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Hamiltonian formulation of the standard $\mathcal{PT}$-symmetric nonlinear Schrödinger dimer

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The standard $\mathcal{PT}$-symmetric dimer is a linearly coupled two-site discrete nonlinear Schrödinger equation with one site losing and the other one gaining energy at the same rate. We show that despite gain and loss, the standard $\mathcal{PT}$ dimer is a Hamiltonian system. We also produce a Lagrangian formulation for the dimer.

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I. INTRODUCTION

The Schrödinger dimer is a two-site discrete nonlinear Schrödinger equation of the form

$$i\frac{du}{dz} + v + F(u,u^*,v,v^*) = 0,$$
$$i\frac{dv}{dz} + u + G(u,u^*,v,v^*) = 0.$$

The physically interesting situation pertains to the functions F and G being covariant under the simultaneous phase shifts in u and v; that is,

$$F(e^{i\alpha}u,e^{-i\alpha}u^*,e^{i\alpha}v,e^{-i\alpha}v^*) = e^{i\alpha}F(u,u^*,v,v^*),$$
$$G(e^{i\alpha}u,e^{-i\alpha}u^*,e^{i\alpha}v,e^{-i\alpha}v^*) = e^{i\alpha}G(u,u^*,v,v^*)$$

for any real α.

The Schrödinger dimers with various polynomial nonlinearities are workhorses of photonics, where they serve to model stationary light beams in coupled optical waveguides [1,2]. The variables u and v represent the corresponding complex beam amplitudes, and z measures the distance along the parallel cores.

Dimers also occur in the studies of the Bose-Einstein condensate trapped in a double-well potential [3,4]. Here, u and v are the complex amplitudes of the mean-field condensate wave functions localized in the left and right well, respectively [3], or the amplitudes of the ground and the first excited state [4]. The nonlinear Schrödinger dimers were employed, extensively, in the solid state physics—where they give the simplest discrete self-trapping equations [5–7]—and in the context of electric lattices [7].

With the advent of the parity-time ($\mathcal{PT}$) symmetry [8], the studies of the optical couplers expanded to include structures consisting of a waveguide with loss and a guide with an equal amount of optical gain. While a variety of cubic nonlinearities were considered, the most frequently used nonlinear model has the form [2,9–13]

$$i\frac{du}{dz} + v + |u|^2u = i\gamma u,$$
$$i\frac{dv}{dz} + u + |v|^2v = -i\gamma v. \tag{1}$$

This discrete nonlinear Schrödinger equation is occasionally referred to as the *standard* $\mathcal{PT}$-symmetric dimer. Here, the quantities $P_1 = |u|^2$ and $P_2 = |v|^2$ give the powers carried by the active and lossy channel, respectively, and $\gamma > 0$ is the corresponding gain-loss rate. The model also has an interpretation in the matter-wave context where it emulates a $\mathcal{PT}$-symmetric arrangement of two boson-condensate traps with gain and loss of particles [14]. There is substantial mathematics literature on the standard $\mathcal{PT}$-symmetric dimer, concerning stationary points [15], periodic orbits [12], conserved quantities [11], the blow-up phenomena [16,17], and the geometry of the phase space [17,18].

Equations with parity-time symmetry combine properties of dissipative and conservative systems. A surprising recent result is that despite the presence of gain and loss of energy, a $\mathcal{PT}$-symmetric system may admit the Hamiltonian formulation [19,20]. In particular, there are Hamiltonian $\mathcal{PT}$-symmetric dimers; an example was produced in [20]:

$$i\frac{du}{dz} + v + (|u|^2 + 2|v|^2)u + v^2u^* = i\gamma u,$$
$$i\frac{dv}{dz} + u + (|v|^2 + 2|u|^2)v + u^2v^* = -i\gamma v. \tag{2}$$

The system (2) can be written in the cross-gradient form

$$i\frac{du}{dz} = \frac{\partial\mathcal{H}}{\partial v^*}, \quad i\frac{dv}{dz} = \frac{\partial\mathcal{H}}{\partial u^*}, \tag{3}$$

with the Hamilton function

$$\mathcal{H} = -(|u|^2 + |v|^2)(1 + u^*v + uv^*) + i\gamma(uv^* - u^*v). \tag{4}$$

This observation leads one to wonder about the Hamiltonian structure of the most important two-site nonlinear Schrödinger equation, namely, the standard dimer (1). The standard dimer is known to have two functionally independent integrals of motion [11,16–18]—yet no explicit Hamiltonian formulation has been put forward so far.

The Hamiltonian structure is a fundamental property of a dynamical system. Physically, it establishes regularity of motion (conservation of phase volume and in some cases, compactness of the accessible part of the phase space) and paves the way for quantization. Mathematically, the Hamiltonian structure implies a deep symmetry of the system which leads to considerable analytical simplifications and allows one to use powerful methods, e.g., the Hamilton-Jacobi approach and the Liouville integrability.

The aim of this Brief Report is to establish the Hamiltonian formulation for the standard dimer. We also provide the Lagrangian framework for this system.

II. THE HAMILTONIAN FORMALISM

The canonical formulation for the standard lossless dimer is well known (see, e.g., [5,21,22]). The system (1) with $\gamma = 0$ can be written as

$$i\frac{du}{dz} = \frac{\partial \mathfrak{H}}{\partial u^*}, \quad i\frac{dv}{dz} = \frac{\partial \mathfrak{H}}{\partial v^*}, \tag{5}$$

where the Hamilton function is

$$\mathfrak{H} = -(uv^* + u^*v) - \frac{|u|^4 + |v|^4}{2}. \tag{6}$$

The two pairs of canonical variables are u and u^*, and v and v^*. We emphasize the difference between Eq. (5) and the Hamiltonian structure of the nonlinearly coupled dimer, Eq. (3), where the canonical pairs were u and v^*, and v and u^*.

Unlike Eq. (3), the structure (5) does not survive the addition of the gain-loss terms—that is, u and u^*, and v and v^* are *not* the canonical variables for the system (1) with $\gamma \neq 0$. That being the case, we attempt to transform both dimers to a new set of coordinates where the similarity of the two systems would allow one to model the Hamiltonian structure of the standard $\mathcal{PT}$-symmetric dimer on the canonical formulation of the system (2).

A natural choice of coordinates is furnished by the Stokes variables,

$$X = \frac{u^*v + uv^*}{2}, \quad Y = i\frac{u^*v - uv^*}{2}, \quad Z = \frac{|u|^2 - |v|^2}{2}. \tag{7}$$

(These variables have proved useful in the whole range of bimodal contexts; see a remark in the concluding section.) We also introduce the notation

$$\phi = \phi_1 + \phi_2, \tag{8}$$

where ϕ_1 and ϕ_2 are the arguments of the complex variables u and v:

$$u = |u|e^{i\phi_1}, \qquad v = |v|e^{i\phi_2}.$$

In terms of X, Y, Z, and ϕ, the standard dimer (1) acquires the form

$$\dot{\phi} = \left(r + \frac{X}{r}\right)\cosh\psi, \tag{9a}$$

$$\dot{X} = -YZ, \quad \dot{Y} = (X-1)Z, \quad \dot{Z} = \gamma\mathcal{R} + Y. \tag{9b}$$

In Eq. (9a), r is the magnitude of the two-component vector (X,Y):

$$r = \sqrt{X^2 + Y^2}, \tag{10}$$

whereas in (9b), $\mathcal{R}$ denotes the length of the vector (X,Y,Z):

$$\mathcal{R} = \sqrt{X^2 + Y^2 + Z^2}.$$

The hyperbolic angle ψ is defined by

$$\mathcal{R} = r\cosh\psi, \quad Z = r\sinh\psi, \tag{11}$$

and the overdot stands for the derivative with respect to $t = 2z$.

Using the same set of variables, the anharmonically coupled dimer (2) becomes

$$\dot{\phi} = \left[2r + (1+2X)\frac{X}{r}\right]\cosh\psi, \tag{12a}$$

$$\dot{X} = 0, \qquad \dot{Y} = -(1+2X)Z, \tag{12b}$$

$$\dot{Z} = \gamma\mathcal{R} + (1+2X)Y. \tag{12c}$$

Consider, first, the system (2) which admits the Lagrangian and Hamiltonian formulation. The Lagrange function for the dimer (2) is

$$\mathcal{L} = \frac{i}{4}(u_zv^* - u_z^*v + v_zu^* - v_z^*u) + (1 + uv^* + u^*v)\frac{|u|^2 + |v|^2}{2} + i\frac{\gamma}{2}(u^*v - uv^*).$$

The corresponding Lagrangian for the system in the form (12) results by transforming to X, Y, Z, and ϕ. Dropping a total derivative, we have

$$\mathcal{L} = \dot{X}\phi + \dot{Y}\psi + (1+2X)\mathcal{R} + \gamma Y. \tag{13}$$

We choose X and Y as the coordinates of the fictitious classical particle described by the Lagrangian (13). The canonical momenta are then

$$P_X = \frac{\partial\mathcal{L}}{\partial\dot{X}} = \phi, \quad P_Y = \frac{\partial\mathcal{L}}{\partial\dot{Y}} = \psi,$$

and the Hamiltonian of the particle results by the Legendre transform:

$$\mathcal{H} = -(1+2X)\mathcal{R} - \gamma Y, \tag{14}$$

where

$$\mathcal{R} = \sqrt{X^2 + Y^2}\cosh\psi.$$

The Hamilton equations

$$\dot{X} = \frac{\partial\mathcal{H}}{\partial P_X}, \quad \dot{P}_X = -\frac{\partial\mathcal{H}}{\partial X};$$
$$\dot{Y} = \frac{\partial\mathcal{H}}{\partial P_Y}, \quad \dot{P}_Y = -\frac{\partial\mathcal{H}}{\partial Y}$$

reproduce Eqs. (12).

The Hamiltonian (14) is remindful of the expression for an integral of motion of the standard dimer, $I = -\mathcal{R} - \gamma\theta$, with the role of θ being taken over by Y. Here θ is one of the two polar coordinates on the (X,Y) plane defined by

$$X = 1 + \rho\sin\theta, \quad Y = \rho\cos\theta.$$

(This choice of polar coordinates is crucial for the elucidation of the geometry of the phase space of the standard dimer [17].) Modeling on the Hamiltonian (14) and noting that

$$\mathcal{R} = r\cosh\psi, \quad r = \sqrt{\rho^2 + 1 + 2\rho\sin\theta}, \tag{15}$$

we define

$$P_\theta = \psi \tag{16}$$

and propose the following expression for the Hamiltonian of the standard dimer:

$$H = -\sqrt{\rho^2 + 1 + 2\rho\sin\theta}\cosh P_\theta - \gamma\theta. \tag{17}$$

The Hamiltonian (17) describes another fictitious classical particle—the "*standard*" particle—with the coordinates ρ and θ. The canonical equations are

$$\dot{\theta} = \frac{\partial H}{\partial P_\theta} = -r \sinh\psi, \quad \dot{\rho} = \frac{\partial H}{\partial P_\rho} = 0, \tag{18a}$$

$$\dot{P}_\theta = -\frac{\partial H}{\partial \theta} = \gamma + \frac{\rho \cos\theta}{r} \cosh\psi, \tag{18b}$$

$$\dot{P}_\rho = -\frac{\partial H}{\partial \rho} = \frac{\rho + \sin\theta}{r} \cosh\psi. \tag{18c}$$

The formulas (18a) and (18b) are equivalent to Eqs. (9b) while Eq. (18c) can be used to define the momentum P_ρ. Namely, comparing (18c) to (9a) and using $\dot{\rho} = 0$ yields

$$P_\rho = -\frac{\phi}{\rho} + \frac{2}{\rho}\int_0^t \mathcal{R}(\tau)d\tau. \tag{19}$$

III. THE LAGRANGIAN FORMALISM

The aim of this section is to propose a Lagrangian formulation for the standard $\mathcal{PT}$-symmetric dimer. The Lagrangian formulation complements the Hamilton equations and offers a number of advantages, e.g., the freedom in the coordinate transformations and access to Noether's theorem.

Letting

$$\phi_1 - \phi_2 = \chi,$$

the equations of the standard dimer (1) acquire the form

$$\begin{aligned} \dot{r} &= -r \sin\chi \sinh\psi, \\ \dot{\chi} &= (r - \cos\chi)\sinh\psi, \\ \dot{\phi} &= (r + \cos\chi)\cosh\psi, \\ \dot{\psi} &= \gamma + \sin\chi \cosh\psi. \end{aligned} \tag{20}$$

Here ϕ, r, and ψ are defined by (8), (10), and (11); we remind the reader that r admits a simple expression in polar coordinates, Eq. (15).

To cast the system (20) in the form of the Lagrange-Euler equations for some functional $S = \int L dt$, we start with introducing a new variable μ such that $\dot{\mu} = \mathcal{R}$. The constraint $\dot{\mu} - \mathcal{R} = 0$ can be incorporated in the system by means of a Lagrange multiplier; call it λ. Thus we consider the Lagrangian

$$L = \frac{2\mu - \phi}{\rho}\dot{\rho} + \psi\dot{\theta} + \gamma\theta - \lambda(\dot{\mu} - \mathcal{R}). \tag{21}$$

Here ρ, θ, ϕ, and ψ—as well as μ and λ—are regarded as independent variables, whereas $\mathcal{R}$ is a function of ρ, θ, and ψ given by Eq. (15).

A slightly modified version of (21) is arrived at by dropping a total derivative:

$$\tilde{L} = -(2\dot{\mu} - \dot{\phi})\ln\rho + \psi\dot{\theta} + \gamma\theta - \lambda(\dot{\mu} - \mathcal{R}). \tag{22}$$

The formulation (22) makes it obvious that the variable ϕ is cyclic; this is a consequence of the U(1) phase invariance of the dimer (1). Therefore, $\partial\tilde{L}/\partial\dot{\phi} = \ln\rho$ is a conserved quantity:

$$\dot{\rho} = 0. \tag{23a}$$

The coordinate μ is also cyclic; hence

$$\frac{\partial\tilde{L}}{\partial\dot{\mu}} = -2\ln\rho - \lambda$$

is another integral of motion—and so is λ.

The variation with respect to the remaining four independent coordinates gives

$$\dot{\theta} = -\lambda r \sinh\psi, \tag{23b}$$

$$\dot{\phi} = 2\dot{\mu} - \lambda\frac{\rho}{r}(\rho + \sin\theta)\cosh\psi, \tag{23c}$$

$$\dot{\psi} = \gamma + \lambda\frac{\rho}{r}\cos\theta \cosh\psi, \tag{23d}$$

and $\dot{\mu} = \mathcal{R}$. The constant λ may be chosen arbitrarily; different choices of λ are equivalent up to a rescaling of t and redefinition of γ. Choosing $\lambda = 1$, one can readily verify that four equations (23) (with $\mathcal{R}$ substituted for $\dot{\mu}$) are equivalent to the system (20).

IV. CONCLUDING REMARKS

In this Brief Report, we have revealed the Hamiltonian structure of the standard $\mathcal{PT}$-symmetric dimer, Eq. (1). The Hamilton function is given by (17); the canonical coordinates are ρ and θ, with the canonical momenta defined by (19) and (16), respectively.

We have also proposed the Lagrangian formulation for the standard dimer. The Lagrange function is in (21) and (22). Unlike its Hamiltonian formulation, the Lagrangian description requires the introduction of an auxiliary degree of freedom [accounted for by the variables μ and λ in (21)].

We conclude with two remarks. First, we would like to acknowledge the importance of the Stokes coordinates (7) that were crucial for our construction. Mathematically, the transformation (7) is an example of the Hopf fibration mapping a three-sphere (a hypersphere in the four-dimensional space formed by the real and imaginary components of u and v) onto the two-sphere in the (X,Y,Z) space [23]. In physics, the Hopf map was used to establish the equivalence of two field-theoretic models on the plane: the $\mathbf{CP}^1$ model and the $O(3)$ σ model [23,24]. The same transformation is employed in the studies of quantum two-level systems where it was pioneered by Feynman and coauthors [25]. (Accordingly, the X, Y, and Z are occasionally referred to as the Feynman variables [21].) A closely related object is the Bose-Hubbard dimer; in that context, the X,Y,Z triplet is known as the Bloch vector [22]. The name *Stokes variables* hails from optics where the X, Y, and Z parameters are used to describe the polarization state of electromagnetic radiation. Jensen exploited the Stokes parameters for the analysis of his two-waveguide optical coupler [1].

Our second remark is on the integrability of the standard $\mathcal{PT}$-symmetric dimer. The fact that a system with two degrees of freedom has two integrals of motion is generally insufficient to claim that the system is Liouville integrable. Indeed, assigning particular values to the two integrals reduces the motion to a two-dimensional manifold, e.g., a genus-two Riemann surface, where the flow may happen not to be integrable. However, if the system is known to be Hamiltonian, the existence of the second integral of motion (which is obviously in involution with the Hamilton function) implies the complete integrability of the system. Thus, uncovering the Hamiltonian structure of the standard $\mathcal{PT}$-symmetric dimer completes the proof of its integrability that was suggested when two conserved quantities were found [11].

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