Domain walls and bubble droplets in immiscible binary Bose gases

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The existence and stability of domain walls (DWs) and bubble-droplet (BD) states in binary mixtures of quasi-one-dimensional ultracold Bose gases with inter- and intraspecies repulsive interactions is considered. Previously, DWs were studied by means of coupled systems of Gross-Pitaevskii equations (GPEs) with cubic terms, which model immiscible binary Bose-Einstein condensates (BECs). We address immiscible BECs with two- and three-body repulsive interactions, as well as binary Tonks–Girardeau (TG) gases, using systems of GPEs with cubic and quintic nonlinearities for the binary BEC, and coupled nonlinear Schrödinger equations with quintic terms for the TG gases. Exact DW solutions are found for the symmetric BEC mixture, with equal intraspecies scattering lengths. Stable asymmetric DWs in the BEC mixtures with dissimilar interactions in the two components, as well as of symmetric and asymmetric DWs in the binary TG gas, are found by means of numerical and approximate analytical methods. In the BEC system, DWs can be easily put in motion by phase imprinting. Combining a DW and anti-DW on a ring, we construct BD states for both the BEC and TG models. These consist of a dark soliton in one component (the "bubble"), and a bright soliton (the "droplet") in the other. In the BEC system, these composite states are mobile, too.

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I. INTRODUCTION

Domain walls (DWs) represent the most fundamental type of robust structures which are supported by immiscibility in a variety of binary physical systems. Commonly known are DWs in magnetics [1], ferroelectrics [2], and liquid crystals [3]. In those media, the immiscible species are emulated by different orientations of a vectorial order parameter. Similarly organized are DWs separating temporal-domain regions occupied by light waves with orthogonal circular polarizations in bimodal optical fibers [4,5]. Binary Bose-Einstein condensates (BEC) formed by immiscible atomic species, or immiscible hyperfine states of the same atom, make it possible to study the formation of DW structures in superfluids. These structures were studied theoretically in a number of different settings [6], including the extension to BEC with linear interconversion between the immiscible components [7], condensates with long-range dipole-dipole interactions [8], three-component spinor BEC [9], and the BEC effectively discretized by trapping in a deep optical-lattice potential [10].

In addition to the condensates, degenerate Bose gases have been also experimentally realized in the Tonks-Girardeau (TG) state [11], using tight quasi-one-dimensional traps [12] (for a review of the TG model see Ref. [13]). In this context, a macroscopic (mean-field-like) description of the TG gas was proposed in terms of the quintic nonlinear Schrödinger equations NLSE [14] and adopted for various settings [14–18]. In this framework, it was demonstrated that oscillation frequencies derived from the fermionic hydrodynamic equations, which emulate the hard-core TG gas, are very close to their counterparts predicted by the quintic NLSE [16,19]. The NLSE approach was employed to investigate particular nonlinear features of TG states, including dark solitons [14,17,20], bright solitons formed by the long-range dipole-dipole interac-

tions [21], etc. Exact solutions for the ground state in mixtures of TG and Fermi gases have been found, too [22].

On the other hand, the mean-field approach does not apply beyond the framework of the hydrodynamic regime—in particular, to problems such as the merger of two gas clouds into one [23]. Recently, however, the ground state of the binary TG mixture trapped in the harmonic-oscillator potential was constructed, using the density-functional theory with the local-density approximation [24,25]. Coupled quintic NLSEs emerge in this case, too (although only for particular parameter settings), where they were used to investigate mainly miscible ground states.

One expects that DWs, both in BEC and in TG gases, can exist only in the case of immiscibility. DWs were indeed observed experimentally in immiscible binary BECs [26]. DWs have been also investigated in other physical systems described by systems of continuous or discrete NLSEs, both conservative and dissipative. In particular, DWs were found in one-dimensional (1D) Heisenberg ferromagnets as front patterns, both at the classical [27] and quantum [28] level. Similar structures were also found in a dissipative discrete NLSE [29] and in coupled Ginzburg-Landau equations modeling convection patterns in two dimensions [30]. To the best of our knowledge, however, DWs of immiscible binary gases with quintic interactions, of both BEC and TG types, have not been investigated yet.

The aim of the present paper is to address the existence and stability of DWs and bubble-droplet (BD) states in binary mixtures of immiscible BEC and TG gases. To this end, we introduce a system of nonlinearly coupled cubic-quintic (CQ) NLSEs, which describes, in proper situations, both BEC mixtures and binary TG gases. As concerns the BEC, the quintic nonlinearities model three-particle collisions in the limit when the related losses may be neglected [31]. In this

case, we derive a class of exact DW solutions, with equal background amplitudes and equal intraspecies and interspecies interactions (symmetric DWs).

The existence and stability of asymmetric DWs in the BEC mixtures with dissimilar background amplitudes and interactions in the two components, as well as of symmetric and asymmetric DWs in the binary TG gas, is demonstrated by means of numerical and approximate analytical methods. It is shown, too, that DWs can be easily put in motion by phase imprinting. We also show that, by combining a DW with an anti-DW on a ring, it is possible to construct, for both BEC and TG gases, BD excitations, consisting of a dark (gray) soliton in one component (the "bubble"), and a bright soliton in the other (the "droplet"). In the BEC system, such composite excitations are found to be mobile as well. Finally, our estimates suggest that, using the Feshbach-resonance technique to control the strengths of the inter- and intraspecies repulsion in a binary Bose gas loaded into a quasi-1D trap, the observation of DWs and BDs should be possible in experiments.

The paper is organized as follows. In Sec. II we introduce the model equations, for which exact symmetric DWs and the immiscibility condition are explicitly derived. Section III is focused on symmetric and asymmetric DWs and BD complexes in the BEC mixture. The existence of asymmetric DWs and stability of DWs and BDs is numerically investigated, and their mobility is demonstrated. In Sec. IV we consider DWs and BDs in binary TG gases for the symmetric case of equal masses and equal interactions, as well as for the asymmetric setting. In Sec. V we summarize the paper and briefly discuss possible experimental settings.

II. MODEL EQUATIONS AND EXACT DW SOLUTIONS

We start with the general system of coupled scaled NLSEs with the CQ nonlinearity and equal atomic masses of both components:

$$
\begin{aligned}
i\frac{\partial \psi_j}{\partial t} = &-\frac{1}{2}\frac{\partial^2 \psi_j}{\partial x^2} + [\gamma_j|\psi_j|^2 + \gamma_{12}|\psi_{3-j}|^2]\psi_j \\
&+ [\alpha_j|\psi_j|^4 + \chi(|\psi_{3-j}|^4 + 2|\psi_j|^2|\psi_{3-j}|^2)]\psi_j, \\
&j = 1,2,
\end{aligned} \tag{1}
$$

where positive real parameters $\gamma_j, \gamma_{12}, \alpha_j, \chi$ represent the repulsive interactions. We will focus on two physically relevant cases: (i) $\chi = 0$, and (ii) the quintic-only interactions, $\gamma_{1,2} = \gamma_{12} = 0$. The former case corresponds to the binary BEC with coefficients $\gamma_{1,2}$ and γ_{12} accounting for the intraspecies and interspecies two-body repulsive interactions, respectively, while $\alpha_{1,2}$ add the repulsive three-body interactions in each component [31]. On the other hand, the quintic-only version of Eq. (1) may be used as the model for the binary TG gases.

Obviously, Eq. (1) admits the Hamiltonian representation, in the form of $i\partial\psi_j/\partial t = \delta H/\delta\psi_j^*$, with

$$
\begin{aligned}
H = \int_{-\infty}^{+\infty} \sum_{j=1}^{2} &\left\{ \frac{1}{2}\left|\frac{\partial \psi_j}{\partial x}\right|^2 + \frac{\gamma_j}{2}|\psi_j|^4 + \frac{\alpha_j}{3}|\psi_j|^6 \right. \\
&\left. + \frac{\gamma_{12}}{2}|\psi_j|^2|\psi_{3-j}|^2 + \chi|\psi_j|^4|\psi_{3-j}|^2 \right\} dx. \tag{2}
\end{aligned}
$$

In particular, this representation explains ratio 1:2 of coefficients in front of the terms multiplied by χ in Eq. (1), which are derived from terms $\chi(|\psi_1|^4|\psi_2|^2 + |\psi_2|^4|\psi_1|^2)$ in the Hamiltonian density of Eq. (2). In addition to H, the system preserves the norm (scaled number of the atoms) of each component,

$$
N_j = \int_{-\infty}^{+\infty} |\psi_j|^2 dx, \tag{3}
$$

and the total momentum, $P = i\int_{-\infty}^{+\infty}\sum_{j=1}^{2}\psi_j(\partial\psi_j^*/\partial x)dx$.

Due to the repulsive nature of the nonlinearity, one may expect that, with the increase of the constants accounting for the interactions between the components, γ_{12} and/or χ, the binary system becomes immiscible, building DWs as interfaces between domains filled by different components. To address this point, we are first looking for particular *exact* DW solutions to Eq. (1). In the case of the coupled stationary NLSEs with the cubic nonlinearity, which corresponds to Eq. (1) with $\alpha_{1,2} = \chi = 0$, an exact DW solution was found in Ref. [30], imposing a special restriction on the cubic coefficients, $\gamma_1 = \gamma_2 = \gamma_{12}/3$. For the single-component NLSE with the CQ nonlinearity, an exact solution, describing a transient layer between zero and constant-amplitude states (a variety of DW), was found in Ref. [32]. In the present context, we are looking for exact DW solutions employing an ansatz suggested by the latter solution:

$$
\psi_1(x,t) = \frac{A_1\,e^{-i\mu_1 t}}{\sqrt{1 + e^{\lambda x}}}, \quad \psi_2(x,t) = \frac{A_2\,e^{-i\mu_2 t}}{\sqrt{1 + e^{-\lambda x}}}, \tag{4}
$$

with background amplitudes $A_{1,2}$, chemical potentials $\mu_{1,2}$, and parameter λ defining the DW width, $W \sim 1/\lambda$.

Substituting this ansatz into Eq. (1), one arrives at a system of algebraic equations:

$$
\begin{aligned}
&4\left(A_j^2\gamma_j + A_{3-j}^2\gamma_{12} + 2\chi A_j^2 A_{3-j}^2\right) - 8\mu_j + \lambda^2 = 0, \\
&8A_j^2\left(\gamma_{12} + \chi A_j^2\right) - 8\mu_j - \lambda^2 = 0, \\
&A_j^2\left(\gamma_j + \alpha_j A_j^2\right) - \mu_j = 0, \quad j = 1,2.
\end{aligned} \tag{5}
$$

One can readily check that for symmetric intraspecies interactions, i.e.,

$$
\gamma_1 = \gamma_2 \equiv \gamma, \quad \alpha_1 = \alpha_2 \equiv \alpha, \tag{6}
$$

and equal amplitudes (the symmetric DW), Eq. (5) admits a nontrivial solution with $\mu_1 = \mu_2 \equiv \mu$, $A_1 = A_2 \equiv A$, and

$$
\begin{aligned}
A^2 &= \frac{3}{4}\frac{\gamma_{12} - \gamma}{\alpha - \chi}, \\
\lambda &= \pm(\gamma_{12} - \gamma)\sqrt{\frac{3}{2(\alpha - \chi)}}, \\
\mu &= \frac{3}{16}\frac{\gamma_{12} - \gamma}{(\alpha - \chi)^2}(\alpha\gamma + 3\alpha\gamma_{12} - 4\gamma\chi).
\end{aligned} \tag{7}
$$

From this we conclude that symmetric DWs are generic solutions, provided that conditions $\alpha > \chi$ and $\gamma_{12} > \gamma$ are satisfied. Note that in the decoupling limit, $\gamma_{12} \to 0, \chi \to 0$, and for the symmetric interactions, Eq. (1) reduces to a single-component CQ NLSE. In this case, the exact DW

reproduces the one found in Ref. [32] for the particular case of $\gamma = -2, \alpha = 1$.

A. The immiscibility condition and asymptotic relations

It is interesting to relate the existence of generic DW solutions to the immiscibility in the two-component NLSE system (1). Due to the repulsive nature of all the interactions and the absence of any trapping potential, the ground state must be obviously spatially uniform in the miscible case, hence the DW cannot exist in the miscible system. In the case of immiscibility, generic stable DW can exist if its free energy is lower than the one of the corresponding uniformly mixed state. Then, the immiscibility condition is written in terms of the free energy, $F = H[\psi_1, \psi_2] - \sum_j \mu_j N_j$, as

$$F_{\text{DW}} - F_{\text{UB}} \leqslant 0, \tag{8}$$

where $F_{\text{DW}} - F_{\text{UB}}$ is the difference in the free energies between the DW and of the corresponding uniform background (each free energy diverges in the infinite system, but the difference is finite).

Defining background amplitudes A_j, $j = 1, 2$, of the two DW components as in ansatz (4),

$$\begin{Bmatrix} |\psi_1(x = -\infty)|^2 = A_1^2 \\ |\psi_2(x = -\infty)|^2 = 0 \end{Bmatrix}, \quad \begin{Bmatrix} |\psi_1(x = +\infty)|^2 = 0 \\ |\psi_2(x = +\infty)|^2 = A_2^2 \end{Bmatrix} \tag{9}$$

(the corresponding densities in the uniformly mixed state obviously being $|\psi_j|^2 = A_j^2/2$), one can readily write the immiscibility condition (8), with the aid of Eq. (2), in the explicit form:

$$\sum_{j=1,2} \left(\gamma_j A_j^4 + \alpha_j A_j^6 \right) \leqslant \left[2\gamma_{12} + \chi \left(A_1^2 + A_2^2 \right) \right] A_1^2 A_2^2, \tag{10}$$

which is exact for the infinite domain, and approximated for a finite domain of length L, the error coming from the gradient-energy terms being estimated as $\sim 1/L$. For the exact symmetric DW solution given by Eq. (7), condition (8) is always satisfied. Indeed, from Eq. (7) we obtain

$$A^2 = \frac{3}{4} \frac{\gamma_{12} - \gamma}{\alpha - \chi} < \frac{\gamma_{12} - \gamma}{\alpha - \chi},$$

this being in agreement with Eq. (10), taking Eq. (6) into regard. It is also relevant to note that the equality of the free-energy densities at $x = \pm\infty$ implies the following relation between the chemical potentials and asymptotic densities of the numbers of atoms, fixed as per Eq. (9):

$$\sum_{j=1,2} (-1)^j \left[\mu_j - \frac{1}{2} \gamma_j A_j^2 - \frac{\alpha_j}{3} A_j^4 \right] A_j^2 = 0. \tag{11}$$

For $\chi = \alpha$ and $\gamma = \gamma_{12} = 0$, which corresponds to the two-component TG gas described by the coupled quintic equations, exact solution (7) degenerates into a uniform one. Different solutions for this case are given below.

III. REPULSIVE BEC MIXTURES WITH CUBIC AND QUINTIC INTERACTIONS

For the binary BEC mixture, we consider a reduced form of the NLSE system (1), in which the interspecies repulsion
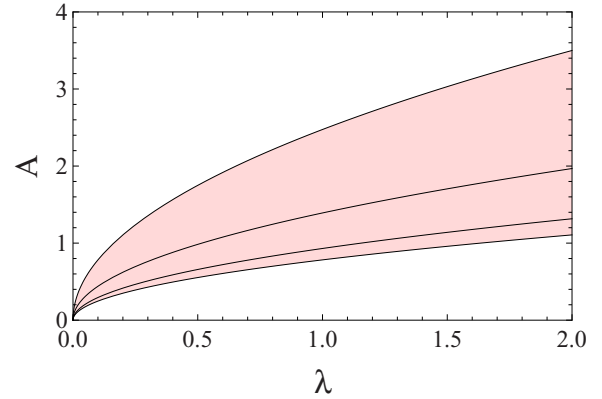

FIG. 1. (Color online) The existence region (shaded) for families of stationary exact domain-wall solutions given by Eqs. (4) and (7) under condition (6), with the quintic coefficient, α, varying in the interval of $[0.01, 1]$. Continuous curves from top to bottom refer to $\alpha = 0.01, 0.1, 0.5, 1.0$, respectively.

is accounted for by the cubic terms, while the quintic ones contribute solely to the self-repulsion ($\chi = 0$):

$$i \frac{\partial \psi_1}{\partial t} = -\frac{1}{2} \frac{\partial^2 \psi_1}{\partial x^2} + (\gamma_1 |\psi_1|^2 + \gamma_{12} |\psi_2|^2 + \alpha_1 |\psi_1|^4) \psi_1,$$
$$i \frac{\partial \psi_2}{\partial t} = -\frac{1}{2} \frac{\partial^2 \psi_2}{\partial x^2} + (\gamma_2 |\psi_2|^2 + \gamma_{12} |\psi_1|^2 + \alpha_2 |\psi_2|^4) \psi_2. \tag{12}$$

A. Symmetric and asymmetric DWs in the infinite domain

For the symmetric system, as defined in Eq. (6), parameters of the exact DW solution given by Eqs. (4) and (7) with $\chi = 0$ are

$$\mu = \frac{\gamma + 3\gamma_{12}}{4} A^2, \quad \lambda = \sqrt{\frac{8\alpha}{3}} A^2, \quad A^2 = \frac{3}{4} \frac{\gamma_{12} - \gamma}{\alpha}. \tag{13}$$

In Fig. 1 we show the existence region of this solution in the plane of (A, λ) (the amplitude and inverse width of the DW), as the three-body coefficient α varies in the interval of $[0.01, 1]$. Typical profiles of the corresponding stationary domain walls for different values of the three-body interaction parameter α and the background amplitude fixed to be $A = 1$, i.e.,

$$\gamma_{12} = \gamma + (4/3)\alpha, \tag{14}$$

as it follows from Eqs. (4) and (5), are depicted in Fig. 2. In this case, as seen from Eq. (13), the inverse width of the DW is determined solely by the quintic parameter, $\lambda = \sqrt{8\alpha/3}$.

As said above, for the exact DW solutions the immiscibility condition (10) always holds, hence the stability of these solutions is expected. This has been checked by simulations of Eq. (12), with small noise added to the initial DW profile (not shown here in detail). We have found that the DW remains stable also when it is put in motion by means of the standard phase-imprinting method, i.e., multiplying the quiescent DW by $\exp{(ivx)}$, as shown in Fig. 3.

The symmetry restrictions, $\gamma_1 = \gamma_2$, $\alpha_1 = \alpha_2$, adopted above for obtaining the exact DW solution, is not a limitation for the existence of DWs in real condensates. Indeed, it is possible to demonstrate that DWs are generic states in immiscible
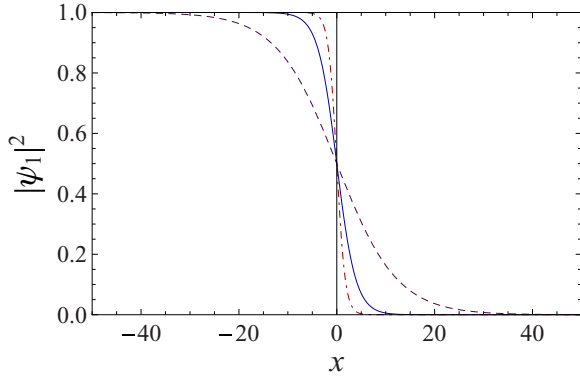
FIG. 2. (Color online) Typical profiles of stationary domain walls in the binary immiscible BEC generated by Eq. (12) with values of the quintic coefficient $\alpha = 0.01, 0.1$, and 0.05 (purple dashed, blue continuous, and red dotted curves, respectively). The parameters are taken as per Eq. (14), so as to have $A = 1$ in Eq. (7). We only show the profiles of the first component, the other one being produced by specular reflection with respect to the vertical axis.

two-component BECs with repulsive interactions, including asymmetric settings with unequal background densities in the two components at $x = \pm\infty$. A numerically found example of such a stable asymmetric DW is depicted in the left panel of Fig. 4 (see also Sec. V).

Another relevant physical situation, in which strongly asymmetric DW solutions naturally appear, is the one with the two-body and three-body intraspecies interactions acting only in the first and second components, respectively, the coupling being accounted for by two-body interactions. This corresponds to $\gamma_2 = 0, \alpha_1 = 0$ in Eq. (12), which can be experimentally implemented by enhancing the three-body interaction in one component via Efimov states [33], simultaneously tuning the two-body scattering length in the same component to zero by means of the Feshbach resonances. For this case, typical asymmetric DWs are displayed in the right panel of Fig. 4. One can check, using parameter values given in the caption to the figure, that the respective immiscibility condition
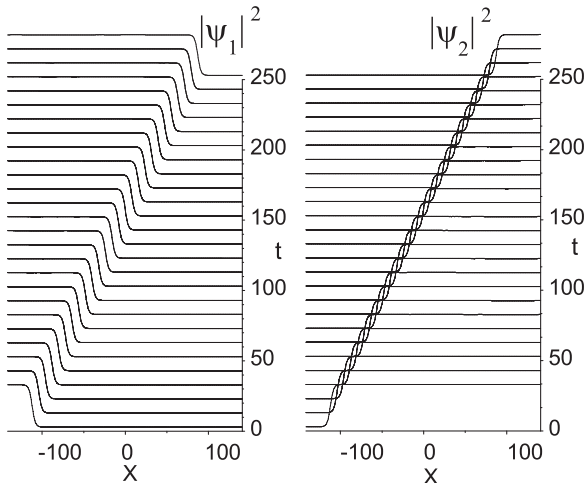

FIG. 3. A moving domain wall produced by phase imprinting with initial velocity $v = 0.8$ (see the text).
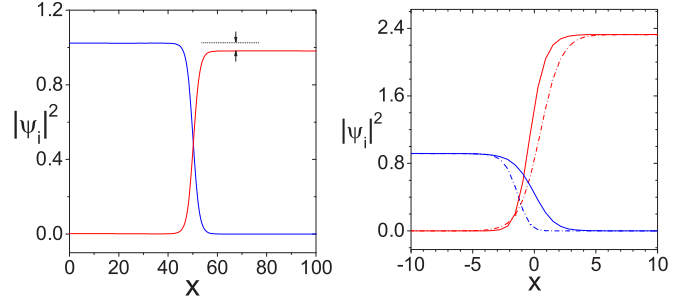

FIG. 4. (Color online) (Left) Typical density profiles of an asymmetric stationary domain wall in the binary immiscible BEC with $\gamma_1 = 1, \gamma_2 = 0.9, \gamma_{12} = 1.08$ and equal coefficients of the quintic nonlinearity, $\alpha_1 = \alpha_2 = 0.1$. The short horizontal line with the vertical arrow shows the mismatch between the left and right background levels. (Right) The same as in the left panel, but for $\gamma_1 = 1$, $\gamma_2 = 0$, $\alpha_1 = 0$, $\alpha_2 = 0.05$ in Eq. (12), and different strengths of the interspecies cubic interaction: $\gamma_{12} = 0.5$ (continuous curves), $\gamma_{12} = 1.1$ (dash-dotted curves), and asymptotic (left and right) densities $A_1 = 0.916$, $A_2 = 2.325$. In both panels, the blue and red lines depict profiles of the first and second components, respectively.

[setting $\chi = 0, \alpha_1 = 0, \gamma_2 = 0$ in Eq. (10)] for these solutions,

$$\gamma_1 A_1^4 + \alpha_2 A_2^6 - 2\gamma_{12} A_1^2 A_2^2 \leqslant 0, \tag{15}$$

is satisfied; hence, despite the large mismatch between the backgrounds, one can expect this asymmetric DW to be stable. This was confirmed by direct simulations of Eq. (12) (not shown here). The respective relation between asymptotic amplitudes A_j and chemical potentials [setting $\chi = 0, \alpha_1 = 0, \gamma_2 = 0$ in Eq. (11)],

$$\mu_1 A_1^2 - \frac{\gamma_1}{2} a_1^4 = \mu_2 A_2^2 - \frac{\alpha_2}{3} a_1^6, \tag{16}$$

was also confirmed numerically. Note that, while Eq. (16) does not contain γ_{12}, the immiscibility condition (15) explicitly depends on it. From the experimental point of view, this suggests controlling the immiscibility condition by keeping parameters γ_j, α_j, $j = 1, 2$, fixed and changing the interspecies two-body scattering length by means of the Feshbach resonance. The variation of γ_{12} mainly affects the shape of the DW interface, while the asymptotic values A_j remain unaltered. It is easy to find, from Eq. (15), the critical value at which the mixture becomes miscible in this case:

$$\tilde{\gamma}_{12} = \frac{\gamma_1 A_1^4 + \alpha_2 A_2^6}{2 A_1^2 A_2^2}. \tag{17}$$

This prediction will be numerically checked in the next subsection (see Fig. 7 below for DWs in finite-length rings).

B. BEC mixtures on a ring: BD states

The DW solutions considered above refer to infinite domains. DWs in finite-length ring regions, subject to periodic boundary conditions (b.c.), are relevant for the realization of the BEC in toroidal traps [34,35], including binary condensates [36]. Such patterns can be constructed by combining a DW and the respective anti-DW on the ring, to satisfy the b.c.
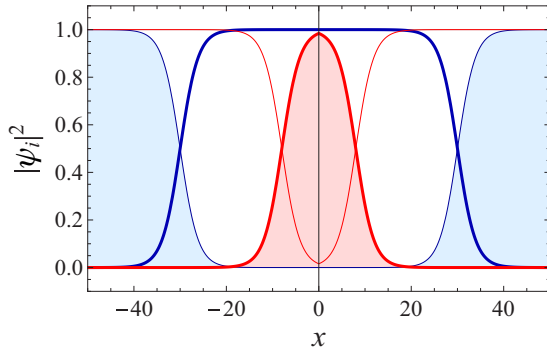
FIG. 5. (Color online) The DW-anti-DW state in Eq. (18) of the repulsive binary BEC with $\alpha = 0.1, A = 1$, $\lambda = 2\sqrt{2\alpha/3}$ and separation $x_0 = 30$ (blue lines) or $x_0 = 8$ (red lines). Thin and thick lines refer to the first and second components, respectively. Filled blue and red regions depict, severally, the bubble and droplet parts of the patterns with $x_0 = 30$ and $x_0 = 8$ (they are shown for different values of x_0 to avoid confusing overlap between them).

More precisely, we consider trial solutions of the type,

$$\psi_1(x,t) = \begin{cases} \dfrac{Ae^{-i\mu t}}{\sqrt{1 + e^{\lambda(x + x_0)}}} & \text{at} \quad x \leqslant 0 \\[2ex] \dfrac{Ae^{-i\mu t}}{\sqrt{1 + e^{-\lambda(x - x_0)}}} & \text{at} \quad x > 0 \end{cases},$$

$$\psi_2(x,t) = \begin{cases} \dfrac{Ae^{-i\mu t}}{\sqrt{1 + e^{-\lambda(x + x_0)}}} & \text{at} \quad x \leqslant 0 \\[2ex] \dfrac{Ae^{-i\mu t}}{\sqrt{1 + e^{\lambda(x - x_0)}}} & \text{at} \quad x > 0 \end{cases},$$

(18)

where $4x_0$ is the perimeter of the underlying ring, and $2x_0$ is separation between the DW and anti-DW placed at diametrically opposite positions.

Typical profiles of such numerically found DW-anti-DWs patterns are displayed in Fig. 5 for two different values of x_0. One can expect that ansatz (18) is a virtually exact solution when DW and anti-DW are well separated, e.g., $x_0 \gg \lambda^{-1}$. This is indeed what one observes from the numerical solutions. More remarkable is the fact that the ansatz provides an almost exact solution even when the DW and anti-DW are relatively close to each other, as one can see in Fig. 5 . In this case, the complex composed of the DW and anti-DW profiles may be considered as a *bubble* (sort of a dark soliton [37]) in field ψ_2 coupled to a localized bright profile (*droplet*) of field ψ_1, which we refer to as BDs. The stability of these complexes was verified by direct simulations of their perturbed simulations in the framework of Eq. (12) (not shown here in detail).

The possibility of setting a DW in motion by phase imprinting a velocity onto it is relevant to the BD states, too. This is shown in Fig. 6, where simulations of Eq. (12) for moving DBs are reported. This figure makes it clear that the BD structures are very robust ones not only in the stationary form, but also when they are set in motion, thanks to the repulsive character of all the interactions.

BD solutions can be constructed as well by combining an asymmetric DW with the corresponding anti-DW on the ring (not shown here, as their shape is quite obvious). The asymmetric complexes are stable, too.
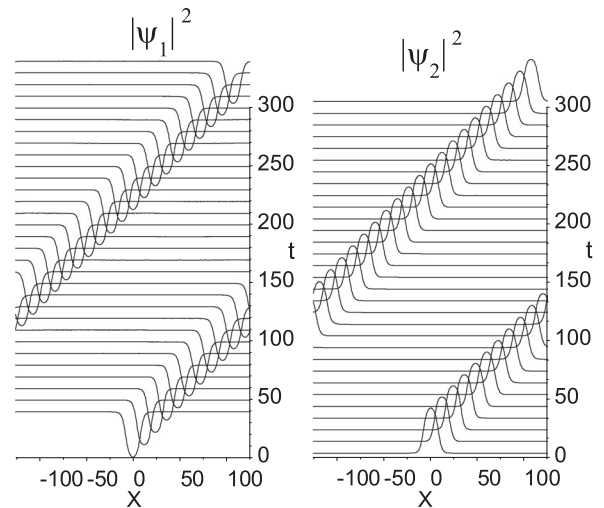

FIG. 6. A moving bubble-droplet solution obtained by the phase imprinting, with initial velocity $v = 1.2$ velocity, onto the stationary profiles shown in Fig. 5 for $x_0 = 8$.

It is also interesting to check, in terms of the BD solutions, the miscibility condition discussed above. For the parameter values given in the caption of Fig. 4 (left panel), the miscibility threshold is predicted by Eq. (17) to occur at $\tilde{\gamma}_{12} \approx 0.344$. In Fig. 7 we display the evolution of the density profiles produced by the simulations of Eq. (12) for γ_{12} taken just above and below the threshold, i.e., in the regions of weak immiscibility and miscibility, respectively, taking, as initial conditions, a BD profile constructed from the asymmetric DW corresponding to the continuous curves in the left panel of Fig. 4. It is seen that, for $\gamma_{12} = 0.4$, the BD solution (and its DW and anti-DW constituents) quickly adapt, by emitting small-amplitude matter waves, to the new value of the interspecies interaction, keeping the immiscibility, while for $\gamma_{12} = 0.3$ the mixing of the two components sets in, generating strong density waves in the two components.

IV. DOMAIN WALLS AND BUBBLE DROPLET COMPLEXES IN BINARY TONKS-GIRARDEAU GASES

As said in the Introduction, the NLSE with the quintic nonlinearity emerges in connection to the ground-state properties of TG gases, both for the single-component ones [14–18] and binary mixtures [24,25]. In this section we investigate the existence of DW states at the interface of two interacting TG gases by means of the NLSE system (1) with only quintic terms included. The respective equations for stationary states follow from Eq. (1), substituting $\psi_{1,2}(x,t) = \exp(-i\mu_{1,2}t)\phi_{1,2}(x)$:

$$\left[-\frac{\hbar^2}{2m_j}\frac{d^2}{dx^2} + \alpha_j|\phi_j|^4 + \chi(|\phi_{3-j}|^4 + 2|\phi_j|^2|\phi_{3-j}|^2)\right]\phi_j$$
$$= \mu_j\phi_j,$$
(19)

with $m_{1,2}$ being the atomic masses of the two bosonic species.

For equal masses $m_1 = m_2 = m$ and fully symmetric interactions, $\alpha_1 = \alpha_2 = \chi$, this model for the TG mixture can be justified in terms of the density-functional theory, as discussed in Ref. [25]. In this respect, we recall that the
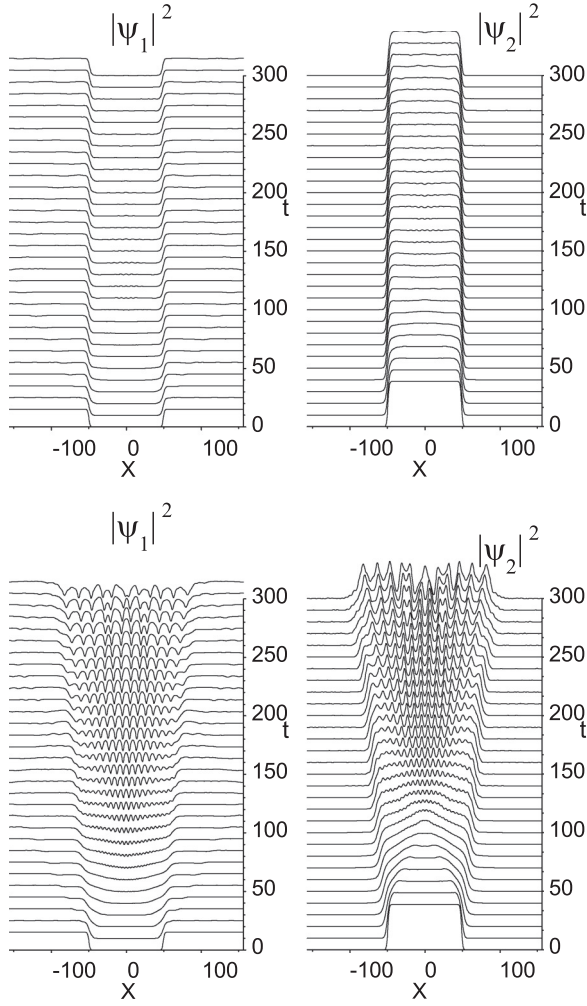
FIG. 7. (Top panels) The evolution of densities of the first and second (left and right columns) components of the BD state for the same parameters as in the right panel of Fig. 4, except for $\gamma_{12} = 0.4$, which is taken above the miscibility threshold. (Bottom panels) The same as above, but for $\gamma_{12} = 0.3$, taken just below the miscibility threshold.

1D quantum model of two boson species interacting via the hard-core repulsion is exactly solvable by means of the Bethe-ansatz method [38]. From that solution, one obtains the ground-state energy density (in physical units),

$$\varepsilon(\rho_T) = \frac{\pi^2 \hbar^2}{6m} \rho_T^2,$$

with $\rho_T = \sum_{j=1,2} |\phi_j|^2$ being the total density of the mixture. In terms of the density-functional theory with the local-density approximation, this amounts to the consideration of the following energy functional:

$$E[\rho_T] = \int_{-\infty}^{+\infty} dx \left\{ \sum_{j=1}^{2} \phi_j^* \left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \right] \phi_j + \rho_T \varepsilon(\rho_T) \right\},$$

$$(20)$$

where chemical potentials μ_j are introduced as Lagrangian multipliers to guarantee the conservation of the numbers of atoms in the two species. In the case of equal masses and *fully symmetric* interactions,

$$\alpha_1 = \alpha_2 = \chi = \pi^2/(2m), \qquad (21)$$

Eq. (19), i.e., in this case,

$$\tilde{\mu}_j \phi_j = -\frac{d^2 \phi_j}{dx^2} + \pi^2 (|\phi_j|^4 + |\phi_{3-j}|^4$$
$$+ 2|\phi_j|^2 |\phi_{3-j}|^2) \phi_j, \quad j = 1,2, \qquad (22)$$

coincide with the equations which provide for the minimization of energy functional (20).

In the following we investigate DW and BD solutions of Eq. (1), both for the fully symmetric interaction strengths and relatively small deviations from this special case.

A. The variational approach

DWs of Eq. (19) with $m_1 = m_2 \equiv m, \alpha_1 = \alpha_2 \equiv \alpha$, $\chi_1 = \chi_2 \equiv \chi$, but $\alpha \neq \chi$, can be studied by means of a variational approximation (VA) based on the corresponding Lagrangian,

$$L = \frac{1}{2} \int_{-\infty}^{+\infty} \left\{ \left[\frac{1}{2} \left(\frac{d\phi_1}{dx} \right)^2 + \left(\frac{d\phi_2}{dx} \right)^2 \right] - \mu(\phi_1^2 + \phi_2^2) \right.$$
$$\left. + \frac{\alpha}{3} (\phi_1^6 + \phi_2^6) + \chi (\phi_1^4 \phi_2^2 + \phi_1^2 \phi_2^4) + \frac{2}{3} \mu^{3/2} \right\} dx, \quad (23)$$

where constant $(2/3) \mu^{3/2}$ is the *counterterm*, which is added to cancel the divergence of the integral at $|x| \to \infty$. The DW solution may be approximated by an ansatz similar to the one adopted for the BEC mixture in Eq. (4), i.e.,

$$\phi_1(x) = \sqrt{\frac{\sqrt{\mu}}{1 + e^{\lambda x}}}, \quad \phi_2(x) = \sqrt{\frac{\sqrt{\mu}}{1 + e^{-\lambda x}}}, \qquad (24)$$

with λ considered as a free variational parameter. The substitution of Eq. (24) into Lagrangian (23) yields the corresponding effective Lagrangian,

$$L_{\text{eff}} = \frac{\sqrt{\mu}}{2} \left[\frac{\lambda}{8} + (\chi - \alpha) \frac{\mu}{\lambda} \right]. \qquad (25)$$

Finally, the variational equation, $\partial L_{\text{eff}} / \partial \lambda = 0$, produces the main result of the VA,

$$\lambda^2 = 8\mu(\chi - \alpha). \qquad (26)$$

This analysis suggests that, in the TG mixtures with equal masses and fully symmetric interactions ($\chi = \alpha$), DW states cannot exist, i.e., they should degenerate into a mixed uniform background, as one can see from Eq. (26). This prediction correlates with the fact that the immiscibility threshold in Eq. (10), in the absence of cubic interactions and for equal asymptotic densities, reduces to $\alpha = \chi$, hence the TG mixture with the fully symmetric interactions sits precisely at the miscibility threshold.

The numerical analysis of the model based on Eq. (22) with the full interaction symmetry show that DW and BD patterns indeed decay into uniformly mixed backgrounds. An example of this is shown in Fig. 8, where a stationary numerical BD solution and its perturbed evolution are depicted, respectively,
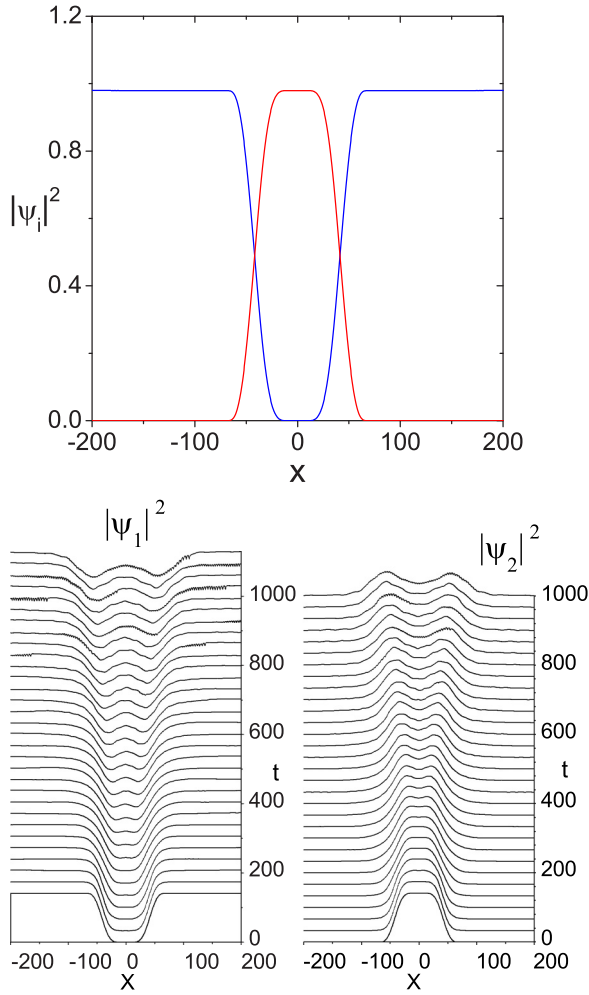
FIG. 8. (Color online) (Top) Density profiles of a bubble-drop state, produced by Eq. (22) as a DW-anti-DW complex, in the binary TG gas with equal masses and fully symmetric interactions [see Eq. (21)], i.e., exactly at the miscibility threshold. (Bottom) The stability test by real-time simulations of the configuration depicted in the top panel. The density profiles remain stationary up to $t \simeq 400$. Subsequently, they slowly decay into the uniformly mixed background with oscillations.

in the top and bottom panels. As seen, such a stationary state exists at the miscibility threshold, but it turns out to be unstable. In this connection, it is relevant to mention that, although for the TG mixture the time evolution governed by the quintic NLSE may have no direct physical meaning, it can be used as a mathematical tool to verify if a stationary DW or BD state is well defined, and also to check if the mixture indeed sits at the miscibility threshold. This can be done, as usual, by adding a small random perturbation to the initial stationary profiles and letting the state evolve according to the time-dependent quintic NLSEs associated with stationary equations (22). The bottom panel of Fig. 8 shows that, although the BD remains stationary for a relatively long time ($t \approx 400$), still later the profiles slowly decay into the uniform mixed background state. This behavior is consistent with the fact that the TG-TG mixture finds itself precisely at the miscibility threshold, as predicted by the above analysis.

However, the simulations demonstrate that, as soon as one deviates from the fully symmetric-interaction limit defined by Eq. (21), the DW and BD states become well-defined ground states, depending on the b.c. This point is further investigated in the next section.

B. The asymmetric setting

The results of the previous section can be extended for the binary TG gas with asymmetric interspecies and intraspecies interaction strengths and/or unequal masses, provided that the deviation from the fully symmetric setting is not too large. As the corresponding quantum problem is not exactly solvable in the absence of the full symmetry, such a model is substantiated less rigorously than the full-symmetry limit.

In the following, we assume an asymmetric system with different coefficients of the quintic self-interaction in Eq. (1), $\alpha_1 \neq \alpha_2$. The accordingly modified system obviously keeps Lagrangian (23), making it possible to perform a similar VA as in the previous section. Due to the asymmetry, however, the DW solution should be looked for with unequal chemical potentials, $\mu_1 \neq \mu_2$. Then, the DW solution is defined by the following b.c.:

$$\phi_1^2(x = -\infty) = \sqrt{\mu_1/\alpha_1}, \quad \phi_2^2(x = -\infty) = 0,$$
$$\phi_1^2(x = +\infty) = 0, \quad \phi_2^2(x = +\infty) = \sqrt{\mu_2/\alpha_2}. \tag{27}$$

It is worth mentioning here that direct substitution demonstrates that the simplest ansatz based on Eq. (4) does not produce any exact DW solution of Eq. (1). Nevertheless, it is possible to find an exact relation between μ_1 and μ_2 which is necessary for the existence of the DW solution. Indeed, Eq. (1) may be formally considered as equations of motion for a mechanical system with two degrees of freedom, $\phi_1(x)$ and $\phi_2(x)$, where x plays the role of formal time. The Hamiltonian of this mechanical system is

$$H_{\text{mech}} = \sum_{n=1,2} \left[\frac{1}{4m_n} \left(\frac{d\phi_n}{dx} \right)^2 + \frac{1}{2} \mu_n \phi_n^2 \right.$$
$$\left. - \frac{\alpha_n}{6} \phi_n^6 - \frac{\chi}{2} \phi_n^2 \phi_{3-n}^4 \right]. \tag{28}$$

Because H_{mech} must take identical values at $x = \pm\infty$, b.c. (27) demonstrate that μ_1 and μ_2 are related by

$$\alpha_2 \mu_1^3 = \alpha_1 \mu_2^3. \tag{29}$$

By means of obvious rescalings, we can eventually set

$$\alpha_2 = m_2 = \mu_2 \equiv 1, \tag{30}$$

and Eq. (29) then yields, for the DW solution, the single admissible value of the chemical potential of species ϕ_1:

$$\mu_1 = \alpha_1^{1/3}. \tag{31}$$

Thus there remain three free parameters: coefficient χ of the interspecies quintic interaction, along with m_1 and α_1.

We have checked by means of numerical methods that DW solutions exist for a wide range of values of parameters $\chi, \alpha_{1,2}$ away from the fully symmetric-interaction limit considered in the previous section. Two typical examples of such numerically
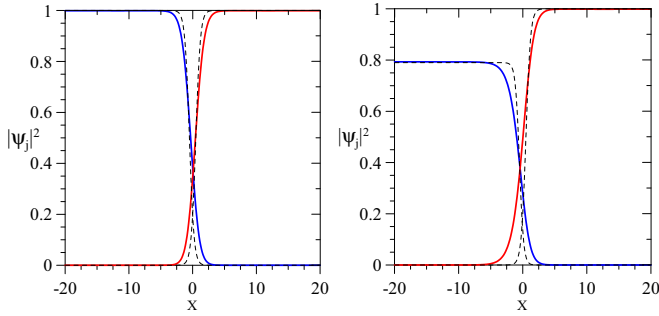
FIG. 9. (Color online) DW density profiles produced by Eq. (19) for equal masses $m_1 = m_2 = 1$ and interspecies interaction strength $\chi = 2$. Other parameters are fixed as $\alpha_1 = \alpha_2 = 1$ (left panel) and $\alpha_1 = 2$, $\alpha_2 = 1$ (right panel). Blue and red lines refer to first and second components, respectively, while the black dashed lines denote VA results.

found DWs are depicted in Fig. 9 for $\chi \neq \alpha_1 = \alpha_2 \equiv \alpha$ (left panel) and $\chi = \alpha_1 \neq \alpha_2$ (right panel). Note that in the former (respectively, latter) case the DW components have equal (respectively, unequal) backgrounds, as a consequence of the equality (respectively, inequality) of the intraspecies interaction strengths. In the same figure are also depicted, by dashed lines, the predictions of the VA for the two cases, which demonstrate reasonable agreement with the numerical findings.

In addition to the DW, a physically relevant state may be the BD complex similar to those studied above for the BEC mixture. In the general case, the BD is a spatially even solution, with $\phi_{1,2}(-x) = \phi_{1,2}(x)$, which obeys the following b.c.:

$$\phi_1^2(x = +\infty) = 0, \quad \phi_2^2(x = +\infty) = \sqrt{\mu_2/\alpha_2}, \quad (32)$$

$$\phi_1'(x = 0) = \phi_2'(x = 0) = 0. \quad (33)$$

The extreme case of the BD complex is the one with

$$\phi_2(x = 0) = 0, \quad (34)$$

when field ϕ_1 completely ousts ϕ_2 at the central point, $x = 0$.

The conservation of H_{mech} along x [see Eq. (28)] has its bearing for the BD solution, too. Indeed, condition $H_{\text{mech}}(x = +\infty) = H_{\text{mech}}(x = 0)$ yields the following relation for the densities of the two species at the central point, $x = 0$ [see Eq. (33)]:

$$\sum_{n=1,2} \left[\mu_n \phi_n^2(x = 0) - \frac{\alpha_n}{3} \phi_n^6(x = 0) \right] - \chi \phi_1^2(x = 0)$$

$$\times \phi_2^2(x = 0) \left[\phi_1^2(x = 0) + \phi_2^2(x = 0) \right] = \frac{2\mu_2^{3/2}}{3\sqrt{\alpha_2}}. \quad (35)$$

This cumbersome relation simplifies for the extreme type of the BD defined above by condition (34):

$$\mu_1 \phi_1^2(x = 0) - \frac{\alpha_1}{3} \phi_1^6(x = 0) = \frac{2\mu_2^{3/2}}{3\sqrt{\alpha_2}}. \quad (36)$$

The same normalization (30) as used above for the DW may be adopted for the BD solutions. Then, the general family of

such solutions depends on four independent parameters: χ, m_1, α_1, and μ_1 [the latter constant is no longer determined by an additional condition, such as Eq. (31)].

V. DISCUSSION AND CONCLUSIONS

In this work, we have introduced the concept of DWs (domain walls) in the two-component TG (Tonks-Girardeau) gas, as well as in the binary BEC described by coupled GPEs (Gross-Pitaevskii equations) with CQ (cubic-quintic) nonlinearities. Thus, we have extended the concept of DW previously elaborated for immiscible binary BEC, described by the system of two coupled GPEs with cubic terms.

In several cases, exact or approximate analytical solutions have been found. In particular, for the binary BEC with the CQ nonlinearity, exact solutions for DWs were obtained in the symmetric system, with equal intraspecies scattering lengths. For the existence of these solutions, the presence of the quintic interactions is necessary. Numerical analysis was used to prove the existence and stability of asymmetric DWs in the BEC mixtures with asymmetric CQ interactions in the two components.

In addition to the DWs, we have also investigated DB (droplet-bubble) complexes in the same settings, which consist of a dark (gray) soliton in one component (the "bubble"), and a bright soliton (the "droplet") in the other. In the BEC system, the DWs and DB states are mobile, keeping their shape in the state of motion.

We have also introduced symmetric and asymmetric DWs in the binary TG gas, for which the system of two coupled NLSEs (nonlinear Schrödinger equations) with quintic-only repulsive terms was adopted. In particular, we have shown that a TG mixture with equal atomic masses and fully symmetric interactions [as per Eq. (21)] sits precisely at the immiscibility threshold, therefore DWs and BDs exist only as metastable states in this case. The VA (variational approximation) and numerical analysis, however, demonstrate the existence of stable DWs ground states at a small deviation from the full symmetry.

Stability of DWs and BD complexes in BEC mixtures with symmetric and asymmetric CQ interactions in the two components has also been demonstrated. In this respect, we have derived a general condition for the immiscibility, which is valid for both the BEC and TG settings. The stability of the DW and BD states is secured if the immiscibility condition is satisfied.

It is relevant to discuss possible implementations of the above results in experimental settings. DWs and anti-DWs can be observed (either being separated, or merged into BD complexes) in toroidal quasi-one-dimensional traps for gases with scattering lengths tuned so as to satisfy the immiscibility condition. Note that toroidal traps are routinely created in laboratories with the aid of magnetic fields [34], and, as said above, the scattering length can be easily changed via the Feshbach-resonance technique. To create BD complexes in the ring, one can load the gases into two different semicircles of the trap, initially kept separated by laser sheets, which are slowly removed after the loading.

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