c_0| > |c_1| > \cdots > |c_{(N-1)/2}|$ and for the single-qubit states $|\phi_m\rangle$, we use the specific parametrization $|\phi_m\rangle = \cos(\theta_m/2)|0\rangle + e^{i\varphi_m}\sin(\theta_m/2)|1\rangle$. We also note that in the general case $\langle\phi_n|\phi_m\rangle \neq 0$.

The proof of Eq. (11), which is presented in [25], provides the steps for identifying the parameters of decomposition [states $|\phi_m\rangle$ and coefficients c_m] for a given generic state. Specifically, it gives us a recipe to obtain these paramaters as a function of the roots of the Majorana polynomial, Eq. (4), associated with $|\Psi_{(\text{odd})}\rangle$.

If we exploit the normalization condition and the freedom of choice of the global phase, we can rewrite the state in the following more convenient form for our purposes:

$$|\Psi_{(\text{odd})}\rangle = A\left(|\Phi_0\rangle + \sum_{m=1}^{(N-1)/2} y_m e^{ik_m}|\Phi_m\rangle\right), \tag{12}$$

where $y_m = |c_m/c_0| < 1$, $e^{ik_m} = c_m(c_0 y_m)^{-1}$, and A the normalization factor.

For the case of an *even* number of qubits, the decomposition is slightly different. We prove in [25] that the following unique decomposition exists:

$$|\Psi_{(\text{even})}\rangle = c_0|\Phi_0\rangle + c_1|\Phi_0^\perp\rangle + \sum_{\substack{m=2\\(N>2)}}^{N/2} c_m|\Phi_m\rangle,$$
$$|\Phi_m\rangle = |\phi_m\rangle \otimes |\phi_m\rangle \cdots |\phi_m\rangle. \tag{13}$$

The complex amplitudes c_m satisfy now the following conditions: $|c_2| > \cdots > |c_{N/2}|$ and $|c_0| > |c_1|$. In addition $\langle\phi_0|\phi_0^\perp\rangle = 0$. For the single-qubit states we use as before the convention $|\phi_m\rangle = \cos(\theta_m/2)|0\rangle + e^{i\varphi_m}\sin(\theta_m/2)|1\rangle$. We suggest the more convenient form

$$|\Psi_{(\text{even})}\rangle = A\left(|\Phi_0\rangle + y_1 e^{ik_1}|\Phi_0^\perp\rangle + \sum_{\substack{m=2\\(N>2)}}^{N/2} y_m e^{ik_m}|\Phi_m\rangle\right), \tag{14}$$

where $y_m = |c_m/c_0|$, $e^{ik_m} = c_m(c_0 y_m)^{-1}$, and A the normalization factor.

Now, performing local $\mathbf{SU(2)}$ or $\mathbf{SL(2,\mathbb{C})}$ on Eqs. (12) and (14), the canonical forms given by Eqs. (7) and (8) and Eqs. (9) and (10) are obtained.

In order to show this, let us start with the case of an odd number of qubits and consider the state $|\Psi_{(\text{odd})}\rangle$ in Eq. (12). We denote $|\Psi'_{(\text{odd})}\rangle = \hat{U}|\Psi_{(odd)}\rangle$ with $\hat{U} \in \mathbf{SU(2)}$ and introduce the new phases k'_m defined as $e^{ik'_m}|\Phi'_m\rangle = \hat{U}e^{ik_m}|\Phi_m\rangle$ such that the single-qubit parametrization remains as $|\phi'_m\rangle = \cos(\theta'_m/2)|0\rangle + e^{i\varphi'_m}\sin(\theta'_m/2)|1\rangle$. The new representation for the rotated state $|\Psi'\rangle$ is

$$|\Psi'_{(\text{odd})}\rangle = A\left(|\Phi'_0\rangle + \sum_{m=1}^{(N-1)/2} y_m e^{ik'_m - ik'_0}|\Phi'_m\rangle\right). \tag{15}$$

In analogy, for an even number of qubits and starting from the state represented in Eq. (14) we arrive at

$$|\Psi'_{(\text{even})}\rangle = A\left(|\Phi'_0\rangle + y_1 e^{ik'_1 - ik'_0}|\Phi_0'^\perp\rangle + \sum_{\substack{m=2\\(N>2)}}^{N/2} y_m e^{ik'_m - ik'_0}|\Phi'_m\rangle\right). \tag{16}$$

We note that for both cases (N even or odd) parameters y_m are not affected and all the one-qubit states $|\Phi_m\rangle$ are rotated in the same way by the LU transformation. It is now clear that choosing the appropriate $\mathbf{SU(2)}$ local transformation such that $|\Phi_0\rangle \to |\mathbf{0}\rangle$ (and $|\Phi_0^\perp\rangle \to |\mathbf{1}\rangle$ for N even) canonical LU form Eqs. (7) and (8) are obtained.

The case of IL operations, although more complex, is treated in the same spirit; the details are given in [25].

It is worth noting here that the partial invariance of Eqs. (12) and (14) under LU transformations suggests a useful geometric interpretation of generic symmetric states. Indeed, we can represent Eq. (12) on the Bloch ball with one normalized vector $|\phi_0\rangle$ and $(N-1)/2$ unnormalized vectors $y_m|\phi_m\rangle$ of length $\mathbf{l} \leq 1$. The only ingredient missing in the picture is the $(N-1)/2$ real phases k_m. Similarly, in the case of an even number of qubits, one can attribute a geometrical picture to the state in Eq. (14), with normalized vector $|\phi_0\rangle$, the unnormalized vector $y_1|\phi_0^\perp\rangle$, and $N/2-1$ unnormalized vectors $y_m|\phi_m\rangle$. According to Eqs. (15) and (16) the vectors $\{y_m|\phi_m\rangle\}$ of the initial state [Eq. (14) or Eq. (12)] represented on the Bloch ball simply undergo a uniform rotation under the action of LU operators. In other words the suggested geometric representation of the decomposition rotates as a rigid body.

These observations lead naturally to two additional criteria offered by our representation:

(a) If two symmetric states are convertible among each other via LU rotations, their representations on the Bloch ball are identical up to global rotations of the ball.

(b) An overcomplete set of LU invariants (independent from the LU invariants offered by canonical forms) is formed by the complex numbers $e^{ik_m - ik_n}\langle\Phi_n|\Phi_m\rangle$, the real positive numbers $\{y_m\}$, and the normalization factor A.

Finally, it is important to note that the decomposition given by Eqs. (11) and (13) provides straightforwardly the Schmidt measure [12] for every generic symmetric state. Indeed, if we note r the number of nonzero y_m coefficients, then the Schmidt measure is given by $P = \log_2(r)$. So, as a by-product our decomposition provides for free a method to quantify and classify entanglement according to this widely used measure.

We now illustrate the different aspects of our representation by discussing in detail the three-qubit case.

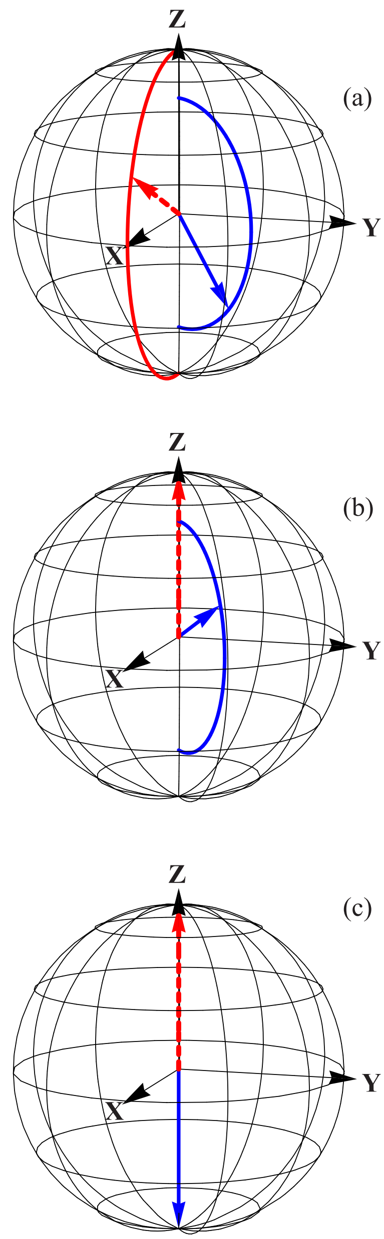

FIG. 1. (Color online) The geometric representation (a) for a general symmetric state of three qubits, Eq. (17). The normalized vector $|\phi_0\rangle$ is represented by the (red) dotted vector while the unnormalized vector $y|\phi_1\rangle$ is represented by a (blue) solid vector. (b) The LU-canonic form of a state as in Eq. (19). (c) The geometric representation of a GHZ state for any number of qubits.

According to Eq. (12) a generic state of three qubits can be written as

$$|\Psi\rangle = A(|\Phi_0\rangle + ye^{ik}|\Phi_1\rangle), \tag{17}$$

where A is the normalization factor. The obtained form, Eq. (17), corresponds to the previously derived three-qubit extension of Schmidt decomposition [19,26].

Using our results, the three-qubit symmetric state is represented by the two vectors $|\phi_0\rangle$ and $y|\phi_1\rangle$ in the Bloch ball as in Fig. 1(a). For the state of maximum tripartite entanglement, i.e., the Greenberger-Horne-Zeilinger (GHZ) state $(|\mathbf{0}\rangle + |\mathbf{1}\rangle)/\sqrt{2}$, we have $\langle\phi_0|\phi_1\rangle = 0$ and $y = 1$. Therefore, a GHZ state is geometrically represented by two normalized and orthogonal vectors [see Fig. 1(c)]. It is easy to check that this geometric representation (by two orthogonal vectors) holds true for all GHZ states independently of the number (N) of qubits.

Now let us investigate how the three-tangle τ [27], a widely applied measure of entanglement, is related to the LU invariant characteristics of the representation: y, $e^{-ik}\langle\Phi_1|\Phi_0\rangle$, and A. For state $|\Psi\rangle$ in Eq. (17) we have

$$\tau = 4y^2(1 - |\langle\Phi_0|\Phi_1\rangle|^{2/3})^{3/2}A^4. \tag{18}$$

Furthermore, we may compare three-tangle with the invariant set of parameters deduced by the -canonical form of the given state $|\Psi\rangle$, i.e.,

$$|\Psi^{\text{LU}}\rangle = A(|\mathbf{0}\rangle + y|\mathbf{X}\rangle), \tag{19}$$

where $|\mathbf{X}\rangle = |\chi\rangle|\chi\rangle|\chi\rangle$ with $|\chi\rangle = [\cos(\varepsilon/2)|0\rangle + e^{i\varphi}\sin(\varepsilon/2)|1\rangle]$ and $A = 1/\sqrt{[1 + y^2 + 2y\cos^3(\varepsilon/2)]}$. According to Eq. (7) the complete set of LU invariants is formed by the three real numbers $\{y,\varepsilon,\varphi\}$ with $0 < y \leqslant 1$, $0 < \varepsilon \leqslant \pi$, and $0 < \varphi \leqslant 2\pi$. In addition, for this three-qubit case, the geometric representation for $|\Psi_{\text{LU}}\rangle$ in Eq. (19) is faithful and one may visualize the set of invariants on the Bloch Ball by the length and position of the vector $y|\chi\rangle$ [see Fig. 1(b)]. The three-tangle depends only on two of them, $\{y,\varepsilon\}$ via the simple and intuitive relation

$$\tau = 4y^2\sin^3(\varepsilon/2)A^4. \tag{20}$$

The three-tangle is therefore an increasing function of y and ε and the geometric representation, Fig. 1(b), permits us to compare the amount of tripartite entanglement among different states.

We have derived a complete solution to the problem of entanglement classification for the vast majority of symmetric states composed by an arbitrary number of qubits. The results have been obtained via a representation for generic symmetric states of qubits which can be considered as a generalized Schmidt decomposition. We believe that the suggested decomposition is a general tool with potential applications in other fields of quantum physics, including quantum optics with collective spin states.

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