Parametric control in coupled fermionic oscillators

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A simple model of parametric coupling between two fermionic oscillators is considered. Statistical properties, in particular the mean and variance of quanta for a single mode, are described by means of a time-dependent reduced density operator for the system and the associated P function. The density operator for fermionic fields as introduced by Cahill and Glauber [K. E. Cahill and R. J. Glauber, Phys. Rev. A **59**, 1538 (1999)] thus can be shown to provide a quantum mechanical description of the fields closely resembling their bosonic counterpart. In doing so, special emphasis is given to population trapping, and quantum control over the states of the system.

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I. INTRODUCTION

Two modes of harmonic oscillators, which are parametrically coupled, form the basic paradigm for the treatment of parametric amplification over the decades [1–3]. The model and several of its variants have been extensively used to describe several physical situations in quantum optics and laser physics, such as the coherent Raman effect, Brillouin scattering, frequency splitting of light beams in nonlinear media, low-noise amplifier in the radio-frequency region, and so on [4–10]. We begin with an inquiry on the fermionic counterpart of the model, i.e., a fermionic oscillator coupled parametrically to another fermionic oscillator. However, since fermions obey the Pauli exclusion principle, the fermionic oscillator cannot accommodate an infinite number of levels [11]. It therefore follows that if two interacting modes of harmonic oscillators are replaced by fermionic oscillators, the thermal behavior will differ quite significantly due to severe restriction on the possibility for thermal excitation in the latter. These differences leading to a distinctive behavior of the amplification with respect to the traditional parametric amplifier have been elucidated by several authors in a completely different context in condensed-matter physics, particularly in connection with quantum coherence in the presence of dissipation [12–15]. Enormous progress has been achieved in quantum mesoscopic physics [16,17] in the last 15 years, and the close correspondence between fermionic modes and qubit systems [18] has resulted in significant progress in quantum amplification and the control of quantum coherent media [19,20]. This advancement in the field of digital and analog devices [16–20] has been the motivation for the study of parametrically coupled fermionic oscillators as undertaken in this present work.

The method we use as a basis of our analysis is the density operator expansion in terms of coherent states with quasiprobability functions as weight factors. The scheme is well adopted for the treatments of bosonic fields [10,21–23]; an extension of the scheme to fermionic fields, however, is not straightforward. The reason, as noted by Schwinger in the early 1950s [24], is the anticommuting properties of the fermionic operators for which the eigenvalues are anticommuting Grassmann numbers. In order to overcome this difficulty, Cahill and Glauber [25] have used Grassmann variables [26] to show that in spite of substantial mathematical differences, many close parallels can be established between fermionic fields and more familiar bosonic fields. This, in particular, indicates that the density operator and P representation for the boson have interesting fermionic analogs and thus allow us to calculate a broad range of correlation functions using a grand canonical density operator for fermionic fields, which can be measured in experiments involving the counting of fermions [27]. In the last few years, several investigations of fermionic systems have been carried out by adopting Grassmann variables. Among them are the counting of fermions in strongly correlated systems [27], the non-Markovian stochastic Schrödinger equation for open quantum systems [28], the study of decoherence and dissipation in fermionic bath [29], and the characterization of qubit quantum channels [20], to name just a few [30].

The basic development, as outlined above [25], is primarily centered around the equilibrium density operator for fermionic fields. In this paper, we look for the time development of the density operator and its corresponding P representation. The present analysis reveals that the quantum statistical mean and the variance of the number of quanta for a particular mode possess the same structure as those of bosonic fields. As for experimentally relevant quantities, explicit expressions are obtained for the mean and variance for a variety of initial states of the system. Specifically, we have explored the possibility of vacuum amplification, trapping, and control over the quantum states of the system.

The layout of the paper is as follows: In Sec. II, we introduce the model and the dynamical equations of motion for the operators. The basic aspects of Grassmann algebra and the reduced density operator for the fermionic field are reviewed in Sec. III in order to make the presentation self-contained. In Sec. IV, we have considered the statistical description of the mean and variance for a single mode of the system, and important physical situations such as vacuum amplification, trapping, and quantum control are discussed in greater detail. Possible applications for experimental realizations of the theoretical scheme are discussed in Sec. V. The paper is concluded in Sec. VI.

II. PARAMETRICALLY COUPLED FERMIONIC OSCILLATORS

A. The model

We begin by introducing the dynamical behavior of a single mode of a fermionic field in terms of a fermionic oscillator.

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This is analogous to the way one introduces a harmonic oscillator to represent a traditional bosonic or electromagnetic field. The Hamiltonian of a fermionic oscillator with frequency ω_0 is given by [11]

$$\hat{H}_F = \hbar\omega_0 \hat{a}_F^\dagger \hat{a}_F, \tag{2.1}$$

where the annihilation ($\hat{a}_F$) and creation ($\hat{a}_F^\dagger$) operators satisfy the anticommutation relations

$$\{\hat{a}_F, \hat{a}_F^\dagger\} = 1, \tag{2.2}$$

$$\{\hat{a}_F, \hat{a}_F\} = 0, \ \ \{\hat{a}_F^\dagger, \hat{a}_F^\dagger\} = 0, \tag{2.3}$$

instead of commutation relations obeyed by bosonic operators. These anticommutation relations [Eqs. (2.2) and (2.3)] have the immediate consequence that fermions obey the Pauli exclusion principle as well as Fermi-Dirac statistics. We may define $\hat{a}_F^\dagger \hat{a}_F$ as the number operator $\hat{N}_F$ which satisfies the eigenvalue equation $\hat{N}_F|n\rangle = n|n\rangle$, with eigenvalues $n = 0$ and 1. Thus, in contrast to the harmonic oscillator, the Hilbert space of the fermionic oscillator is two dimensional. The state with no quantum is denoted by $|0\rangle$ and fulfills $\hat{N}_F|0\rangle = 0$, while the state with one quantum, which is expressed as $|1\rangle$, satisfies $\hat{N}_F|1\rangle = |1\rangle$. Furthermore, we should keep in mind that a state is physical if it remains invariant under 2π rotation about any axis. For fermions, since the odd quantum state ($|1\rangle$) changes by a phase factor of -1 under 2π rotations, only the vacuum or the zero quantum state fulfills the invariance criterion.

Now we assume two such fermionic oscillators (A and B) are coupled by a parameter which oscillates at a frequency ω equal to the sum of the frequencies ω_a and ω_b of the individual modes so that they undergo a closely coupled forced oscillation and we have

$$\omega = \omega_a + \omega_b. \tag{2.4}$$

The uncoupled A and B modes have the dynamical behavior of fermionic oscillators which are described by the annihilation ($\hat{a}$ and $\hat{b}$) and creation ($\hat{a}^\dagger$ and $\hat{b}^\dagger$) operators, respectively. These operators obey the following relations:

$$\{\hat{a}, \hat{a}^\dagger\} = \{\hat{b}, \hat{b}^\dagger\} = 1, \tag{2.5}$$

$$\{\hat{a}, \hat{b}\} = \{\hat{a}, \hat{b}^\dagger\} = 0. \tag{2.6}$$

The Hamiltonian for these two coupled modes may be described by

$$\hat{H} = \hbar\omega_a \hat{a}^\dagger \hat{a} + \hbar\omega_b \hat{b}^\dagger \hat{b} - \hbar\kappa[\hat{a}^\dagger \hat{b}^\dagger e^{-i\omega t} + \hat{a}\hat{b}e^{i\omega t}], \tag{2.7}$$

where κ is the coupling constant. The form of the Hamiltonian describes only the behavior of the two modes which are resonantly coupled, while the nonresonant coupling to other modes is ruled out. The external "pump" field, which oscillates at a frequency equal to the sum of the frequencies of the two modes, has been assumed strong enough to be described in classical terms. Lastly, the model is free from any kind of dissipation and the coupled oscillation it describes therefore continues indefinitely without quenching.

The Hamiltonian [Eq. (2.7)] may be useful in describing quantum synchronization in mesoscopic or nanoscale devices where two coupled qubits are driven by an ac signal with a frequency in resonance with interlevel transitions of the system [20]. Such coupled qubits also can act as a quantum controlling device where one qubit is used to control the state of the other qubit via dynamical coupling [19]. Again, several single and coupled two-qubit systems have been proposed recently as a quantum amplifier to amplify the weak signal at the nanoscale level [16–20]. Further, the close correspondence between the isomorphic Hilbert space associated with m-fermionic modes (Fock space) and the m-qubit space allows us to propose that the two-state fermionic oscillator could be a very promising candidate in understanding the behavior of such new quantum devices [18].

We now examine that the Hamiltonian is invariant under the group of transformation defined by

$$\hat{J}^{-1}(\theta)\hat{H}\hat{J}(\theta) = \hat{H}, \tag{2.8}$$

where the unitary operator $\hat{J}(\theta)$ is given by

$$\hat{J}(\theta) = e^{i\theta[\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}]}. \tag{2.9}$$

The Hamiltonian remains unchanged under the transformation $\hat{a} \to \hat{a}e^{i\theta}$ and $\hat{b} \to \hat{b}e^{-i\theta}$ generated by the relations

$$\hat{J}^{-1}(\theta)\hat{a}\hat{J}(\theta) = \hat{a}e^{i\theta}, \tag{2.10}$$

$$\hat{J}^{-1}(\theta)\hat{b}\hat{J}(\theta) = \hat{b}e^{-i\theta}. \tag{2.11}$$

This implies that $\hat{H}$ commutes with the generator of the group, i.e.,

$$[\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}, \hat{H}] = 0. \tag{2.12}$$

Equation (2.12), in other words, indicates that the generator is a constant of motion, so that we can write

$$\hat{a}^\dagger(t)\hat{a}(t) - \hat{b}^\dagger(t)\hat{b}(t) = \hat{a}^\dagger(0)\hat{a}(0) - \hat{b}^\dagger(0)\hat{b}(0) = \hat{M} \tag{2.13}$$

or

$$\hat{N}_a(t) - \hat{N}_b(t) = \hat{N}_a(0) - \hat{N}_b(0) = \hat{M}. \tag{2.14}$$

Here, we define $\hat{M}$ as a time-independent operator. The above relation specifies a conservation law between the number of quanta present in the A and B modes.

B. Dynamical equations of motion

With the help of Eqs. (2.5) and (2.6), Heisenberg equations of motion for the operators may be written down in the following form:

$$\dot{\hat{a}}(t) = -i\omega_a\hat{a}(t) + i\kappa[1 - 2\hat{a}^\dagger(t)\hat{a}(t)]\hat{b}^\dagger(t)e^{-i\omega t}, \tag{2.15}$$

$$\dot{\hat{b}}^\dagger(t) = i\omega_b\hat{b}^\dagger(t) - i\kappa\hat{a}(t)[1 - 2\hat{b}^\dagger(t)\hat{b}(t)]e^{i\omega t}, \tag{2.16}$$

and the adjoint of Eqs. (2.15) and (2.16). The second term of Eqs. (2.15) and (2.16) contains nonlinear terms which make the equations apparently complicated for an exact solution. To proceed further, we make use of the relation (2.13), which simplifies Eqs. (2.15) and (2.16) into a linear set of equations as follows:

$$\dot{\hat{a}}(t) = -i\omega_a\hat{a}(t) - i\kappa(1 + 2\hat{M})\hat{b}^\dagger(t)e^{-i\omega t}, \tag{2.17}$$

$$\dot{\hat{b}}^\dagger(t) = i\omega_b\hat{b}^\dagger(t) - i\kappa\hat{a}(t)(1 + 2\hat{M})e^{i\omega t}, \tag{2.18}$$

together with their adjoint equations. The exact solution to the coupled Eqs. (2.17) and (2.18) may be obtained in the following form:

$$\hat{a}(t) = \hat{a}(0)\hat{C}_a(t) + \hat{b}^\dagger(0)\hat{S}_a(t), \tag{2.19}$$

$$\hat{b}^\dagger(t) = \hat{b}^\dagger(0)\hat{C}_b^*(t) + \hat{a}(0)\hat{S}_b^*(t). \tag{2.20}$$

Here we have defined the operator functions by the following expressions:

$$\begin{aligned} \hat{C}_a(t) &\equiv e^{-i\omega_a t}\cos\hat{\Omega}t,\\ \hat{S}_a(t) &\equiv -ie^{-i\omega_a t}\sin\hat{\Omega}t,\\ \hat{C}_b^*(t) &\equiv e^{i\omega_b t}\cos\hat{\Omega}t,\\ \hat{S}_b^*(t) &\equiv ie^{i\omega_b t}\sin\hat{\Omega}t, \end{aligned} \tag{2.21}$$

where $\hat{\Omega}$ refers to the frequency operator $\hat{\Omega} = \kappa(1+2\hat{M})$.

To get a better insight into the way the number of quanta in a particular mode changes with time, we consider a second-order rate equation involving the occupation number of the A and B modes. With the help of Eqs. (2.19) and (2.20) and the relation (2.4), the second-order time derivative of $\hat{a}^\dagger(t)\hat{a}(t)$ can be expressed as

$$\begin{aligned} \frac{d^2}{dt^2}\{\hat{a}^\dagger(t)\hat{a}(t)\} &= 2\kappa^2(1+2\hat{M})^2[1-\hat{a}^\dagger(t)\hat{a}(t)-\hat{b}^\dagger(t)\hat{b}(t)]\\ &= 2\kappa^2(1+2\hat{M})^2[1-\hat{N}_a(t)-\hat{N}_b(t)]. \end{aligned} \tag{2.22}$$

We now make use of the conservation law [Eq. (2.13) or Eq. (2.14)] to eliminate $\hat{N}_b(t)$ from Eq. (2.22). The rate equation of the operator $\hat{N}_a(t)$ for the A mode therefore reduces to

$$\frac{d^2}{dt^2}\hat{N}_a(t) = 2\kappa^2(1+2\hat{M})^2[(1+\hat{M})-2\hat{N}_a(t)]. \tag{2.23}$$

The solution of Eq. (2.23) for $\hat{N}_a(t)$ in terms of initial values of $\hat{N}_a(0)$, $\dot{\hat{N}}_a(0)$ takes the form of

$$\hat{N}_a(t) = \frac{\dot{\hat{N}}_a(0)}{2\hat{\Omega}}\sin 2\hat{\Omega}t + \left[\hat{N}_a(0) - \frac{1}{2}(1+\hat{M})\right]\cos 2\hat{\Omega}t + \frac{1}{2}(1+\hat{M}). \tag{2.24}$$

A similar solution with the sign of $\hat{M}$ reversed also holds for the operator $\hat{N}_b(t)$ of the B mode. An explicit use of the solutions to these equations may be applied to find out various time-dependent expectation values or moments of the respective field modes. However, approaches based on density operator formalism offer a more compact way of evaluating such averages, which have been well known for bosonic fields for a long time [2,10,21–23].

Particularly in this connection, in the following section we find it convenient to construct a reduced form of a time-dependent density operator for either one of the two modes. The fermionic density operator may also be expressed as a statistical mixture of pure coherent states of the corresponding mode and a suitable weight factor, P function [25]. We have shown that this representation may be a good source of insight as it describes the quantum states in particular. Although the method and several of its variants have been used extensively for bosonic fields and form the basis for understanding the phase space of electromagnetic fields of parametric amplifiers [2,10,23], a straightforward extension of the scheme to its fermionic counterpart is difficult. The main reason, as pointed out by Schwinger [24], is the anticommuting nature of fermionic field operators for which the eigenvalues must be anticommuting numbers. In what follows, in the next section we first briefly review the relevant parts of the algebra of anticommuting numbers [11,26] for the density operator of fermionic fields as developed by Cahill and Glauber [25] and, in the process, explicit solutions of the reduced form of the density operator and its corresponding P-distribution functions are obtained for a variety of initial states. A major interest of the present work is therefore the determination of the P representation in a dynamical context and exploitation of its usefulness for fermionic modes.

III. DENSITY OPERATOR AND *P* REPRESENTATION FOR THE A MODE

A. Fermionic density operator

In the spirit of quasiprobability functions for bosonic fields, Cahill and Glauber have shown that the P function has its interesting counterpart for fermionic fields and, analogous to bosonic fields, the density operator $\hat{\rho}$ can be expanded in terms of the coherent-state dyadic [25],

$$\hat{\rho} = \int d^2\alpha\, P(\alpha,\alpha^*)|\alpha\rangle\langle-\alpha|. \tag{3.1}$$

The fermionic coherent state $|\alpha\rangle$ also acts as an eigenstate of the annihilation operator $\hat{a}$ similar to its bosonic counterpart [21],

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \tag{3.2}$$

with an eigenvalue α. Since fermionic operators anticommute with each other, their eigenvalues are anticommuting numbers [11,26]. They satisfy very unusual properties; for example, let $\{\alpha_i\}$, $i = 1,2,\ldots,n$, represent a set of generators which obeys anticommuting properties,

$$\alpha_i\alpha_j + \alpha_j\alpha_i \equiv \{\alpha_i,\alpha_j\} = 0 \quad \forall i,j. \tag{3.3}$$

Equation (3.3), in particular, implies that for any given i, $\alpha_i^2 = 0$. This, in other words, implies that for fermionic fields, the vacuum state is the only physically realizable eigenstate of the annihilation operator $\hat{a}$ [Eq. (3.2)] with eigenvalue zero. However, it is possible to define such eigenstate $|\alpha\rangle$ [Eq. (3.2)] in a formal way so that they can be used for the same analytical purposes as are made in the case of bosonic fields. The essential difference between the ordinary variables and Grassmann variables has far-reaching consequences as, for example, integration is identical to differentiation for Grassmann variables [26]. They also anticommute with their fermionic operators,

$$\{\alpha_i,\hat{a}\} = 0, \ \ \{\alpha_i,\hat{a}^\dagger\} = 0. \tag{3.4}$$

The adjoint of the coherent state $|\alpha\rangle$ obeys the relation $\langle\alpha|\hat{a}^\dagger = \langle\alpha|\alpha^*$, where α^* is the complex conjugate of α. We should keep in mind that α and α^* are independent numbers and satisfy

$$\alpha\alpha^* + \alpha^*\alpha \equiv \{\alpha,\alpha^*\} = 0. \tag{3.5}$$

The inner product of coherent states obeys

$$\langle\alpha|\beta\rangle = \exp\left[\alpha^*\beta - \tfrac{1}{2}(\alpha^*\alpha + \beta^*\beta)\right]. \quad (3.6)$$

Although the coherent states lack orthogonality, they do form a complete set of states with the completeness relation,

$$\int d^2\alpha |\alpha\rangle\langle\alpha| = I. \quad (3.7)$$

In Eqs. (3.1) and (3.7), we are typically concerned with integration over pairs of anticommuting variables α and α^*, and for such pairs we confine ourselves to a typical notation,

$$\int d^2\alpha = \int d\alpha^* d\alpha. \quad (3.8)$$

At this point, we want to make an important note. It is worth pointing out that the minus sign in Eq. (3.1) results from our convention that we have chosen $d^2\alpha$ as $d\alpha^* d\alpha$. If we had chosen the differential $d^2\alpha$ as $d\alpha d\alpha^*$, the sign would have been positive [29]. Finally, for a system described by the density operator $\hat{\rho}$, one may also define the characteristic function $\chi(\eta,\eta^*)$ of Grassmann arguments η and η^* as the mean value [25],

$$\chi(\eta,\eta^*) = \mathrm{Tr}[\hat{\rho}\exp(\eta\hat{a}^\dagger - \hat{a}\eta^*)], \quad (3.9)$$

and the Fourier transform of Eq. (3.9) gives the P function.

B. Reduced density operator for the A mode

Now to evaluate statistical averages of time-dependent operators in the Heisenberg picture, for example $\hat{a}(t)$ and $\hat{b}^\dagger(t)$, we must make explicit use of the solutions given by Eqs. (2.19) and (2.20). The Schrödinger picture, on the other hand, offers a more compact way of evaluating such averages as it combines the dynamical part with the statistical part by describing the total system Hamiltonian [Eq. (2.7)] in terms of a time-dependent density operator $\hat{\rho}(t)$. The density operator in the Schrödinger picture is like a state vector and thus becomes a time-dependent quantity. However, the Schrödinger density operator $\hat{\rho}(t)$ is related to the time-independent Heisenberg density operator $\hat{\rho}$ for fermionic fields [Eq. (3.1)], as developed by Cahill and Glauber [25], by the relation

$$\hat{\rho}(t) = \hat{U}(t)\hat{\rho}\hat{U}^{-1}(t), \quad (3.10)$$

where $\hat{U}(t)$ refers to the unitary time translation operator that connects the Heisenberg and Schrödinger pictures of equation of motion of the system. The formal solution of Heisenberg operators $\hat{a}(t)$ and $\hat{b}^\dagger(t)$ [Eqs. (2.19) and (2.20)] is therefore given in terms of their initial conditions as follows:

$$\hat{a}(t) = \hat{U}^{-1}(t)\hat{a}(0)\hat{U}(t) \equiv \hat{U}^{-1}(t)\hat{a}\hat{U}(t), \quad (3.11)$$

$$\hat{b}^\dagger(t) = \hat{U}^{-1}(t)\hat{b}^\dagger(0)\hat{U}(t) \equiv \hat{U}^{-1}(t)\hat{b}^\dagger\hat{U}(t). \quad (3.12)$$

Since the two representations coincide at $t = 0$, we denote the initial values of the operators as

$$\hat{a}(0) \equiv \hat{a} \quad \text{and} \quad \hat{b}^\dagger(0) \equiv \hat{b}^\dagger, \quad (3.13)$$

and from now on we will adhere to this notation for all future purposes. Carrying out the trace over the B mode, the time-dependent reduced density operator for the A mode may be given by

$$\hat{\rho}_A(t) = \mathrm{Tr}_B[\hat{\rho}(t)], \quad (3.14)$$

where $\hat{\rho}(t)$ is the total density operator for the system and Tr_B denotes the trace over the initial states of the B mode. The mean value of an arbitrary operator $(\hat{A})$ for the A mode can be calculated as

$$\begin{aligned}\mathrm{Tr}[\hat{\rho}(t)\hat{A}] &= \mathrm{Tr}[\hat{\rho}\hat{A}(t)] = \mathrm{Tr}_A\mathrm{Tr}_B[\hat{\rho}(t)\hat{A}] \\ &= \mathrm{Tr}_A[\hat{\rho}_A(t)\hat{A}], \end{aligned} \quad (3.15)$$

with the help of the relations (3.10) and (3.11).

In order to evaluate the P function for the A mode, we have to introduce the time-dependent form of the normally ordered characteristic function $\chi_N(\eta,\eta^*,t)$ for the A mode, which is given by

$$\chi_N(\eta,\eta^*,t) = \mathrm{Tr}_A[\hat{\rho}_A(t)\exp(\eta\hat{a}^\dagger)\exp(-\hat{a}\eta^*)] \quad (3.16)$$

$$= \mathrm{Tr}[\hat{\rho}(t)\exp(\eta\hat{a}^\dagger)\exp(-\hat{a}\eta^*)]. \quad (3.17)$$

According to Eq. (3.16), the function $\chi_N(\eta,\eta^*,t)$ is defined in terms of the reduced density operator $\hat{\rho}_A(t)$ by an expansion analogous to the definition of the ordinary characteristic function $\chi(\eta,\eta^*)$ [Eq. (3.9)], with the exponential written in normally ordered form. In deriving Eq. (3.17) from Eq. (3.16), we have used the relation (3.14). It may be worth emphasizing at this point that special care must be taken to the ordering of all fermionic quantities, i.e., both the operators and the anticommuting numbers. Apart from these ordering prescriptions, we can easily verify that Eq. (3.17) looks very similar to their bosonic characteristic function. By further substitution of Eq. (3.10) for $\hat{\rho}(t)$ and making use of the cyclic property of the traces of products, Eq. (3.17) may be rewritten as

$$\chi_N(\eta,\eta^*,t) = \mathrm{Tr}[\hat{\rho}\hat{U}^{-1}(t)\exp(\eta\hat{a}^\dagger)\exp(-\hat{a}\eta^*)\hat{U}(t)]. \quad (3.18)$$

Now, using Eq. (3.11) and its adjoint, we can write

$$\chi_N(\eta,\eta^*,t) = \mathrm{Tr}\{\hat{\rho}\exp[\eta\hat{a}^\dagger(t)]\exp[-\hat{a}(t)\eta^*]\}. \quad (3.19)$$

Equation (3.19) expresses $\chi_N(\eta,\eta^*,t)$ in terms of the initial density operator $\hat{\rho}$ for the joint system of the A and B modes, and the time-dependent operator $\hat{a}(t)$ and its adjoint. From Eq. (3.19), a formal solution of $\chi_N(\eta,\eta^*,t)$ may be calculated by using the solution of Eq. (2.19) and its adjoint.

Now to obtain P representation for the A mode at time t, the reduced density operator $\hat{\rho}_A(t)$ should be expressed in the following form [29]:

$$\hat{\rho}_A(t) = \int d^2\alpha P(\alpha,\alpha^*,t)|\alpha\rangle\langle-\alpha|. \quad (3.20)$$

From Eqs. (3.19) and (3.20), it is evident that $P(\alpha,\alpha^*,t)$ may be calculated as the Fourier transform of the characteristic function $\chi_N(\eta,\eta^*,t)$ as

$$P(\alpha,\alpha^*,t) = \int d^2\eta \exp(\alpha\eta^* - \eta\alpha^*)\chi_N(\eta,\eta^*,t). \quad (3.21)$$

So, the characteristic function $\chi_N(\eta,\eta^*,t)$ obtained from Eq. (3.19), in turn, gives $P(\alpha,\alpha^*,t)$ and thereby the form of the reduced density operator $\hat{\rho}_A(t)$. Equations (3.19)–(3.21) together with Eq. (3.15) form the basis of our analysis in the next section for a variety of initial preparations of the system.

IV. STATISTICAL DESCRIPTION OF THE A MODE

A. Mean and variance; parametric amplification

Now it is straightforward to calculate the expectation value and variance of the number of quanta present in the A mode for a variety of initial states using Eqs. (2.19) and (3.15). We are particularly interested in the situation when both the A and B modes are initially in pure coherent states. Then the initial state of the system may be taken as $|\alpha_0,\beta_0\rangle$, where α_0 and β_0 are the Grassmann amplitudes of the A and B modes, respectively. For such initial state, the mean value of $\hat{a}(t)$ is given by

$$\begin{aligned}\overline{\alpha}(t) &= \mathrm{Tr}\{\hat{\rho}\hat{a}(t)\}\\ &= \langle\alpha_0,\beta_0|\hat{a}\hat{C}_a(t)+\hat{b}^\dagger\hat{S}_a(t)|\alpha_0,\beta_0\rangle\\ &= \alpha_0 C_a(t)+\beta_0^* S_a(t),\end{aligned} \tag{4.1}$$

where $C_a(t)$ and $S_a(t)$ are, respectively, the c-number functions corresponding to the operator functions $\hat{C}_a(t)$ and $\hat{S}_a(t)$ as

$$C_a(t) \equiv e^{-i\omega_a t}\cos\Omega t, \tag{4.2}$$

$$S_a(t) \equiv -ie^{-i\omega_a t}\sin\Omega t, \tag{4.3}$$

with an effective frequency $\Omega=\kappa(1+2M)$. Equation (4.1) for $\overline{\alpha}(t)$ with Grassmann field amplitudes α_0 and β_0^* has the same form as those of bosonic fields with complex mode amplitudes. Similarly, the variance of the quanta for the A mode can be calculated as

$$\begin{aligned}\mathrm{var} &= \langle\alpha_0,\beta_0|[\hat{a}^\dagger(t)-\overline{\alpha}^*(t)][\hat{a}(t)-\overline{\alpha}(t)]|\alpha_0,\beta_0\rangle\\ &= \left|S_a^2(t)\right| = \sin^2\Omega t.\end{aligned} \tag{4.4}$$

It is evident from Eq. (4.4) that the variance of the Grassmann field amplitudes exhibits amplification followed by deamplification for fermionic fields, which is in sharp contrast with exponential enhancement for bosonic or electromagnetic fields with complex field amplitudes [1–3]; see Fig. 1. However, it is clear from Eq. (4.4) and from Fig. 1 that the state which evolves from an initially coherent state does not retain its coherent character for all times.

At this point, let us make a few remarks on fermionic field amplitudes (or Grassmann amplitudes) in the physical context [29]. Since the Grassmann variables obey the anticommutation relation, one may conclude that they do not bear any classical analogy. This may lead to a misunderstanding and requires further clarification. To be classically measurable, a field amplitude has to be strong enough. It is only possible when a large number of particles are accommodated in the same state so that the fields get summed up coherently. Thus, for a field amplitude to be classically measurable, the particles have to obey Bose-Einstein statistics, e.g., light quanta are bosons because strong electromagnetic fields can be produced and measured classically. On the other hand, for fermionic fields obeying Fermi-Dirac statistics, quantities which are only bilinear in field variables $\hat{a}$ and $\hat{a}^\dagger$ can be measured classically. The mean number of quanta in Eq. (4.1) is linear in $\hat{a}$ and $\hat{a}^\dagger$ and hence linear in Grassmann variables, represents the "amplitude" of the fermionic field mode, and

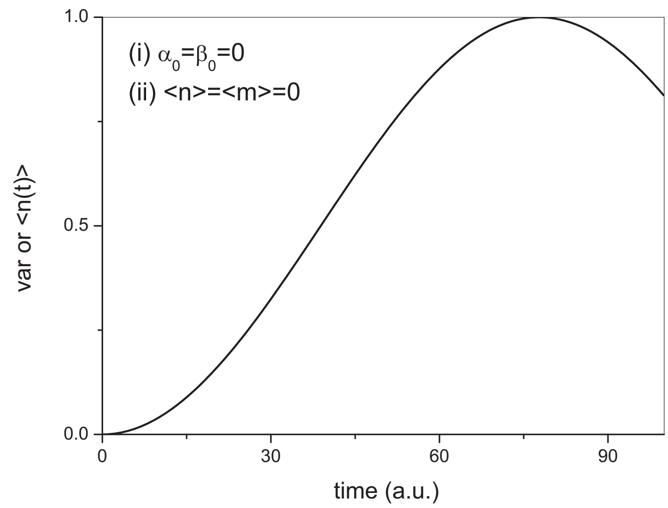

FIG. 1. Variance of the number of the quanta present in the A mode is plotted against time (in arbitrary units). This is identical to the mean number of quanta $\langle n(t)\rangle$ of the A mode for $\alpha_0=\beta_0=0$, which corresponds to the vacuum amplification [case (i)]. It also happens for a chaotic mixture [Eq. (4.23)] with $\langle n\rangle=\langle m\rangle=0$ [case (ii)]. For both of the plots (i) and (ii), we have used the parameter $\Omega=0.01$.

is not an experimentally relevant quantity. The variance of quanta present in the A mode, on the other hand, is bilinear in Grassmann amplitudes, which makes it experimentally measurable.

B. *P* representation of the A mode; trapping and quantum control

1. *Initial coherent state*

In the previous section, we have considered the case in which the joint system of the A and B modes is initially described by the pure coherent state $|\alpha_0,\beta_0\rangle$. Evaluating the variance of the amplitudes of the Grassmann field for the A mode, we have shown that such state does not remain coherent for all times. In this section, we consider initial coherent state $|\alpha_0,\beta_0\rangle$ with greater detail to solve for the P representation in order to obtain a better description for the A mode.

Now the initial density operator for the joint system in this case is given by

$$\hat{\rho} = |\alpha_0,\beta_0\rangle\langle\alpha_0,\beta_0|. \tag{4.5}$$

To evaluate $\chi_N(\eta,\eta^*,t)$ for the A mode, we write the exponentials of the normal ordered form of Eq. (3.19) as follows:

$$\begin{aligned}&\exp[\eta\hat{a}^\dagger(t)]\exp[-\hat{a}(t)\eta^*]\\ &\quad= \exp[\eta\hat{a}^\dagger(t)-\hat{a}(t)\eta^*]\exp(\eta^*\eta/2).\end{aligned} \tag{4.6}$$

Equation (4.6) follows from the well-known Baker-Hausdroff operator identity, which holds whenever $[[\hat{A},\hat{B}],\hat{A}]=[[\hat{A},\hat{B}],\hat{B}]=0$ [2,25]. Equation (3.19), together with Eqs. (4.5) and (4.6), may then be cast into

$$\begin{aligned}\chi_N(\eta,\eta^*,t) = \exp(-\eta\eta^*/2)\langle\alpha_0,\beta_0|\exp\{\eta[\hat{a}^\dagger\hat{C}_a^*(t)+\hat{b}\hat{S}_a^*(t)]\\ -[\hat{a}\hat{C}_a(t)+\hat{b}^\dagger\hat{S}_a(t)]\eta^*\}|\alpha_0,\beta_0\rangle.\end{aligned} \tag{4.7}$$

In deriving Eq. (4.7) with the rules of anticommuting Grassmann numbers [Eq. (3.5)], we have used Eq. (2.19) for $\hat{a}(t)$ and adjoint of Eq. (2.19). From Eq. (4.7), after appropriate rearrangement, we find

$$\chi_N(\eta,\eta^*,t) = \exp\left\{-\frac{\eta\eta^*}{2}[1-|C_a(t)|^2+|S_a(t)|^2]\right\} \times\langle\alpha_0,\beta_0|\exp[\eta\hat{a}^\dagger\hat{C}_a^*(t)-\hat{b}^\dagger\eta^*\hat{S}_a(t)] \times\exp[-\hat{a}\eta*\hat{C}_a(t)+\eta\hat{b}\hat{S}_a^*(t)]|\alpha_0,\beta_0\rangle \quad (4.8)$$

$$= \exp[-\eta\eta^*|S_a(t)|^2+\eta\alpha_0^*C_a^*(t)-\beta_0^*\eta^*S_a(t) -\alpha_0\eta^*C_a(t)+\eta\beta_0S_a^*(t)] \quad (4.9)$$

$$= \exp[-\eta\eta^*|S_a(t)|^2+\eta\overline{\alpha}^*(t)-\eta^*\overline{\alpha}(t)], \quad (4.10)$$

where $\overline{\alpha}(t)$ is given by Eq. (4.1) and $\overline{\alpha}^*(t)$ refers to its complex conjugate.

Substituting Eq. (4.10) for $\chi_N(\eta,\eta^*,t)$ into Eq. (3.21), we find out that $P(\alpha,\alpha^*,t)$ is given by the complex Fourier integral in Grassmann variables,

$$P(\alpha,\alpha^*,t) = \int d^2\eta\exp\{-\eta\eta^*|S_a(t)|^2-\eta[\alpha^*-\overline{\alpha}^*(t)] +[\alpha-\overline{\alpha}(t)]\eta^*\}. \quad (4.11)$$

This integral may be evaluated with the help of Fourier transform of a Gaussian function in Grassmann variables, which is defined by [11,26]

$$\int d^2\eta\exp[\lambda\eta\eta^*+\alpha\eta^*-\eta\alpha^*] = \lambda\exp\left(\frac{\alpha\alpha^*}{\lambda}\right), \quad (4.12)$$

where λ is an arbitrary complex number. Making use of the above identity, we find out from Eq. (4.11) that $P(\alpha,\alpha^*,t)$ takes the form

$$P(\alpha,\alpha^*,t) = -|S_a(t)|^2\exp\left\{-\frac{[\alpha-\overline{\alpha}(t)][\alpha^*-\overline{\alpha}^*(t)]}{|S_a(t)|^2}\right\}. \quad (4.13)$$

The function $P(\alpha,\alpha^*,t)$ for the A mode is thus a Gaussian function in the complex Grassmann plane about the mean value $\overline{\alpha}(t)$ and $\overline{\alpha}^*(t)$. The variance of the distribution is $|S_a(t)|^2 = \sin^2\Omega t$, which was also obtained from the solution of the operator equations of motion [Eq. (4.3)].

The minus sign in front of Eq. (4.13) may appear surprising since their bosonic counterparts are, in general, positive in character [25,29]. The Hermiticity of $\hat{\rho}$ and condition of $\mathrm{Tr}\hat{\rho} = 1$, however, implies that the $P(\alpha,\alpha^*,t)$ satisfy

$$\int d^2\alpha\, P(\alpha,\alpha^*,t) = 1. \quad (4.14)$$

Equation (4.14) implies P has some characteristics of a probability distribution. But the nonorthogonality of the projection operators $|\alpha\rangle\langle\alpha|$ [Eq. (3.6)] makes it impossible to interpret the function $P(\alpha,\alpha^*,t)$ as a probability density. For the bosonic case, however, it becomes a probability density in an asymptotic sense, but no such correspondence can be made for its fermionic counterpart. We can, at best, think of the function P simply as a weight factor in an expansion of this sort. No physical measurement is possible which corresponds to the function P. The physical constraint such as positive definiteness imposed on the density operator may lead to P functions which may freely take negative values even for a perfectly well-behaved density operator [25,29]. However, the fermionic P representation has major advantages over the bosonic P representation because of the conspicuous properties of the Grassman calculus. Since the integration is identical to differentiation for Grassmann algebra, the fermionic P function is not affected by the mathematical limitations that somewhat restrict the use of bosonic P representation [25] to calculate statistical averages of various normally ordered operators.

2. *Initial vacuum state*

Equation (4.13) has been derived for the initial coherent state $|\alpha_0,\beta_0\rangle$. If the initial state is chosen to be $\alpha_0 = \beta_0 = 0$, then it corresponds to the absence of any quanta in the system or the system is free from any initial excitation. Even then, a field is generated due to vacuum fluctuation, and the corresponding $P(\alpha,\alpha^*,t)$ function is obtained by setting $\overline{\alpha}(t) = \overline{\alpha}^*(t) = 0$ as

$$P(\alpha,\alpha^*,t) = -|S_a(t)|^2\exp\left[-\frac{\alpha\alpha^*}{|S_a(t)|^2}\right]. \quad (4.15)$$

This function also describes a chaotic mixture with variance $|S_a(t)|^2$ [Eq. (4.23)]. Since $\overline{\alpha}(t) = \overline{\alpha}^*(t) = 0$, the variance is equal to the mean number of quanta present in the mode, i.e.,

$$\langle n(t)\rangle = |S_a(t)|^2 = \sin^2\Omega t. \quad (4.16)$$

In Fig. 1, we have shown that the variance is identical to the mean number of quanta $\langle n(t)\rangle$ present in the A mode for $\alpha_0 = \beta_0 = 0$ [case (i)]. This happens also for a chaotic mixture [Eq. (4.23)] with $\langle n\rangle = \langle m\rangle = 0$ [case (ii)], as illustrated in Fig. 1.

Following Cahill and Glauber, it may be shown that a density operator with a Gaussian P representation may be written in the n-fermionic states with characteristic thermal distribution. The density operator $\hat{\rho}_A(t)$ may then be given by [25]

$$\hat{\rho}_A(t) = [1-\langle n(t)\rangle]\left[\frac{\langle n(t)\rangle}{1-\langle n(t)\rangle}\right]^{\hat{a}^\dagger\hat{a}}. \quad (4.17)$$

For the case of vacuum fluctuation, $\hat{\rho}_A(t)$ may be formally given by substituting $\langle n(t)\rangle$ with $|S_a(t)|^2$,

$$\hat{\rho}_A(t) = |C_a(t)|^2\left[\frac{|S_a(t)|^2}{|C_a(t)|^2}\right]^{\hat{a}^\dagger\hat{a}} = |C_a(t)|^2\exp\left[\hat{a}^\dagger\hat{a}\ln\left\{\frac{|S_a(t)|^2}{|C_a(t)|^2}\right\}\right]. \quad (4.18)$$

3. *Initial thermal or chaotic state*

The cases in which one of the modes in a chaotically mixed state (e.g., thermal equilibrium distribution) are important from a practical point of view [2]. We start with when both of the modes are of independent chaotic mixtures and in the process examine situations of particular interest that characterize population trapping and coherent control of particular modes.

Now, if the initial states of the A and B modes are independent chaotic mixtures with mean quantum number $\langle n\rangle$ and $\langle m\rangle$, respectively, the initial density operator for the system may be written in the form

$$\hat{\rho}_c = \int d^2\alpha_0 d^2\beta_0 P_c(\alpha_0,\alpha_0^*;\beta_0,\beta_0^*)|\alpha_0,\beta_0\rangle\langle\alpha_0,\beta_0|, \quad (4.19)$$

where

$$P_c(\alpha_0,\alpha_0^*;\beta_0,\beta_0^*) = \langle n\rangle\langle m\rangle \exp\left[-\frac{\alpha_0\alpha_0^*}{\langle n\rangle} - \frac{\beta_0\beta_0^*}{\langle m\rangle}\right]. \quad (4.20)$$

The function $P_c(\alpha,\alpha^*,t)$ may then be obtained by taking an average of Eq. (4.9) over the weight function $P_c(\alpha_0,\alpha_0^*;\beta_0,\beta_0^*)$. We then have

$$\chi_{N,c}(\eta,\eta^*,t) = \int d^2\alpha_0 d^2\beta_0 \times \exp[-\eta\eta^*|S_a(t)|^2 + \eta\alpha_0^* C_a^*(t) - \beta_0^*\eta^* S_a(t) - \alpha_0\eta^* C_a(t) + \eta\beta_0 S_a^*(t)] \times P_c(\alpha_0,\alpha_0^*;\beta_0,\beta_0^*). \quad (4.21)$$

By substituting the Gaussian form of $P_c(\alpha_0,\alpha_0^*;\beta_0,\beta_0^*)$ given by Eq. (4.20) into this expression and performing the integration with the help of identity relation (4.12), we obtain

$$\chi_{N,c}(\eta,\eta^*,t) = e^{-\eta\eta^* N(t)}, \quad (4.22)$$

where

$$N(t) = \langle n\rangle|C_a(t)|^2 + (1-\langle m\rangle)\,|S_a(t)|^2. \quad (4.23)$$

The $P(\alpha,\alpha^*,t)$ for the A mode is then evaluated as the Fourier transform of $\chi_{N,c}(\eta,\eta^*,t)$ as

$$P_c(\alpha,\alpha^*,t) = \int d^2\eta e^{-\eta\eta^* N(t)+\alpha\eta^*-\eta\alpha^*} = -N(t)\exp\left[-\frac{\alpha\alpha^*}{N(t)}\right]. \quad (4.24)$$

The reduced operator $\hat{\rho}_{A,c}(t)$ for the A mode thus corresponds to a chaotic mixture with mean number $N(t)$. For $\langle n\rangle = \langle m\rangle = 0$, the joint system is initially in the vacuum state, for which $N(t) = |S_a(t)|^2$. Equation (4.24) then becomes identical to Eq. (4.15), as obtained earlier for vacuum amplification. The effect of chaotic fields initially present in both the A and B modes thus modifies the field strength of the A mode at time t, from $|S_a(t)|^2$ to $N(t)$. Equation (4.23) bears the characteristic of fermionic nature of the model which carries important consequences. In Fig. 2(a), we have plotted $N(t)$ for nonzero values of $\langle n\rangle$ and $\langle m\rangle$. From Fig. 2(a), it is obvious that one can always prepare the initial states of the coupled modes so that population in a given mode remains constant. The situation is reminiscent of population trapping in quantum multilevel systems [31]. In Fig. 2(b), for three sets of parameter values, we have plotted the constant population for the A mode to stay in its ground, excited, or superposed state. Thus the present scheme may be applied as an efficient technique to control the states of coupled qubit systems to store quantum information.

Finally, the choice of $\langle n\rangle = 0$ implies that the initial state of the A mode is the vacuum state $|0\rangle_A$. The initial density operator for the joint system, in this case, may be described by

$$\hat{\rho} = |0\rangle_{A\,A}\langle 0|\,\hat{\rho}_{B,\langle m\rangle}, \quad (4.25)$$

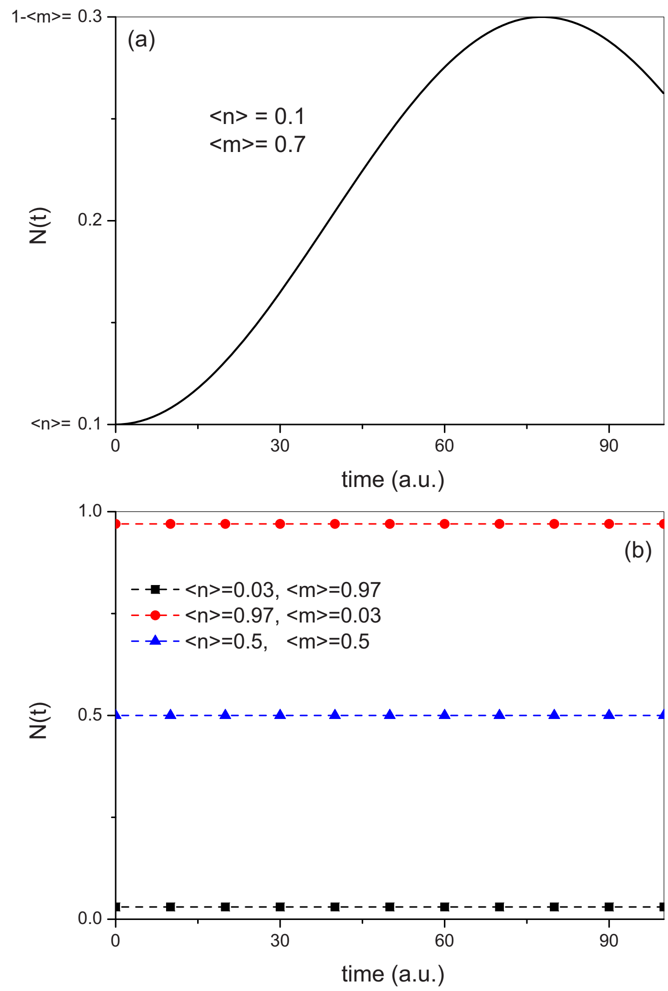

FIG. 2. (Color online) Variation of the number of quanta $N(t)$ in the A mode with time (a.u.) is plotted when (a) the initial states of the A and B modes are independent chaotic mixtures with nonzero mean quantum number $\langle n\rangle$ and $\langle m\rangle$, respectively, and (b) illustrates the population trapping in the ground (black), excited (red), and superposed state (blue) for three sets of parameters $\langle n\rangle = 0.03$, $\langle m\rangle = 0.97$; $\langle n\rangle = 0.97$, $\langle m\rangle = 0.03$; and $\langle n\rangle = 0.5$, $\langle m\rangle = 0.5$. For both (a) and (b), we have used $\Omega = 0.01$.

where the density operator for the chaotic B mode is

$$\hat{\rho}_{B,\langle m\rangle} = \langle m\rangle \int d^2\beta_0 \exp\left[-\frac{\beta_0\beta_0^*}{\langle m\rangle}\right]|\beta_0\rangle\langle\beta_0|. \quad (4.26)$$

The function $P(\alpha,\alpha^*,t)$ which corresponds to the initial state [Eq. (4.25)] may be obtained by substituting $\langle n\rangle = 0$ in Eqs. (4.23) and (4.24), and is given by the form

$$P(\alpha,\alpha^*,t) = -(1-\langle m\rangle)\,|S_a(t)|^2 \exp\left[-\frac{\alpha\alpha^*}{(1-\langle m\rangle)\,|S_a(t)|^2}\right]. \quad (4.27)$$

The variance in this case may be illustrated by the same plot as shown in Fig. 2(a), with the replacement of the mean quantum number $\langle n\rangle = 0$.

V. POSSIBLE APPLICATIONS

We point out that our theoretical findings are well adopted for the treatment of quantum optical experiments with atoms which are fermionic in nature [32–35]. The fermionic atom optics counterpart of parametric down-conversion with photons has been recently reported [36]. The dissociation of $^{40}K_2$ molecules near magnetic Feshbach resonance reveals an interesting twist as the constituent atoms in this case obey fermionic statistics [37]. Based on a fermionic analog of the squeezing Hamiltonian of standard quantum optics, efforts have been made [36] to correlate the spatial correlation measurement performed at JILA [37]. Their observations correspond to our results, i.e., the average mode occupancy of any one of the resonant modes undergoes oscillation characteristic of fermionic statistics [36], which is in complete contrast to exponential enhancement of the boson case [3]. To set up the connection, we may note that one can generate two-mode squeezed fermionic states [38] by applying a two-mode squeezing operator $\hat{S}(\xi) = \exp(\xi\hat{a}^\dagger\hat{b}^\dagger - \xi^*\hat{a}\hat{b})$ with real number $\xi = re^{i\phi}$, on the two-mode vacuum $|0,0\rangle$ as $|\xi\rangle = \hat{S}(\xi)|0,0\rangle$, $\hat{a}|0,0\rangle = \hat{b}|0,0\rangle = 0$ [25]. The squeezing operator is unitary and resembles a unitary evolution operator in quantum mechanics; therefore, $\hat{S}(\xi)$ can be expressed as $\hat{S}(\xi) = \exp(-\frac{i}{\hbar}\hat{H}t)$ [39] where $\hat{H} = i\hbar g(e^{i\phi}\hat{a}^\dagger\hat{b}^\dagger - e^{-i\phi}\hat{a}\hat{b})$, with $gt = r$. Clearly this Hamiltonian can be identified as the Hamiltonian for the three-wave interaction term $\hbar\kappa[\hat{a}^\dagger\hat{b}^\dagger e^{-i\omega t} + \text{H.c.}]$ given by Eq. (2.7). It may be noted that in experiment, the correlation measurements were made by using absorption images after a time-of-flight expansion [40–42]. These detection techniques allow us to calculate full atom counting distributions with spatial resolution and provide an efficient way of detecting strongly correlated systems, both in and out of thermal equilibrium [43]. Another area of application is the mesoscopic systems which have been proposed and implemented as parametric amplifiers [16–20]. A two-qubit coupled parametric amplifier can amplify a weak signal about a hundredfold [20]. It has been achieved when the two qubits are biased simultaneously by a weak signal and a strong pump frequency. Harmonic mixing in two coupled qubits can be used to control one driven qubit by applying an additional ac signal to the other qubit. Such combined coupled qubits can act as a quantum amplifier as well as frequency shifter [19]. Advancement in such quantum qubit systems together with the current investigation on fermionic four-wave mixing [44] and association of fermionic atoms into molecules [45,46] expand the paradigm of *fermionic quantum atom optics* [47]. The present work sets up a general framework in this context from a theoretical point of view.

VI. CONCLUSION

Based on a model of two fermionic oscillators which are parametrically coupled to each other, we have explored vacuum amplification, population trapping, and quantum control over the quantum states of systems. The key elements of our analysis are an expansion of the density operator for fermionic fields in terms of Glauber representation or P function and a reduced form of the time-dependent density operator of a particular mode. Since fermionic operators anticommute, it is necessary to work with anticommuting numbers or Grassmann variables to calculate the statistical properties of the coupled fermionic modes. We have shown that the fermionic density operator and the P representation may be used to calculate the statistical mean and the variance of quanta which obey similar structures as their bosonic counterpart. Although the anticommuting nature of Grassmann variables precludes the possibility of interpreting the fermionic field amplitude in physical terms, the variance or the mean number of quanta for vacuum fluctuation can be interpreted as experimentally relevant quantities. It is interesting to note that vacuum amplification for a fermionic field possesses an interesting bosonic analog. Explicit solutions are obtained for the P representation for a variety of initial states of the modes which facilitates the calculation of the mean number of quanta in that given mode. The initial states considered are coherent states with particular emphasis on vacuum amplification and initial chaotic or thermal states. Thermal initial states provide an interesting insight into the phenomena of quantum control and population trapping for a single mode. Lastly, due to the peculiar properties of the Grassmann algebra, the fermionic P representation has the extra advantage over the bosonic P distribution since it is not affected by the mathematical limitations that sometimes restrict the use of bosonic P function to calculate statistical averages. It is always possible to calculate, therefore, the mean and correlation functions using the fermionic P function by simple means.

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