Decay of hydrodynamic modes in dilute Bose-Einstein condensates

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Expressions for the speed and lifetimes of the sound modes in a dilute Bose-Einstein condensate are obtained using Bogoliubov mean field theory. The condensate has two pairs of sound modes which undergo an avoided crossing as the equilibrium temperature is varied. The two pairs of sound modes decay at very different rates, except in the neighborhood of the avoided crossing, where the identity of the longest-lived mode switches. The predicted speed and lifetime of the longest-lived sound mode are consistent with recent experimental observations on sound in an ^{87}Rb Bose-Einstein condensate.

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I. INTRODUCTION

At very low temperature, the dynamical description of a dilute Bose gas involves a kinetic equation for the dynamics of thermal excitations coupled to the equation of motion of the macroscopic phase of the Bose-Einstein condensate (BEC) [1,2]. The thermal excitations of this system are found from Bogoliubov mean field theory and can be interpreted as an ideal gas of weakly interacting excitations or "bogolons." The bogolon energy spectrum is phononlike at low momentum and particlelike at high momentum, with the extent of phononlike region increasing at lower temperatures.

Spatial variations in the macroscopic phase of the condensate determine the local value of the superfluid velocity, which influences the dynamics of the bogolon gas. Correspondingly, spatial variations in the density, which are partly determined by distribution of bogolons, act to accelerate the superfluid flow.

The existence of a macroscopic condensate phase (or superfluid velocity) adds an extra degree of freedom to the BEC, allowing it to have six hydrodynamic modes. In terms of plane-wave disturbances, two of the hydrodynamic modes are transverse and associated with shear effects. In Refs. [1,3], we used these modes to obtain values for the shear viscosity of a condensed Bose gas as a function of equilibrium temperature and density.

The remaining four modes are longitudinal propagating modes or sound modes. The condensed Bose gas can support two pairs of sound modes, which are analogous to first and second sound in a superfluid [4]. First sound consists of in-phase oscillations of the thermal excitations and superfluid velocity while second sound is an out-of-phase oscillation. The primary mechanism for decay of sound modes is collision between the bogolons. Since each mode involves a different amount of motion in bogolon gas, the sound modes may decay at strikingly different rates.

The kinetic equation derived in Ref. [1], and used here, differs from the Zaremba-Nikuni-Griffin (ZNG) kinetic equation [5,6] and is closer to the KD kinetic equation derived by Kirkpatrick and Dormann [7]. The ZNG kinetic equation simplifies the description of the thermal component of the BEC by using an approximate particlelike Hartree-Fock excitation spectrum. This approximation creates problems for ZNG predictions for transport properties at low temperature. For example, the ZNG kinetic equation does not give the correct sound speeds at $T = 0$ [8].

A further difference with both the ZNG and KD kinetic equations is the collision operator used in these kinetic equations. In both ZNG and KD theories, the collision operator includes $2 \leftrightarrow 2$ and $2 \leftrightarrow 1$ type collisions between excitations, but does not include $3 \leftrightarrow 1$ type collisions, which may sensibly affect transport properties. In [2], we show that $3 \leftrightarrow 1$ collisions give a large contribution to the decay rate of excitation number. In the sections below, we compute transport properties for the BEC using the full spectrum of the linearized collision operator with no approximations made to its form. To our knowledge, this level of calculation has not been performed with either the ZNG or KD kinetic equations.

In the following sections, we calculate the decay rates of the sound modes as a function equilibrium temperature, density, particle mass, and interaction strength. This requires the computation of all of the current correlation functions that normally determine transport coefficients [9]. Computation of these correlation functions requires solving the eigenvalue problem of the linearized collision operator [2]. We obtain expressions for the decay rates of sound modes that can be applied to any monatomic dilute Bose gas and we also explicitly compare our results to a recent experiment on a BEC of ^{87}Rb atoms [10].

We begin in Sec. II by deriving the form of the kinetic equation that will be used in subsequent sections. In Sec. III, we solve the kinetic equation to second order in the wave vector (a small parameter) for spatial variations of thermodynamic variables in the BEC. In Sec. IV, we obtain expressions for the decay rates in a form that can be compared to a variety of monatomic BECs, and plot them as a function of equilibrium temperature and density. In Sec. V, we compare our results to a recent experiment. Finally, in Sec. VI, we make some concluding remarks on possible tests and applications of this calculation.

II. BOGOLON KINETIC EQUATION

Mean field theory describes the dynamics of a Bose-Einstein condensate in terms of excitations (bogolons) and in terms of the macroscopic phase that results from the broken gauge symmetry. In a BEC, particle number is conserved

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but bogolon number is not conserved. The linearized (about equilibrium) kinetic equation for a dilute atomic BEC in a box can be written [1]

$$\frac{\partial \delta\mathcal{N}_{\mathbf{k}}(\mathbf{q},t)}{\partial t} = i\frac{\hbar}{m}\mathbf{k}\cdot\mathbf{q}\frac{(\epsilon_k+\Delta)}{E_k}\delta\mathcal{N}_{\mathbf{k}}(\mathbf{q},t) + i\mathbf{q}\cdot\mathbf{v}_s(\mathbf{q},t)\mathcal{N}_{\mathbf{k}}^{\mathrm{eq}} - \mathcal{G}_{\mathbf{k}}\{h\}, \quad (1)$$

where $\hbar$ is Planck's constant, $\hbar\mathbf{k}$ is the bogolon momentum, $\epsilon_k = \frac{\hbar^2k^2}{2m}$, Δ is the equilibrium condensate order parameter, $E_k = \sqrt{(\epsilon_k+\Delta)^2-\Delta^2}$ is the bogolon energy, and $\mathcal{N}_{\mathbf{k}}^{\mathrm{eq}} = (e^{E_k/k_BT}-1)^{-1}$ is the equilibrium Bose-Einstein distribution for bogolons at temperature T. The distribution $\mathcal{N}_{\mathbf{k}}^{\mathrm{eq}}$ is a stationary state of Eq. (1). The quantity $\delta\mathcal{N}_{\mathbf{k}}(\mathbf{q},t)$ gives the deviation of the bogolon distribution from equilibrium for spatial variations with wave vector $\mathbf{q}$. It can be written in the following form:

$$\delta\mathcal{N}(\mathbf{k},\mathbf{q},\omega) = \mathcal{N}_k^{\mathrm{eq}}\mathcal{F}_k^{\mathrm{eq}}h(\mathbf{k},\mathbf{q},\omega), \quad (2)$$

where $h(\mathbf{k},\mathbf{q},t)$ is a small quantity that decays to zero as the gas relaxes to equilibrium. The linearized collision operator $\mathcal{G}_{\mathbf{k}}\{h\}$ is defined in terms of $h(\mathbf{k},\mathbf{q},t)$ (see Appendix A). In the hydrodynamic regime where spatial variations have very long wavelength, the wave vector $q = |\mathbf{q}|$ is a very small parameter.

The broken gauge symmetry in the Bose-Einstein condensate gives rise to a macroscopic phase $\phi(\mathbf{q},t)$ whose spatial variation determines the superfluid velocity. As shown in [1], the equation governing the dynamics of the microscopic phase takes the form

$$\frac{\partial^2\phi(\mathbf{q},t)}{\partial t^2} = -i\frac{g}{m}\frac{1}{(2\pi)^3}\int d\mathbf{k}\,\mathbf{q}\cdot\mathbf{k}\,\delta\mathcal{N}_{\mathbf{k}}(\mathbf{q},t) - i\frac{g}{\hbar}\mathbf{q}\cdot\mathbf{v}_s(\mathbf{q},t)n^{\mathrm{eq}}, \quad (3)$$

where $g = 4\pi\hbar^2 a/m$ is the coupling constant, a is the s-wave scattering length of the atoms in the gas, and n^{eq} is the total particle-number density.

In order to obtain the dispersion relation for the hydrodynamic modes of the Bose-Einstein condensate, we consider one frequency component of the kinetic equations and write

$$\delta\mathcal{N}_{\mathbf{k}}(\mathbf{q},t) \sim e^{i\omega t}\delta\mathcal{N}_{\mathbf{k}}(\mathbf{q},\omega). \quad (4)$$

Then, the bogolon kinetic equation (1) takes the form

$$\omega\delta\mathcal{N}_{\mathbf{k}}(\mathbf{q},\omega) = \frac{\hbar}{m}\mathbf{k}\cdot\mathbf{q}\frac{(\epsilon_k+\Delta)}{E_k}\delta\mathcal{N}_{\mathbf{k}}(\mathbf{q},\omega) - i\frac{\hbar}{m}q^2\phi(\mathbf{q},\omega)\mathcal{N}_{\mathbf{k}}^{\mathrm{eq}} + i\mathcal{G}_{\mathbf{k}}\{h\}, \quad (5)$$

where we have used the fact that $\mathbf{v}_s(\mathbf{q},\omega) = -i\frac{\hbar}{m}\mathbf{q}\phi(\mathbf{q},\omega)$. Equation (3) takes the form

$$\omega^2\phi(\mathbf{q},\omega) = i\frac{g}{m}\frac{1}{(2\pi)^3}\int d\mathbf{k}\,\mathbf{q}\cdot\mathbf{k}\,\delta\mathcal{N}_{\mathbf{k}}(\mathbf{q},\omega) + q^2\frac{g}{m}\phi(\mathbf{q},\omega)n^{\mathrm{eq}}. \quad (6)$$

Equations (5) and (6) are the bogolon kinetic equations that describe hydrodynamic behavior of a dilute BEC.

If we eliminate the phase between Eqs. (5) and (6), the kinetic equation for the bogolon distribution can be written

$$\begin{aligned}\omega h(\mathbf{k},\mathbf{q},\omega) &= \mathbf{q}\cdot\mathbf{k}\mathcal{B}_k h(\mathbf{k},\mathbf{q},\omega) + \frac{q^2}{\omega^2 - v_B^2q^2}\frac{g\hbar}{m^2}\frac{1}{\mathcal{F}_k^{\mathrm{eq}}} \\ &\quad\times\frac{1}{(2\pi)^3}\int d\mathbf{k}_1\,\mathbf{q}\cdot\mathbf{k}_1\mathcal{N}_{k_1}^{\mathrm{eq}}\mathcal{F}_{k_1}^{\mathrm{eq}}h(\mathbf{k}_1,\mathbf{q},\omega) \\ &\quad + i\int_0^\infty dk_1\int d\Omega_1\sqrt{\frac{k_1^2\mathcal{N}_{k_1}^{\mathrm{eq}}\mathcal{F}_{k_1}^{\mathrm{eq}}}{k^2\mathcal{N}_k^{\mathrm{eq}}\mathcal{F}_k^{\mathrm{eq}}}}\,\mathcal{C}(\mathbf{k},\mathbf{k}_1)h(\mathbf{k}_1,\mathbf{q},\omega),\end{aligned} \quad (7)$$

where $\mathcal{C}(\mathbf{k},\mathbf{k}_1)$ can be expanded in spherical harmonics $\mathrm{Y}_\ell^m(\hat{\mathbf{k}})$ and takes the form

$$\mathcal{C}(\mathbf{k},\mathbf{k}_1) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\mathcal{C}_\ell(k,k_1)\mathrm{Y}_\ell^m(\hat{\mathbf{k}})\mathrm{Y}_\ell^{m*}(\hat{\mathbf{k}}_1). \quad (8)$$

This form of the collision operator is discussed in Appendix A. The collision operator $\mathcal{C}(\mathbf{k},\mathbf{k}_1)$ is a symmetric operator and has a complete set of orthonormal eigenfunctions. Also, we have introduced the quantities

$$v_B = \sqrt{\frac{gn^{\mathrm{eq}}}{m}} \quad \text{and} \quad \mathcal{B}_k = \frac{\hbar}{m}\frac{(\epsilon_k+\Delta)}{E_k}. \quad (9)$$

The equilibrium particle density, in the so-called Popov approximation (see [6], Chap. 5) can be written

$$n^{\mathrm{eq}} \approx n_0^{\mathrm{eq}} + \frac{1}{(2\pi)^3}\int d\mathbf{k}\frac{(\epsilon_k+\Delta)}{E_k}\mathcal{N}_{k_1}^{\mathrm{eq}}, \quad (10)$$

where $\mathrm{n}_0^{\mathrm{eq}}$ is the density of particles that have condensed into the state with $k = 0$. We will use this relation between the particle density and the condensate density in our computations of equilibrium quantities. There is some experimental evidence [5] that the approximation in Eq. (10) limits our theory to temperatures below about $0.6T_C$. In order to simplify notation, and without loss of generality, we shall assume that the superfluid velocity $\mathbf{v}_s$ and the wave vector $\mathbf{q}$ are directed along the z axis.

The definition of hydrodynamic modes is determined by properties of the collision operator. This becomes clear if we consider Eq. (7) for a spatially homogeneous gas ($q = 0$). Let us expand $h(\mathbf{k},q,\omega)$ in the form

$$h(\mathbf{k},q,\omega) = \frac{1}{\sqrt{k^2\mathcal{N}_k^{\mathrm{eq}}\mathcal{F}_k^{\mathrm{eq}}}}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}h_\ell^m(k,q,\omega)\mathrm{Y}_\ell^m(\hat{\mathbf{k}}). \quad (11)$$

Then, for $q = 0$ and Eq. (7) reduces to

$$\omega h_\ell^m(k,0,\omega) = i\int_0^\infty dk_1\mathcal{C}_\ell(k,k_1)h_\ell^m(k_1,0,\omega). \quad (12)$$

The right-hand side of Eq. (12) behaves as a linear operator, and we denote its eigenfunctions and eigenvalues as $\psi_{\beta,\ell}(k)$ and $\lambda_{\beta,\ell}$, respectively (Appendix A). Since the eigenfunctions $\psi_{\beta,\ell}(k)$ form a complete set, $h_\ell(k_1,0,\omega)$ can be expanded in terms of them. For the case when $h_\ell(k,0,\omega) = \psi_{\beta,\ell}(k)$, where $\psi_{\beta,\ell}(k)$ is an eigenvector of $\mathcal{C}_\ell(k,k_1)$ with eigenvalue $\lambda_{\beta,\ell}$, then $\omega = i\lambda_{\beta,\ell}$ and the mode $h_\ell^m(k,0,t) \sim \psi_{\beta,\ell}(k)\mathrm{e}^{-\lambda_{\beta,\ell}t}$ decays at a rate determined by the eigenvalue of the collision operator. The hydrodynamic modes are modes with eigenvalue $\lambda_{\beta,\ell} = 0$ and to zeroth order in q, they do not decay at all. The decay rate of the hydrodynamic modes is determined by terms that depend on q. The derivation of the decay rates of hydrodynamic modes is the topic of subsequent sections.

III. EXPAND KINETIC EQUATION IN POWERS OF q

The hydrodynamic modes in a BEC describe the dynamics of quantities that are conserved during collisions (bogolon momentum and energy in the BEC) and they describe the dynamics of the macroscopic phase. The wavelength of hydrodynamic modes is long compared to the microscopic scattering lengths in the gas and, for a hydrodynamic mode, the rate of decay depends on the wavelength of the inhomogeneity. For long-wavelength inhomogeneities (small q), we can obtain the speed and lifetime of a hydrodynamic mode by expanding the dispersion relation for the mode in powers of q. Thus, we write ω and $h(\mathbf{k},q,\omega)$ in the form

$$\omega = \omega^{(0)} + q\omega^{(1)} + q^2\omega^{(2)} + \cdots \tag{13}$$

and

$$h(\mathbf{k},q,\omega) = \frac{1}{\sqrt{k^2\mathcal{N}_k^{\mathrm{eq}}\mathcal{F}_k^{\mathrm{eq}}}}[\Psi^{(0)}(\mathbf{k},\omega) + q\Psi^{(1)}(\mathbf{k},\omega) + q^2\Psi^{(2)}(\mathbf{k},\omega) + \cdots]. \tag{14}$$

Since the hydrodynamic modes are fourfold degenerate for $q=0$, the zeroth-order term $\Psi^{(0)}(\mathbf{k},\omega)$ will, in general, be a superposition of all four hydrodynamic eigenfunctions of the collision operator $\mathcal{C}(\mathbf{k},\mathbf{k}_1)$ discussed in Appendix A. Considering only hydrodynamic modes (the nonhydrodynamic modes decay on a faster timescale), the zeroth-order term takes the form

$$\Psi^{(0)}(\mathbf{k},\omega) = \sum_{\ell=0}^{1}\sum_{m=-\ell}^{\ell}\Gamma_{0,\ell,m}\psi_{0,\ell}(k)\mathrm{Y}_\ell^m(\hat{\mathbf{k}}), \tag{15}$$

where

$$\psi_{0,0}(k) = D_{0,0}E_k\sqrt{k^2\mathcal{N}_k^{\mathrm{eq}}\mathcal{F}_k^{\mathrm{eq}}} \quad \text{and} \quad \psi_{0,1}(k) = D_{0,1}k\sqrt{k^2\mathcal{N}_k^{\mathrm{eq}}\mathcal{F}_k^{\mathrm{eq}}}. \tag{16}$$

The coefficients $\Gamma_{0,\ell,m}$ are determined by the perturbation theory and $D_{0,0}$ and $D_{0,1}$ are normalization coefficients. When Eqs. (13) and (14) are substituted into Eq. (7), we obtain

$$\begin{aligned}(\omega^{(0)} + q\omega^{(1)} + \cdots)&[\Psi^{(0)}(\mathbf{k},\omega) + q\Psi^{(1)}(\mathbf{k},\omega) + \cdots] \\ &= qk_z\mathcal{B}_k[\Psi^{(0)}(\mathbf{k},\omega) + q\Psi^{(1)}(\mathbf{k},\omega) + \cdots] + \frac{q^3}{(\omega^{(0)} + q\omega^{(1)} + \cdots)^2 - v_B^2q^2}\frac{g\hbar k}{m^2}\sqrt{\frac{\mathcal{N}_k^{\mathrm{eq}}}{\mathcal{F}_k^{\mathrm{eq}}}}\frac{1}{(2\pi)^3}\int_0^\infty dk_1\int d\Omega_1 k_{1,z}\sqrt{k_1^2\mathcal{N}_{k_1}^{\mathrm{eq}}\mathcal{F}_{k_1}^{\mathrm{eq}}} \\ &\quad\times[\Psi^{(0)}(\mathbf{k}_1,\omega) + q\Psi^{(1)}(\mathbf{k}_1,\omega) + \cdots] + i\int_0^\infty dk_1\int d\Omega_1\,\hat{\mathcal{C}}(\mathbf{k},\mathbf{k}_1)[\Psi^{(0)}(\mathbf{k}_1,\omega) + q\Psi^{(1)}(\mathbf{k}_1,\omega) + \cdots].\end{aligned} \tag{17}$$

We can now examine this equation at each order in q.

A. Zeroth order

When we obtain the dispersion relation of the hydrodynamic modes as a perturbation expansion in powers of q, we must take into account the fact that at zeroth order the modes are fourfold degenerate. To zeroth order in q, Eq. (17) takes the form

$$\omega^{(0)}\Psi^{(0)}(\mathbf{k}_1,\omega) = i\int_0^\infty dk_1\int d\Omega_1\,\mathcal{C}(\mathbf{k},\mathbf{k}_1)\Psi^{(0)}(\mathbf{k}_1,\omega). \tag{18}$$

The degeneracy of the zeroth-order dispersion relation is lifted by terms of higher order in q in the perturbation expansion. We will find that the matrix that is needed to lift the degeneracy at first order in q is not symmetric. Therefore, we need to introduce "left" and "right" zeroth-order eigenstates. We denote the zeroth-order right eigenstates as

$$\Psi_R^{(0)}(\mathbf{k},\omega) = \sum_{\ell=0}^{1}\sum_{m=-\ell}^{\ell}\Gamma_{0,\ell,m}^{R}\psi_{0,\ell}(k)\mathrm{Y}_\ell^m(\hat{\mathbf{k}}), \tag{19}$$

and the left eigenstates as

$$\Psi_L^{(0)}(\mathbf{k},\omega) = \sum_{\ell=0}^{1}\sum_{m=-\ell}^{\ell}\Gamma_{0,\ell,m}^{L}\psi_{0,\ell}(k)\mathrm{Y}_\ell^m(\hat{\mathbf{k}}). \tag{20}$$

In Eq. (18), $\Psi^{(0)}(\mathbf{k},\omega) = \Psi_R^{(0)}(\mathbf{k},\omega)$, and we find that

$$\int_0^\infty dk_1\int d\Omega_1\mathcal{C}(\mathbf{k},\mathbf{k}_1)\Psi_R^{(0)}(\mathbf{k}_1,\omega) = 0 \tag{21}$$

because the states $\psi_{0,\ell}(k)$ (for $\ell = 0,1$) are eigenstates of $\mathcal{C}_\ell(k,k_1)$ with eigenvalues $\lambda_{0,\ell} = 0$. Therefore, $\omega^{(0)} = 0$ for these hydrodynamic modes, as expected.

B. First order

If we keep terms that are first order in q in Eq. (17), we obtain

$$\begin{aligned}\omega^{(1)}\Psi_R^{(0)}(\mathbf{k},\omega) &= k\mathcal{B}_k\left[\sqrt{\frac{4\pi}{3}}\mathrm{Y}_1^0(\hat{\mathbf{k}})\right]\Psi_R^{(0)}(\mathbf{k},\omega) \\ &\quad+\frac{1}{(\omega^{(1)})^2 - v_B^2}\frac{g\hbar k}{m^2}\sqrt{\frac{\mathcal{N}_k^{\mathrm{eq}}}{\mathcal{F}_k^{\mathrm{eq}}}}[\sqrt{4\pi}\mathrm{Y}_0^0(\hat{\mathbf{k}})] \\ &\quad\times\frac{1}{(2\pi)^3}\int_0^\infty dk_1k_1\int d\Omega_1\left[\sqrt{\frac{4\pi}{3}}\mathrm{Y}_1^0(\hat{\mathbf{k}}_1)\right] \\ &\quad\times\sqrt{k_1^2\mathcal{N}_{k_1}^{\mathrm{eq}}\mathcal{F}_{k_1}^{\mathrm{eq}}}\Psi_R^{(0)}(\mathbf{k}_1,\omega) \\ &\quad+i\int_0^\infty dk_1\int d\Omega_1\mathcal{C}(\mathbf{k},\mathbf{k}_1)\Psi^{(1)}(\mathbf{k}_1,\omega),\end{aligned} \tag{22}$$

where we used the fact that $k_z = k\sqrt{\frac{4\pi}{3}}\mathrm{Y}_1^0(\hat{\mathbf{k}})$ and $1 = \sqrt{4\pi}\mathrm{Y}_0^0(\hat{\mathbf{k}})$. We can then use the orthonormality of spherical harmonics to simplify subsequent calculations.

If we multiply on the left by $\Psi_R^{(0)}(\mathbf{k},\omega)$ and integrate over dk and $d\Omega$, we can use the orthonormality of the spherical harmonics and of the states $\psi_{\beta,\ell}(k)$ (see Appendix A) to obtain

$$\begin{aligned}\omega^{(1)}&[\Gamma^L_{0,0,0}\Gamma^R_{0,0,0}+\Gamma^L_{0,1,0}\Gamma^R_{0,1,0}+\Gamma^L_{0,1,1}\Gamma^R_{0,1,1}+\Gamma^L_{0,1,-1}\Gamma^R_{0,1,-1}]\\&=\frac{1}{\sqrt{3}}\langle\psi_{0,1}|\hat{B}\hat{k}|\psi_{0,0}\rangle\Gamma^L_{0,1,0}\Gamma^R_{0,0,0}+\frac{1}{\sqrt{3}}\langle\psi_{0,0}|\hat{B}\hat{k}|\psi_{0,1}\rangle\Gamma^L_{0,0,0}\Gamma^R_{0,1,0}\\&\quad+\frac{1}{\sqrt{3}}\frac{1}{2\pi^2}\left[\frac{1}{(\omega^{(1)})^2-v_B^2}\right]\frac{g\hbar}{m^2}\langle\psi_{0,0}|\hat{k}|\Xi\rangle\frac{1}{D_{0,1}}\Gamma^L_{0,0,0}\Gamma^R_{0,1,0},\end{aligned}\tag{23}$$

where $\langle k|\Xi\rangle = \sqrt{\frac{\mathcal{N}_k^{\mathrm{eq}}}{\mathcal{F}_k^{\mathrm{eq}}}}$ and $\langle\psi_{0,1}|\hat{B}\hat{k}|\psi_{0,0}\rangle \equiv \int_0^\infty dk\,\psi_{0,1}(k)\mathcal{B}_k k\psi_{0,0}(k)$. Note that the terms involving B_k require an integration over a product of three spherical harmonics.

We can now write Eq. (23) in the form

$$\bar{\Gamma}_L\begin{pmatrix}-\omega^{(1)} & \alpha+\frac{\gamma}{(\omega^{(1)})^2-v_B^2} & 0 & 0\\ \alpha & -\omega^{(1)} & & 0\\ 0 & 0 & -\omega^{(1)} & 0\\ 0 & 0 & 0 & -\omega^{(1)}\end{pmatrix}\bar{\Gamma}_R^T=0,\tag{24}$$

where

$$\begin{aligned}\bar{\Gamma}_L&=\left(\Gamma^L_{0,0,0},\Gamma^L_{0,1,0},\Gamma^L_{0,1,1},\Gamma^L_{0,1,-1}\right)\\ \text{and}\quad\bar{\Gamma}_R^T&=\left(\Gamma^R_{0,0,0},\Gamma^R_{0,1,0},\Gamma^R_{0,1,1},\Gamma^R_{0,1,-1}\right)^T\end{aligned}\tag{25}$$

are row and column matrices (T denotes transpose), respectively,

$$\alpha=\frac{1}{\sqrt{3}}\langle\psi_{0,1}|\hat{B}\hat{k}|\psi_{0,0}\rangle,\tag{26}$$

and

$$\gamma=\frac{g\hbar}{m^2}\frac{1}{\sqrt{3}}\frac{1}{2\pi^2}\langle\psi_{0,0}|\hat{k}|\Xi\rangle\frac{1}{D_{0,1}}.\tag{27}$$

It is clear from Eq. (24) that the longitudinal modes ($\Gamma_{0,0,0}$ and $\Gamma_{0,1,0}$) decouple from the transverse modes ($\Gamma_{0,1,\pm1}$).

If we set the determinant of the matrix in Eq. (24) to zero, we obtain

$$(\omega^{(1)})^2\left[(\omega^{(1)})^2-\alpha^2+\frac{\gamma\alpha}{(\omega^{(1)})^2-v_B^2}\right]=0.\tag{28}$$

The two solutions with $\omega^{(1)}=0$ correspond to the transverse modes and indicate that transverse modes are nonpropagating modes. In addition, there are four solutions to the equation

$$(\omega^{(1)})^2-\alpha^2+\frac{\gamma\alpha}{(\omega^{(1)})^2-v_B^2}=0.\tag{29}$$

These give the speeds of the two pairs of longitudinal modes. It is straightforward to show that the speeds of the "slow" modes are

$$\omega_2^{(1)}=-\omega_1^{(1)}=\frac{1}{\sqrt{2}}\sqrt{v_B^2+\alpha^2-\sqrt{\left(v_B^2-\alpha^2\right)^2+4\alpha\gamma}}\tag{30}$$

and the speeds of the "fast" modes are

$$\omega_4^{(1)}=-\omega_3^{(1)}=\frac{1}{\sqrt{2}}\sqrt{v_B^2+\alpha^2+\sqrt{\left(v_B^2-\alpha^2\right)^2+4\alpha\gamma}}.\tag{31}$$

In Appendix B, we obtain the limiting value of these sound speeds at low temperature. At $T=0$ K, $\omega_2^{(1)}=\frac{1}{\sqrt{3}}v_B$ and $\omega_4^{(1)}=v_B$, where $v_B=\sqrt{gn^{\mathrm{eq}}/m}$ is the Bogoliubov speed of sound.

From Eq. (24), we see that the zeroth-order modes $\psi_{1,\pm1}$ decouple from the zeroth-order modes $\psi_{0,0}$ and $\psi_{1,0}$. The modes $\psi_{1,\pm1}$ are zeroth-order transverse hydrodynamic modes. The decay rates for these transverse modes are straightforward to obtain and have been discussed in [1,3]. The decay rates for the longitudinal modes are derived and analyzed in subsequent sections.

IV. DECAY RATES FOR LONGITUDINAL MODES

Although there are only two longitudinal eigenfunctions of the collision operator, due to the presence of the macroscopic phase there are four longitudinal hydrodynamic modes whose dispersion relations, to second order in q, are given by $\omega=\pm\omega_j^{(1)}q+\omega_j^{(2)}q^2$ with $j=1,3$ [see Eqs. (30) and (31)]. To compute the second-order contribution $\omega_j^{(2)}$, we first must solve for $\Psi^{(1)}(\mathbf{k},\omega)$.

Let us return to Eq. (22) and write $\Psi_R^{(0)}(\mathbf{k},\omega)$ in terms of the zeroth-order longitudinal modes so that

$$\Psi_R^{(0)}(\mathbf{k},\omega)=\Gamma^R_{0,0,0}\psi_{0,0}(k)\mathrm{Y}_0^0(\hat{\mathbf{k}})+\Gamma^R_{0,1,0}\psi_{0,1}(k)\mathrm{Y}_1^0(\hat{\mathbf{k}}).\tag{32}$$

Expressions for $\Gamma^R_{0,0,0}$ and $\Gamma^R_{0,1,0}$ are given in Appendix C. If we note the form of the collision operator given in Eq. (8), we see that the state $\Psi^{(1)}(\mathbf{k}_1,\omega)$ has contributions from three different spherical harmonics and can be written

$$\begin{aligned}\Psi^{(1)}(\mathbf{k},\omega)&=\Psi^{(1)}_{0,0}(k,\omega)\mathrm{Y}_0^0(\hat{\mathbf{k}})+\Psi^{(1)}_{1,0}(k,\omega)\mathrm{Y}_1^0(\hat{\mathbf{k}})\\&\quad+\Psi^{(1)}_{2,0}(k,\omega)\mathrm{Y}_2^0(\hat{\mathbf{k}}).\end{aligned}\tag{33}$$

We now proceed to obtain these three contributions.

To obtain $\Psi^{(1)}_{0,0}(k)$, first multiply Eq. (22) by $d\Omega_1 \mathrm{Y}_0^{0*}(\hat{\mathbf{k}})$ and integrate to get

$$\omega^{(1)}\psi_{0,0}(k)\Gamma^R_{0,0,0} = \frac{1}{\sqrt{3}}\mathcal{B}_k k\psi_{0,1}(k)\Gamma^R_{0,1,0} + \frac{1}{(\omega^{(1)})^2 - v_B^2}\frac{g\hbar k}{m^2}\sqrt{\frac{\mathcal{N}_k^{\rm eq}}{\mathcal{F}_k^{\rm eq}}}\frac{1}{\sqrt{3}}\frac{1}{D_{0,1}}\frac{1}{2\pi^2}\Gamma^R_{0,1,0} + i\int_0^\infty dk_1 \mathcal{C}_0(k,k_1)\Psi^{(1)}_{0,0}(k_1,\omega). \quad (34)$$

It is useful to write $\Psi^{(1)}_{0,0}(k,\omega)$ and the equation for $\Psi^{(1)}_{0,0}(k,\omega)$ in abstract notation so that $\Psi^{(1)}_{0,0}(k,\omega) = \langle k|\Psi^{(1)}_{0,0}(\omega)\rangle$. Then, Eq. (34) takes the form

$$\omega^{(1)}|\psi_{0,0}\rangle\Gamma^R_{0,0,0} = \frac{1}{\sqrt{3}}\hat{\mathcal{B}}\hat{k}|\psi_{0,1}\rangle\Gamma^R_{0,1,0} + \frac{1}{(\omega^{(1)})^2 - v_B^2}\frac{g\hbar}{m^2}\frac{1}{\sqrt{3}}\frac{1}{2\pi^2}\frac{1}{D_{0,1}}\hat{k}|\Xi\rangle\Gamma^R_{0,1,0} + i\hat{\mathcal{C}}_0\big|\Psi^{(1)}_{0,0}(\omega)\big\rangle. \quad (35)$$

Note that $\hat{\mathcal{C}}_0|\psi_{0,0}\rangle = 0$, so the expression for $|\Psi^{(1)}_{0,0}(\omega)\rangle$ takes the form

$$\big|\Psi^{(1)}_{0,0}(\omega)\big\rangle = \frac{i}{\hat{\mathcal{C}}_0}\left[\frac{1}{\sqrt{3}}\hat{\mathcal{B}}\hat{k}|\psi_{0,1}\rangle + \frac{1}{(\omega^{(1)})^2 - v_B^2}\frac{g\hbar}{m^2}\frac{1}{\sqrt{3}}\frac{1}{D_{0,1}}\frac{1}{2\pi^2}\hat{k}|\Xi\rangle\right]\Gamma^R_{0,1,0}. \quad (36)$$

Next, we obtain an expression for $|\Psi^{(1)}_{1,0}(\omega)\rangle$. Multiply Eq. (22) by $\int d\Omega\, \mathrm{Y}_1^{0*}(\hat{\mathbf{k}})$ and follow steps similar to those used to obtain $|\Psi^{(1)}_{0,0}(\omega)\rangle$. We obtain

$$\big|\Psi^{(1)}_{1,0}(\omega)\big\rangle = \frac{i}{\hat{\mathcal{C}}_1}\left[\frac{1}{\sqrt{3}}\hat{\mathcal{B}}_k\hat{k}|\psi_{0,0}\rangle\Gamma^R_{0,0,0}\right]. \quad (37)$$

Finally, multiply Eq. (22) by $\int d\Omega\, \mathrm{Y}_2^{0*}(\hat{\mathbf{k}})$ and integrate to obtain

$$\big|\Psi^{(1)}_{2,0}(\omega)\big\rangle = \frac{i}{\hat{\mathcal{C}}_2}\left[\frac{2}{\sqrt{15}}\hat{\mathcal{B}}_k\hat{k}|\psi_{0,1}\rangle\Gamma^R_{0,1,0}\right]. \quad (38)$$

Note that the operators $\hat{\mathcal{C}}_\ell^{-1}$, in these expressions, always act on states that are orthogonal to the hydrodynamic eigenfunctions so the expressions are well defined. We can now use these results to find $\omega^{(2)}$.

A. Solve for $\omega^{(2)}$

If we retain terms of order q^2 in Eq. (17), we can write

$$\begin{aligned}
&\omega^{(2)}\Psi^{(0)}(\mathbf{k},\omega) + \omega^{(1)}\Psi^{(1)}(\mathbf{k},\omega)\\
&= \sqrt{\frac{4\pi}{3}}\mathrm{Y}_1^0(\hat{\mathbf{k}})k\mathcal{B}_k\Psi^{(1)}(\mathbf{k},\omega) + \frac{1}{(\omega^{(1)})^2 - v_B^2}\frac{1}{2\pi^2\sqrt{3}}\frac{g\hbar k}{m^2}\mathrm{Y}_0^0(\hat{\mathbf{k}})\sqrt{\frac{\mathcal{N}_k^{\rm eq}}{\mathcal{F}_k^{\rm eq}}}\\
&\quad\times\int_0^\infty dk_1\int d\Omega_1 k_1 \mathrm{Y}_1^0(\hat{\mathbf{k}}_1)\sqrt{k_1^2\mathcal{N}_{k_1}^{\rm eq}\mathcal{F}_{k_1}^{\rm eq}}\Psi^{(1)}(\mathbf{k}_1,\omega) - \frac{\omega^{(1)}\omega^{(2)}}{\left[(\omega^{(1)})^2 - v_B^2\right]^2}\frac{1}{2\pi^2\sqrt{3}}\frac{g\hbar k}{m^2}\mathrm{Y}_0^0(\hat{\mathbf{k}})\sqrt{\frac{\mathcal{N}_k^{\rm eq}}{\mathcal{F}_k^{\rm eq}}}\\
&\quad\times\int_0^\infty dk_1\int d\Omega_1 k_1 \mathrm{Y}_1^0(\hat{\mathbf{k}}_1)\sqrt{k_1^2\mathcal{N}_{k_1}^{\rm eq}\mathcal{F}_{k_1}^{\rm eq}}\Psi^{(0)}(\mathbf{k}_1,\omega) + i\int_0^\infty dk_1\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\mathcal{C}_\ell(k,k_1)\mathrm{Y}_\ell^m(\hat{\mathbf{k}})\Psi^{(2)}_{\ell,m}(k_1,\omega). \quad (39)
\end{aligned}$$

We now multiply Eq. (39), on the left, by

$$\Psi_L^{(0)}(\mathbf{k},\omega) = \Gamma^L_{0,0,0}\psi_{0,0}(k)\mathrm{Y}_0^{0*}(\hat{\mathbf{k}}) + \Gamma^L_{0,1,0}\psi_{0,1}(k)\mathrm{Y}_1^{0*}(\hat{\mathbf{k}}) \quad (40)$$

and integrate over $\int dk\, d\Omega$. Since we require that the zeroth-order states be orthonormal, we have

$$\Gamma^L_{0,0,0}\Gamma^R_{0,0,0} + \Gamma^L_{0,1,0}\Gamma^L_{0,1,0} = 1. \quad (41)$$

Also, note that $\Gamma^L_{0,0,0}\Gamma^R_{0,0,0} = \Gamma^L_{0,1,0}\Gamma^R_{0,1,0} = \frac{1}{2}$, and $\Gamma^L_{0,0,0}\Gamma^R_{0,1,0} = \frac{1}{2}\frac{\omega^{(1)}}{\alpha}$ (see Appendix C). The last term in Eq. (39) drops out because $\hat{\mathcal{C}}_\ell|\psi_{0,\ell}\rangle = 0$ for $\ell = 0,1$. The term that involves $\omega^{(1)}$, on the left-hand side of Eq. (39), makes no contribution for the same reason.

From the definition of the first-order states in Eqs. (36), (37), and (38), we then obtain

$$\begin{aligned}
\omega^{(2)} = \frac{i}{1+\mathcal{S}}\bigg\{&\frac{1}{6}\langle\psi_{0,0}|\hat{\mathcal{B}}_k\,\hat{k}\frac{1}{\hat{\mathcal{C}}_1}\hat{\mathcal{B}}_k\hat{k}|\psi_{0,0}\rangle\\
&+\frac{1}{6}\langle\psi_{0,1}|\hat{\mathcal{B}}_k\hat{k}\frac{1}{\hat{\mathcal{C}}_0}\hat{\mathcal{B}}\hat{k}|\psi_{0,1}\rangle + \frac{2}{15}\langle\psi_{0,1}|\hat{\mathcal{B}}_k\hat{k}\frac{1}{\hat{\mathcal{C}}_2}\hat{\mathcal{B}}_k\hat{k}|\psi_{0,1}\rangle\\
&+\frac{1}{12\pi^2}\frac{1}{D_{0,1}}\frac{1}{(\omega^{(1)})^2 - v_B^2}\frac{g\hbar}{m^2}\langle\psi_{0,1}|\hat{\mathcal{B}}_k\,\hat{k}\frac{1}{\hat{\mathcal{C}}_0}\hat{k}|\Xi\rangle\bigg\},
\end{aligned} \quad (42)$$

where

$$\mathcal{S} = \left[\frac{(\omega^{(1)})^2}{[(\omega^{(1)})^2 - v_B^2]^2}\right] \frac{g\hbar}{m^2\alpha} \frac{1}{\sqrt{3}} \frac{1}{2\pi^2} \frac{1}{D_{0,1}} \langle\psi_{0,0}|\hat{k}|\Xi\rangle. \quad (43)$$

The quantity $(\omega^{(2)}q^2)^{-1}$ gives the lifetime of the sound modes in the BEC. The lifetimes depend on the speed of the sound modes and generally will be different for the two different types of sound that can propagate in a BEC. The first three terms on the right-hand side (rhs) of Eq. (42) are correlation functions of a type similar to those that appear in lifetimes of gases above the condensation temperature T_C. Above T_C, they determine the thermal conduction, bulk viscosity, and shear viscosity, respectively [12]. The last term on the rhs is a consequence of the macroscopic phase due to broken gauge symmetry in the BEC below the condensation temperature.

B. Lifetimes and speeds in dimensionless variables

In order to obtain numerical values for the sound speeds and lifetimes, we rewrite Eqs. (42) and (43) in terms of dimensionless variables. We introduce a dimensionless momentum vector $\mathbf{c}$ so that $\mathbf{k} = \sqrt{\frac{2mk_BT}{\hbar^2}}\mathbf{c}$. Then, $E_k = k_BT\mathcal{E}_c$, where $\mathcal{E}_c = \sqrt{(c^2+b^2)^2 - b^2}$ and $b = \frac{\Delta}{k_BT}$. Also, note that the collision operator and its eigenvalues have units of inverse time. In [2] it was shown that $\lambda_{\beta,\ell} = \gamma_d\lambda^0_{\beta,\ell}$, where $\lambda^0_{\beta,\ell}$ are dimensionless eigenvalues and

$$\gamma_d = \frac{8ma^2(k_BT)^2}{\pi\hbar^3}. \quad (44)$$

In order to preserve the normalization of the eigenfunctions, we must have

$$\int_0^\infty dk\, \psi_{\beta,\ell}(k)\psi_{\beta',\ell}(k) = \int_0^\infty dc\, \psi_{\beta,\ell}(c)\psi_{\beta',\ell}(c) = \delta_{\beta,\beta'}. \quad (45)$$

This requires that the dimensionless eigenfunction $\psi_{\beta,\ell}(c)$ satisfy the relation $\psi_{\beta,\ell}(k) = (\frac{\hbar^2}{2mk_BT})^{1/4}\psi_{\beta,\ell}(c)$. If we note that $\psi_{0,1}(k) = D_{0,1}k^2\sqrt{\mathcal{N}_k^{\rm eq}\mathcal{F}_k^{\rm eq}}$ and $\psi_{0,0}(k) = D_{0,0}kE_k\sqrt{\mathcal{N}_k^{\rm eq}\mathcal{F}_k^{\rm eq}}$ (see Appendix A), then the normalization condition gives

$$D_{0,1} = \left(\frac{2mk_BT}{\hbar^2}\right)^{-5/4}\mathcal{D}_1 \quad \text{with} \quad \mathcal{D}_1 = \left[\int_0^\infty dc\, c^4\mathcal{N}_c^{\rm eq}\mathcal{F}_c^{\rm eq}\right]^{-1/2} \quad (46)$$

and

$$D_{0,0} = \left(\frac{2mk_BT}{\hbar^2}\right)^{-3/4}(k_BT)^{-1}\mathcal{D}_0 \quad \text{with} \quad \mathcal{D}_0 = \left[\int_0^\infty dc\, c^2\mathcal{E}_c^2\mathcal{N}_c^{\rm eq}\mathcal{F}_c^{\rm eq}\right]^{-1/2}. \quad (47)$$

The critical temperature is $T_C = \frac{2\pi\hbar^2}{mk_B}(\frac{n^{\rm eq}}{\zeta(3/2)})^{2/3}$ (we use the ideal gas relation).

1. Sound speeds

The sound speeds are given in Eqs. (30) and (31). We can write

$$\frac{v_{\rm fast}}{v_B} = \frac{1}{\sqrt{2}}\sqrt{1 + \frac{\alpha^2}{v_B^2} + \sqrt{\left(1 - \frac{\alpha^2}{v_B^2}\right)^2 + \frac{4\alpha\gamma}{v_B^4}}} \quad (48)$$

and

$$\frac{v_{\rm slow}}{v_B} = \frac{1}{\sqrt{2}}\sqrt{1 + \frac{\alpha^2}{v_B^2} - \sqrt{\left(1 - \frac{\alpha^2}{v_B^2}\right)^2 + \frac{4\alpha\gamma}{v_B^4}}}. \quad (49)$$

From the definition of α and γ, it can be shown that

$$\alpha = \frac{v_T}{\sqrt{3}}\mathcal{D}_0\mathcal{D}_1\int_0^\infty dc\, \mathcal{B}_c c^4\mathcal{E}_c\mathcal{N}_c^{\rm eq}\mathcal{F}_c^{\rm eq} \quad (50)$$

and

$$\gamma = \frac{1}{\sqrt{3}}\frac{v_Tv_B^2}{n^{\rm eq}\lambda_T^3}\frac{\mathcal{D}_0}{\mathcal{D}_1}\frac{4}{\sqrt{\pi}}\int_0^\infty dc\, c^2\mathcal{E}_c\mathcal{N}_c^{\rm eq}, \quad (51)$$

where $v_T = \sqrt{2k_BT/m}$ is a "thermal" speed and $\mathcal{B}_c = \frac{c^2+b}{\mathcal{E}_c}$. The ratio of v_T to the Bogoliubov speed v_B can be written as

$$\frac{v_T}{v_B} = \frac{1}{(n^{\rm eq}a^3)^{1/6}\zeta(3/2)^{1/3}}\sqrt{\frac{T}{T_C}}. \quad (52)$$

These dimensionless sound speeds in Eqs. (48) and (49) are plotted in Fig. 1 for three different densities: (a) $n^{\rm eq}a^3 = 10^{-4}$, (b) $n^{\rm eq}a^3 = 10^{-5}$, and (c) $n^{\rm eq}a^3 = 10^{-6}$. The speeds in Fig. 1 are scaled in terms of the Bogoliubov speed $v_B = \sqrt{gn^{\rm eq}/m}$.

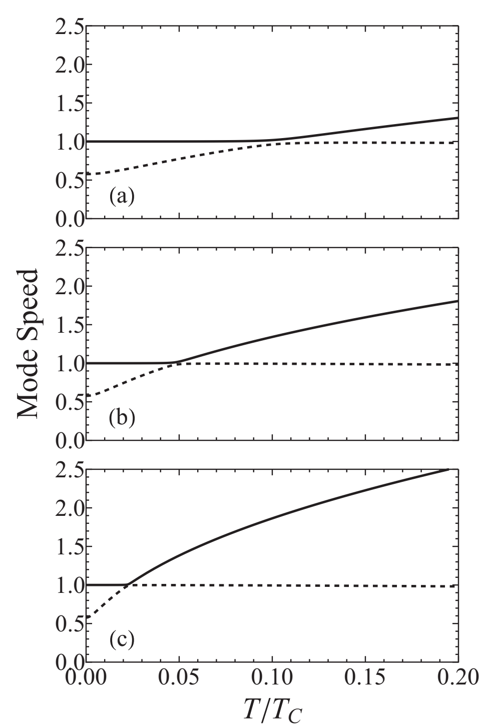

FIG. 1. Propagation speeds of the fast (solid line) and slow (dashed line) longitudinal modes at (a) $na^3 = 10^{-4}$, (b) 10^{-5}, and (c) 10^{-6} in units of $v_B = \sqrt{gn^{\rm eq}/m}$.

The sound speeds undergo an avoided crossing which moves to lower temperature as the density decreases.

2. Decay rate

We can write Eq. (42) in the form

$$\omega^{(2)} = \frac{i}{1+\mathcal{S}}\frac{\hbar}{8m}\left[\frac{\zeta(3/2)}{n^{\rm eq}a^3}\right]^{2/3}\frac{T_C}{T}\left[\frac{1}{6}C_1+\frac{1}{6}C_0+\frac{2}{15}C_2 + \frac{1}{(\omega^{(1)})^2-v_B^2}\frac{2n^{\rm eq}a^3}{3\sqrt{\pi}\zeta(3/2)}\left(\frac{T}{T_C}\right)^{3/2}C_0'\right], \qquad (53)$$

where now

$$\mathcal{S} = \frac{(\omega^{(1)})^2}{\left[(\omega^{(1)})^2-v_B^2\right]^2}\frac{\gamma}{\alpha} \qquad (54)$$

and we have introduced the four dimensionless correlation functions

$$C_1 = \sum_{\beta=1}^{\infty}\frac{1}{\lambda^0_{\beta,1}}\left[\mathcal{D}_0^2\int_0^\infty dc\sqrt{\mathcal{N}_c^{\rm eq}\mathcal{F}_c^{\rm eq}}B_c\mathcal{E}_c c^2\psi_{\beta,1}(c)\right]^2, \qquad (55)$$

$$C_2 = \sum_{\beta=0}^{\infty}\frac{1}{\lambda^0_{\beta,2}}\left[\mathcal{D}_1^2\int_0^\infty dc\sqrt{\mathcal{N}_c^{\rm eq}\mathcal{F}_c^{\rm eq}}B_c c^3\psi_{\beta,2}(c)\right]^2, \qquad (56)$$

$$C_0 = \sum_{\beta=1}^{\infty}\frac{1}{\lambda^0_{\beta,0}}\left[\mathcal{D}_1^2\int_0^\infty dc\sqrt{\mathcal{N}_c^{\rm eq}\mathcal{F}_c^{\rm eq}}B_c c^3\psi_{\beta,0}(c)\right]^2, \qquad (57)$$

and

$$C_0' = \sum_{\beta=1}^{\infty}\frac{1}{\lambda^0_{\beta,0}}\left[\int_0^\infty dc\sqrt{\mathcal{N}_c^{\rm eq}\mathcal{F}_c^{\rm eq}}B_c c^3\psi_{\beta,0}(c)\right] \times\left[\int_0^\infty dc\sqrt{\frac{\mathcal{N}_c^{\rm eq}}{\mathcal{F}_c^{\rm eq}}}c\psi_{\beta,0}(c)\right]. \qquad (58)$$

Above the condensation temperature, C_1 and C_2 determine the thermal conductivity and shear viscosity, respectively, and C_0 determines the bulk viscosity. The correlation function C_0' does not contribute above the transition temperature. In Fig. 2, we plot these four correlation functions for density $n^{\rm eq}a^3 = 10^{-5}$. We find that C_1 and C_2 are two orders of magnitude greater than C_0 and C_0' in the condensate. Therefore, decay processes in the condensate appear to be dominated by the shear viscosity and thermal conductivity.

In Fig. 3, we plot the dimensionless lifetime of the sound modes, scaled to the parameter $\frac{mq^2}{\hbar}$, which has units of time. The dimensionless lifetimes are plotted for the same three densities as the sound modes, and exhibit very interesting behavior. At the temperature of the avoided crossing in the sound speeds, the lifetimes are comparable and at a minimum. On each side of the avoided crossing, one of the modes has a much larger lifetime than the other. We note that the mode with the longest lifetime is always the mode with its speed closest to v_B.

One can gain insight into the behavior of the lifetimes by considering the character of each mode. In Fig. 4, we show the relative amplitude of four hydrodynamic quantities: density, superfluid velocity, bogolon energy density, and bogolon momentum density, for both the fast mode (solid line) and the slow mode (dashed line). In order to show quantities on a comparable scale, they are measured in natural units: $n^{\rm eq}$ for density, (v_B) for superfluid velocity, $g(n^{\rm eq})^2$ for bogolon energy density, and $mn^{\rm eq}v_B$ for bogolon momentum density, and they are normalized so that the sum of all four amplitudes adds to one for each of the two types of sound mode (fast and slow). Negative values indicate out-of-phase motion for that quantity.

The amplitudes shown in Fig. 4 are calculated by inserting the expression for $\delta\mathcal{N}(\mathbf{k},q,\omega)$ given by $\Psi_R^{(0)}(q,\omega)$ for each sound mode [see Eq. (32)] into the expressions for bogolon momentum density $[\mathbf{J}(q,\omega) = \frac{1}{(2\pi)^3}\int d\mathbf{k}\,\hbar\mathbf{k}\delta\mathcal{N}(\mathbf{k},q,\omega)]$, bogolon energy density $[E(q,\omega) = \frac{1}{(2\pi)^3}\int d\mathbf{k}\,E_k\delta\mathcal{N}(\mathbf{k},q,\omega)]$,

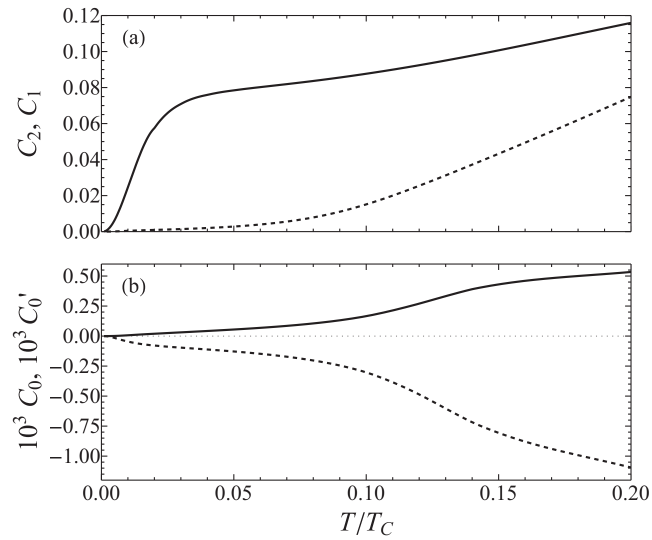

FIG. 2. Dimensionless correlation functions for the BEC. Shown in panel (a) are C_2 (solid line) and C_1 (dashed line). Shown in panel (b) are $10^3 \times C_0$ (solid line) and $10^3 \times C_0'$ (dashed line).

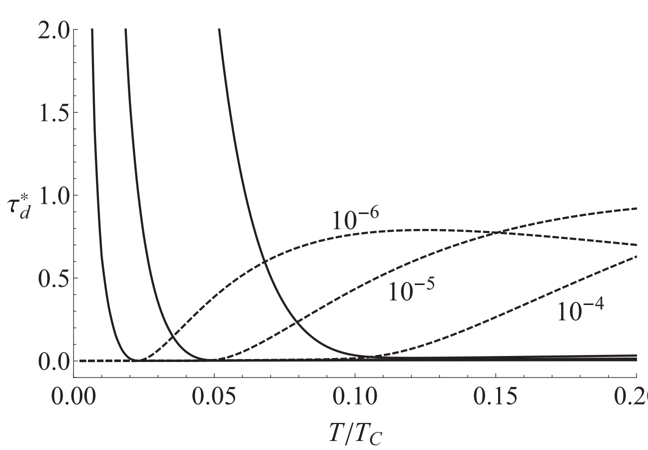

FIG. 3. Dimensionless lifetimes $\tau_d^* = (\hbar q^2/m)\tau_d$ for fast longitudinal modes (solid line) and slow longitudinal (dashed line) modes. Curves are shown for three values of density (na^3). Physical relaxation times may be obtained by $\tau_d = m/(\hbar q^2)\tau_d^*$. For ^{87}Rb at $q = 0.35\ \mu\text{m}^{-1}$, the coefficient is $m/(\hbar q^2) = 11.18$ ms.

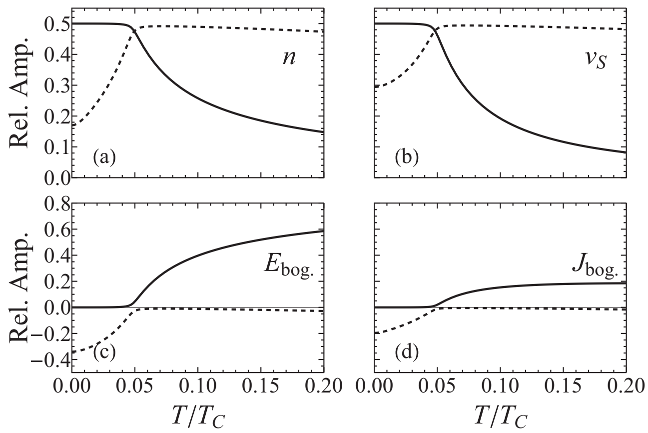

FIG. 4. The relative amplitudes of the (a) density (in units of n^{eq}), (b) superfluid velocity (in units of v_B), (c) bogolon energy density [in units of $g(n^{\text{eq}})^2$], and (d) bogolon momentum density (in units of $mn^{\text{eq}}v_B$) for the fast mode (solid line) and the slow mode (dashed line). For each mode, the sum of these four amplitudes is normalized to one. Positive values indicate in-phase motion and negative values indicate out-of-phase motion. The density in these plots is $n^{\text{eq}}a^3 = 10^{-5}$.

and condensate phase [Eq. (6)]. Superfluid velocity amplitudes are then calculated from the relation $v_s(q,\omega) = -\frac{i\hbar}{m}q\phi(q,\omega)$ and density amplitudes are given by $\delta n(q,\omega) = -\frac{i\hbar}{g}\omega\phi(q,\omega)$, which can be derived from the Hugenholtz-Pines relation [1].

At temperatures below the avoided crossing, the fast mode (solid lines) consists of approximately 50% density waves and 50% superfluid velocity waves, and there is very little participation from the bogolon gas. The fast mode is also very long lived in this region of temperatures with a speed very close to v_B. We interpret this as Bogoliubov sound which is the interplay between density and phase that can occur at zero temperature with no decay.

In contrast, the slow mode (dashed lines) at temperatures below the avoided crossing involves significant bogolon motion in addition to density and superfluid velocity waves. Collisions between bogolons are the primary method of relaxation for the BEC, and so we expect this sound mode to decay rapidly. Indeed, the slow mode has a much smaller lifetime below the avoided crossing, as can be seen in Fig. 3. The slow mode is more similar to classical sound in this region, involving waves in density, momentum density, and energy density.

At temperatures above the avoided crossing, the fast and slow modes essentially exchange characters, the slow mode becoming like Bogoliubov sound, and the fast mode becoming like classical sound. Note however, that in the fast mode, all waves are in phase, while in the slow mode, the bogolon momentum is out of phase with the superfluid velocity, and the bogolon energy density is out of phase with the density. This fact allows one to distinguish the modes at all temperatures. This result, at temperatures above the avoided crossing, is consistent with ZNG theory which is applicable at these higher temperatures (see [6], Chap. 15, and [11]).

The presence of a separate "density" wave and "temperature" wave, such as occurs in superfluid ^{4}He, is not found in the dilute BEC. In superfluid ^{4}He, the clear separation in characteristics of the two types of sound is a consequence of the small compressibility of liquid ^{4}He. In the highly compressible gaseous superfluid, we find instead a separation into "Bogoliubov sound" and "classical sound."

V. COMPARISON TO EXPERIMENT

We now compare our results for the sound speed and lifetime of sound modes to those observed in a ^{87}Rb BEC [10,13]. The wavelength of the sound mode observed in [10] was about 18×10^{-6} m ($q = 0.35\ \mu\text{m}^{-1}$) while the scattering length of ^{87}Rb atoms is about 5.6×10^{-9} m [14]. We will assume the particle density of the ^{87}Rb gas is $n^{\text{eq}} = 9.71 \times 10^{19}\ \text{m}^{-3}$ which, using the expression for the critical temperature of an ideal BEC in a box, gives $T_c = \frac{2\pi\hbar^2}{mk_B}[\frac{n^{\text{eq}}}{\zeta(3/2)}]^{2/3} = 3.90 \times 10^{-7}$ K (this is the condensation temperature reported by [13]). The mean field theory used here is expected to give good results for $T < 0.3T_c$ [1].

Given the value of the density n^{eq} and the scattering length a, the Bogoliubov speed is $v_B \approx 1.887$ mm/s. This agrees with sound speeds observed in the experiment [10], although the dependence of sound speed on temperature was not investigated in the experiment.

The lifetime τ_d of hydrodynamic modes is given by $\tau_d = i/(\omega^{(2)}q^2)$. The lifetimes of the longitudinal modes for the ^{87}Rb BEC are plotted in Fig. 5 as a function of temperature. The dashed line shows the lifetime for modes with speeds $\omega_1^{(1)}$ and $\omega_2^{(1)}$ (the slow modes). The solid line shows the lifetime for modes with speeds $\omega_3^{(1)}$ and $\omega_4^{(1)}$ (the fast modes). We find that the lifetime of one sound mode (dashed line) ranges from 18 μs to 11 ms over the temperature interval 1.0 nK $\leqslant T \leqslant$ 80 nK, while the lifetime of the other sound mode (solid line) ranges from 60 μs to 14 ms in that same temperature interval. The lifetimes of both sound modes drop in the neighborhood of the

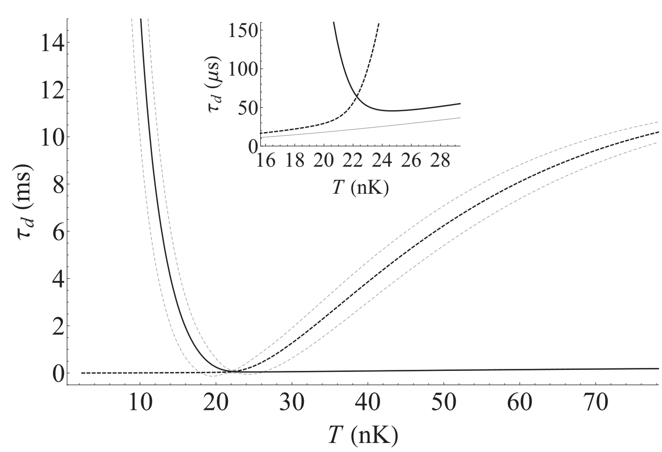

FIG. 5. Physical lifetimes τ_d for ^{87}Rb at $q = 0.35\ \mu\text{m}^{-1}$ in the fast longitudinal (solid line) and slow longitudinal (dashed line) modes. Uncertainty bands are obtained by assuming density $\text{n}^{\text{eq}} = 9.757 \times 10^{19}\ \text{m}^{-3}$ with a 10% uncertainty, $a = (103 \pm 5)a_0$, and assuming a 1% error in the total correlation function. Note the experimental value of 9 ms was obtained for a temperature of $T = 21 \pm 20$ nK. Given our assumed uncertainties, this corresponds to either $T = 11 \pm 1$ nK or $T = 67 \pm 5$ nK.

sound speed avoided crossing and in that temperature regime the identity of the long-lived mode switches.

The lifetimes obtained here for a dilute ^{87}Rb Bose-Einstein condensate use the condensation temperature $T_C = 3.90 \times 10^{-7}$ K reported in [13] and assume the condensate has a spatially uniform density at equilibrium. That is the only input, other than the mass and scattering length of ^{87}Rb. The identity of the longest-lived longitudinal mode switches at temperatures in the neighborhood of the avoided crossing in the sound speeds.

In [10], a one-dimensional (1D) sound mode was created in a harmonic trap with frequencies $f_1 = f_2 = 224$ Hz and $f_3 = 26$ Hz. There were about $\langle N\rangle = 5 \times 10^5$ atoms in the trap giving a critical temperature $T_C = \frac{h}{m}(\frac{\langle N\rangle f_1 f_2 f_3}{1.202})^{1/3} \approx 3.9 \times 10^{-7}$ (the value used here). The sound mode, in the experiment, that we consider here ([10], Fig. 5) had a wave vector $q = 0.35\ \mu\text{m}^{-1}$ and a lifetime of about $\tau_d \sim 9$ ms. The temperature of the BEC was $T = 21 \pm 20$ nK [15], which places the experiment within the temperature regime shown in Fig. 5. The peak density was 3.05×10^{20} particles/m^3 [15], which is slightly higher than that used in our calculation (we chose a density consistent with T_C) for a gas with uniform density). Based on the theoretical results, We estimate that the temperature of the experiment was either $T = 11 \pm 1$ nK or $T = 67 \pm 5$ nK.

Because our computations are done for a BEC in a box with uniform density $n^{\text{eq}} = 9.71 \times 10^{19}\ \text{m}^{-3}$ (and critical temperature $T_C = 3.90 \times 10^{-7}$ K), while the density of a BEC in a trap varies slightly in the region that supports the sound wave, we estimated the change in our decay rates if the density were changed by 10%. The uncertainty in our result, due to a 10% uncertainty in the density, is shown in Fig. 5 by the faint dashed lines that on either side of out result for $n^{\text{eq}} = 9.71 \times 10^{19}\ \text{m}^{-3}$. The uncertainty in the density does not significantly change our prediction for the temperature at which sound waves in [10] were measured.

The value of the sound mode lifetime, predicted by the bogolon kinetic equation (1), is consistent with that reported in [10]. It would be of great interest to explore this temperature regime more thoroughly to determine if there is a variation in lifetime similar to that predicted by the theory. Such a variation could provide a sensitive measure of the temperature for a BEC with a small thermal fraction [15].

VI. CONCLUDING REMARKS

We have used Bogoliubov mean field theory to obtain the speed and lifetime of first and second sound in monatomic BECs. The speeds of the two types of sound undergo an avoided crossing at low temperature and attain limiting values of v_B and $\sqrt{\frac{1}{3}}v_B$ at $T = 0$ K, in agreement with results obtained by Lee and Yang in [8]. We find that the two types of sound have very different lifetimes and that the identity of the long-lived sound mode switches at the temperature of the sound speed avoided crossing.

We find that the two sound modes in the BEC can be interpreted as either a Bogoliubov sound mode that consists primarily of a density wave and a phase (superfluid velocity) wave, or a classical sound mode that consists of waves in density, superfluid velocity, bogolon energy density, and bogolon momentum density. The sharp distinction between a density mode and a temperature mode that occurs in a liquid superfluid is not seen in the gaseous and compressible BEC.

Below the temperature of the avoided crossing, the fast sound mode is similar to Bogoliubov sound and has a long lifetime, while the slow sound mode is similar to classical sound and decays quickly. Above the temperature of the avoided crossing, the fast sound mode is similar to classical sound and, at these low temperatures, has a short lifetime, while the slow sound mode is similar to Bogoliubov sound and, at these low temperatures, is longer lived. We also find that the slow sound mode involves out-of-phase motion at all temperatures, and the fast sound mode consists of in-phase motion.

As the temperature is increased beyond the avoided crossing, the slow sound speed continues to decrease, reaching zero at the transition temperature T_C. The fast sound speed continues to increase as the temperature rises, becoming equal to the speed of sound of a monoatomic Bose gas [12] at temperatures above T_C.

The values of the sound speed and lifetime of the long-lived sound mode are in agreement with those measured in a recent experiment on a ^{87}Rb BEC, although there is a large uncertainty in the temperature of the experiment. The fairly rapid variation that we find in the lifetime of the sound modes with temperature may provide a new means of estimating temperature for systems with very small thermal fraction.

It would also be interesting to see if any of the signatures of the avoided crossing could be observed as the temperature of the BEC is varied. Most prominent would be a sudden inability to sustain any sound modes at the temperature of the avoided crossing. More difficult, although feasible, is the possibility of resolving the two sound-mode frequencies directly by using periodic driving forces.

ACKNOWLEDGMENTS

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APPENDIX A: LINEARIZED BOGOLON COLLISION OPERATOR

From Ref. [2], the collision operator can be written

$$\mathcal{G}_{\mathbf{k}_1}\{h\} = -\mathcal{N}_1^{\text{eq}}\mathcal{F}_1^{\text{eq}}\left[M(\mathbf{k}_1)h_{\mathbf{k}_1} + \int d\mathbf{k}_2\frac{\mathcal{N}_2^{\text{eq}}}{\mathcal{F}_1^{\text{eq}}}K(\mathbf{k}_1,\mathbf{k}_2)h_{\mathbf{k}_2}\right] \tag{A1}$$

with

$$M(\mathbf{k}_1) = \int d\mathbf{k}_3\frac{\mathcal{N}_3^{\text{eq}}}{\mathcal{F}_1^{\text{eq}}}\bigg\{2A_0T_A(1,3) + A_0\frac{\mathcal{F}_3^{\text{eq}}}{\mathcal{N}_3^{\text{eq}}}T_B(1,3) + B_0Q_A(1,3) + B_0Q_B(1,3) + \frac{1}{3}\frac{\mathcal{F}_3^{\text{eq}}}{\mathcal{N}_3^{\text{eq}}}Q_C(1,3)\bigg\}, \tag{A2}$$

$$K(\mathbf{k}_1,\mathbf{k}_2) = \bigg\{2A_0T_A(1,2) - 2A_0\frac{\mathcal{F}_2^{\rm eq}}{\mathcal{N}_2^{\rm eq}}T_B(1,2) - 2A_0\frac{\mathcal{F}_1^{\rm eq}}{\mathcal{N}_1^{\rm eq}}T_B(2,1) + B_0Q_A(1,2) - 2B_0\frac{\mathcal{F}_2^{\rm eq}}{\mathcal{N}_2^{\rm eq}}R_A(1,2) + 2B_0Q_B(1,2) - B_0\frac{\mathcal{F}_2^{\rm eq}}{\mathcal{N}_2^{\rm eq}}Q_C(1,2) - B_0\frac{\mathcal{F}_1^{\rm eq}}{\mathcal{N}_1^{\rm eq}}Q_C(2,1)\bigg\}. \tag{A3}$$

The functions appearing in the above expressions are defined as

$$A_0 = \frac{4\pi N_0 g^2}{(2\pi)^3\hbar V},\quad B_0 = \frac{4\pi g^2}{(2\pi)^6\hbar},$$

$$T_A(\mathbf{k}_1,\mathbf{k}_2) = \int d\mathbf{k}_3\Delta(1+2-3)\big(W^{12}_{1,2,3}\big)^2\mathcal{F}_3^{\rm eq},$$

$$T_B(\mathbf{k}_1,\mathbf{k}_2) = \int d\mathbf{k}_3\Delta(1-2-3)\big(W^{12}_{3,2,1}\big)^2\mathcal{F}_3^{\rm eq},$$

$$T_B(\mathbf{k}_2,\mathbf{k}_1) = \int d\mathbf{k}_3\Delta(2-1-3)\big(W^{12}_{3,1,2}\big)^2\mathcal{F}_3^{\rm eq},$$

$$Q_A(\mathbf{k}_1,\mathbf{k}_2) = \int d\mathbf{k}_3 d\mathbf{k}_4\Delta(1+2-3-4)\big(W^{22}_{1,2,3,4}\big)^2\mathcal{F}_3^{\rm eq}\mathcal{F}_4^{\rm eq},$$

$$R_B(\mathbf{k}_1,\mathbf{k}_2) = \int d\mathbf{k}_3 d\mathbf{k}_4\Delta(1+2-3-4)\big(W^{22}_{1,3,2,4}\big)^2\mathcal{N}_3^{\rm eq}\mathcal{F}_4^{\rm eq},$$

$$Q_B(\mathbf{k}_1,\mathbf{k}_2) = \int d\mathbf{k}_3 d\mathbf{k}_4\Delta(1+2+3-4)\big(W^{31}_{4,3,2,1}\big)^2\mathcal{N}_3^{\rm eq}\mathcal{F}_4^{\rm eq},$$

$$Q_C(\mathbf{k}_1,\mathbf{k}_2) = \int d\mathbf{k}_3 d\mathbf{k}_4\Delta(1-2-3-4)\big(W^{31}_{1,2,3,4}\big)^2\mathcal{F}_3^{\rm eq}\mathcal{F}_4^{\rm eq}, \tag{A4}$$

where $\Delta(1+2-3-4) \equiv \delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\delta(E_1+E_2-E_3-E_4)$ [with similar definitions for $\Delta(1+2-3)$ and $\Delta(1+2+3-4)$, etc.],

$$W^{12}_{1,2,3} = u_1u_2u_3 - u_1v_2u_3 - v_1u_2u_3 + u_1v_2v_3 + v_1u_2v_3 - v_1v_2v_3, \tag{A5}$$

$$W^{22}_{1,2,3,4} = u_1u_2u_3u_4 + u_1v_2u_3v_4 + u_1v_2v_3u_4 + v_1u_2u_3v_4 + v_1u_2v_3u_4 + v_1v_2v_3v_4, \tag{A6}$$

and

$$W^{31}_{1,2,3,4} = u_1u_2u_3v_4 + u_1u_2v_3u_4 + u_1v_2u_3u_4 + v_1v_2v_3u_4 + v_1v_2u_3v_4 + v_1u_2v_3v_4. \tag{A7}$$

The Bogoliubov factors u_k and v_k are given by $u_k = \frac{1}{\sqrt{2}}\sqrt{(1+\frac{(\epsilon_k+\Delta)}{E_k})}$ and $v_k = \frac{1}{\sqrt{2}}\sqrt{(\frac{(\epsilon_k+\Delta)}{E_k}-1)}$. The function $M_1(\mathbf{k}_1) = M_1(k_1)$ and the function $K(\mathbf{k}_1,\mathbf{k}_2)$ depend on k_1, k_2, and $\hat{\mathbf{k}}_1\cdot\hat{\mathbf{k}}_2$. Because of the properties of $K(\mathbf{k}_1,\mathbf{k}_2)$ [2], we can write

$$\mathcal{G}\{\delta\mathcal{N}_{\mathbf{k}_1}(\mathbf{q},\omega)\} = -\mathcal{N}_{k_1}^{\rm eq}\mathcal{F}_{k_1}^{\rm eq}\int_0^\infty dk_2\int d\Omega_2\sqrt{\frac{k_2^2\mathcal{N}_{k_2}^{\rm eq}\mathcal{F}_{k_2}^{\rm eq}}{k_1^2\mathcal{N}_{k_1}^{\rm eq}\mathcal{F}_{k_1}^{\rm eq}}} \times\mathcal{C}(\mathbf{k}_1,\mathbf{k}_2)h_{\mathbf{k}_2}(\mathbf{q},\omega), \tag{A8}$$

where $\mathcal{C}(\mathbf{k_1},\mathbf{k_2})$ is symmetric under the interchange of $\mathbf{k}_1$ and $\mathbf{k}_2$.

The eigenvalues $\lambda_{\beta,\ell}$ and eigenstates $\psi_{\beta,\ell,m}(\mathbf{k}_1)$ of the operator $\mathcal{C}(\mathbf{k}_1,\mathbf{k}_2)$, satisfy the conditions

$$\int d\mathbf{k}_2 C(\mathbf{k}_1,\mathbf{k}_2)\psi_{\beta,\ell,m}(\mathbf{k}_2) = \lambda_{\beta,\ell}\psi_{\beta,\ell,m}(\mathbf{k}_1) \tag{A9}$$

and

$$\int d\mathbf{k}_1\psi_{\beta,\ell,m}(\mathbf{k}_1)C(\mathbf{k}_1,\mathbf{k}_2) = \lambda_{\beta,\ell}\psi_{\beta,\ell,m}(\mathbf{k}_2). \tag{A10}$$

The eigenvalues $\lambda_{\beta,\ell}$ are independent of m due to the angular symmetry of the collision operator.

The collision operator $\mathcal{G}_{\mathbf{k}_1}\{h\}$, acting on four conserved quantities $h = E_k$, $h = k_x$, $h = k_y$, and $h = k_z$, gives zero. We can combine the expressions for k_x and k_y and write the eigenfunctions with specific values of ℓ and m. Note that $k_x + ik_y = -k\sqrt{\frac{8\pi}{3}}\mathrm{Y}_1^1(\hat{\mathbf{k}})$ and $k_x - ik_y = k\sqrt{\frac{8\pi}{3}}\mathrm{Y}_1^{-1}(\hat{\mathbf{k}})$.

The four hydrodynamic eigenfunctions of $C(\mathbf{k}_1,\mathbf{k}_2)$ can be written $\psi_{0,0,0}(\mathbf{k}_1) = \psi_{0,0}(k)\mathrm{Y}_0^0(\hat{\mathbf{k}}_1)$, $\psi_{0,1,0}(\mathbf{k}_1) = \psi_{0,1}(k)\mathrm{Y}_1^0(\hat{\mathbf{k}}_1)$, $\psi_{0,1,1}(\mathbf{k}_1) = \psi_{0,1}(k)\mathrm{Y}_1^1(\hat{\mathbf{k}}_1)$, and $\psi_{0,1,-1}(\mathbf{k}_1) = \psi_{0,1}(k)\mathrm{Y}_1^{-1}(\hat{\mathbf{k}}_1)$, where $\psi_{0,0}(k) = D_{0,0}E_k\sqrt{k^2\mathcal{N}_k^{\rm eq}\mathcal{F}_k^{\rm eq}}$ and $\psi_{0,1}(k) = D_{0,0}k\sqrt{k^2\mathcal{N}_k^{\rm eq}\mathcal{F}_k^{\rm eq}}$. The quantities $D_{\beta,\ell}$ are normalization constants, and the corresponding eigenvalues $\lambda_{\beta,\ell}$ are independent of m and degenerate so that $\lambda_{0,0} = \lambda_{0,1} = 0$ and $\lambda_{0,1}$ is threefold degenerate.

More generally, all the eigenfunctions of $C(\mathbf{k}_1,\mathbf{k}_2)$ can be written in the form

$$\psi_{\beta,\ell,m}(\mathbf{k}) = \psi_{\beta,\ell}(k)\mathrm{Y}_\ell^m(\hat{\mathbf{k}}). \tag{A11}$$

The eigenstates can be orthonormalized so that

$$\int_0^\infty dk_1\int d\Omega_1\psi^*_{\beta_1,\ell_1,m_1}(\mathbf{k}_1)\psi_{\beta_2,\ell_2,m_2}(\mathbf{k}_1) = \delta_{\beta_1,\beta_2}\delta_{\ell_1,\ell_2}\delta_{m_1,m_2} \tag{A12}$$

and $\int_0^\infty dk_1\psi^*_{\beta_1,\ell}(k_1)\psi_{\beta_2,\ell}(k_1) = \delta_{\beta_1,\beta_2}$. We can also write

$$C(\mathbf{k}_1,\mathbf{k}_2) = \sum_{\ell=0}^\infty\sum_{m=-\ell}^{\ell}\mathcal{C}_\ell(k_1,k_2)\mathrm{Y}_\ell^m(\hat{\mathbf{k}}_1)\mathrm{Y}_\ell^{m*}(\hat{\mathbf{k}}_2) = \sum_{\beta=0}^\infty\sum_{\ell=0}^\infty\sum_{m=-\ell}^{\ell}\lambda_{\beta,\ell}\psi_{\beta,\ell,m}(\mathbf{k}_1)\psi^*_{\beta,\ell,m}(\mathbf{k}_2), \tag{A13}$$

where $\mathcal{C}_\ell(k_1,k_2) = \sum_{\beta=0}^\infty\lambda_{\beta,\ell}\psi_{\beta,\ell}(k_1)\psi^*_{\beta,\ell}(k_2)$.

We can express the operator $\mathcal{C}_\ell(k_1,k_2)$ in "bra-ket" notation as

$$\hat{\mathcal{C}}_\ell = \sum_{\beta=0}^\infty\lambda_{\beta,\ell}|\psi_{\beta,\ell}\rangle\langle\psi_{\beta,\ell}|, \tag{A14}$$

so that $\langle k_1|\hat{\mathcal{C}}_\ell|k_2\rangle = \mathcal{C}_\ell(k_1,k_2)$ and $\langle k|\psi_{\beta,\ell}\rangle = \psi_{\beta,\ell}(k)$, where $\langle k_1|k_2\rangle = \delta(k_1-k_2)$ and $\int_0^\infty dk|k\rangle\langle k| = \hat{1}$, where $\hat{1}$ is the unit operator.

APPENDIX B: DIMENSIONLESS VARIABLES AT VERY LOW TEMPERATURES

At very low temperature, most particles have thermal wavelength much longer than the scattering length a. If we let $E_a = k_BT_a \equiv \frac{\hbar^2}{2ma^2}$ denote the energy of a particle with wavelength a, and let T_a denote the corresponding temperature of those particles. The coupling constant g can be written $g = \frac{4\pi\hbar^2 a}{m} = 8\pi k_B T_a a^3$ and $\Delta = g n_0^{\rm eq} = k_B T_a X = 8\pi k_B T_a \sigma$, where $n_0^{\rm eq}$ are the density particles in the condensate, $\sigma = a^3 n_0^{\rm eq}$, and $X = 8\pi\sigma$. The bogolon energy can be written in terms of dimensionless units as $E_k = k_B T_a \sqrt{\kappa^4 + 2\kappa^2 X}$, where $\kappa = ka$ and $X = 8\pi a^3 n_0^{\rm eq}$.

At very low temperatures, only the lowest-energy bogolons contribute to the dynamics of the BEC. For these bogolons, $\kappa \ll X$, and the bogolon energy spectrum is approximately linear in k. We can expand the bogolon energy for $\kappa \ll X$ to obtain $E_k \simeq k_B T_a \sqrt{2X}\kappa$. Using this approximation to E_k we can compute the sound speeds in Eqs. (30) and (31) in the limit $T \to 0$. We find $\alpha = \frac{1}{\sqrt{3}} v_B + O(T^2)$ and $\gamma \sim O(T^4)$. Thus, at $T = 0$ K, $\omega_2^{(1)} = \frac{1}{\sqrt{3}} v_B$ and $\omega_4^{(1)} = v_B$, where $v_B = \sqrt{\frac{g n^{\rm eq}}{m}}$ is the Bogoliubov speed of sound.

APPENDIX C: EIGENVECTORS OF FIRST-ORDER EQUATIONS

It is useful to look at the first-order eigenvalue equation from another point of view. The solutions to Eq. (24) give the eigenvalues of the matrix

$$\mathcal{W} = \begin{pmatrix} 0 & \alpha + \frac{\gamma}{[(\omega^{(1)})^2 - v_B^2]} \\ \alpha & 0 \end{pmatrix}. \tag{C1}$$

This matrix is not symmetric so its left and right eigenvectors will be different, although the eigenvalues will be the same for "left" eigenvalue problem and for the "right" eigenvalue problem. The eigenvalues of $\mathcal{W}$ are

$$\Lambda_1 = -\frac{\sqrt{\alpha}\sqrt{\alpha[v_B^2 - (\omega^{(1)})^2] - \gamma}}{\sqrt{v_B^2 - (\omega^{(1)})^2}}$$
$$\text{and} \quad \Lambda_2 = +\frac{\sqrt{\alpha}\sqrt{\alpha[v_B^2 - (\omega^{(1)})^2] - \gamma}}{\sqrt{v_B^2 - (\omega^{(1)})^2}}. \tag{C2}$$

Since the matrix $\mathcal{W}$ is not a symmetric matrix, its left and right eigenvectors

$$\bar{\Gamma}^L = \begin{pmatrix} \Gamma_i^L & \Gamma_{ii}^L \end{pmatrix} \quad \text{and} \quad \bar{\Gamma}^R = \begin{pmatrix} \Gamma_i^R \\ \Gamma_{ii}^R \end{pmatrix} \tag{C3}$$

will not be the transpose of each other. We can solve the eigenvalue problem for the matrix $\mathcal{W}$ to obtain the left and right eigenvectors, and we find

$$\bar{\Gamma}^L = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\omega^{(1)}}{\alpha} & 1 \end{pmatrix} \quad \text{and} \quad \bar{\Gamma}^R = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\alpha}{\omega^{(1)}}, \\ 1 \end{pmatrix}. \tag{C4}$$

The left and right eigenvectors are normalized so $\bar{\Gamma}^L \bar{\Gamma}^R = 1$.

It is straightforward to show that when $(\omega^{(1)})^2 = (\omega_1^{(1)})^2 = (\omega_2^{(1)})^2$, then $\Lambda_1 = \omega_1^{(1)}$ and $\Lambda_2 = \omega_2^{(1)}$. When $(\omega^{(1)})^2 = (\omega_3^{(1)})^2 = (\omega_4^{(1)})^2$, then $\Lambda_1 = \omega_3^{(1)}$ and $\Lambda_2 = \omega_4^{(1)}$.

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