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Role of correlations in the two-body-marginal problem

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Quantum properties of correlations have a key role in disparate fields of physics including quantum information processing, quantum foundations, and strongly correlated systems. We tackle a specific aspect of the fundamental quantum marginal problem: We address the issue of deducing the global properties of correlations of tripartite quantum states based on the knowledge of their bipartite reductions, focusing on relating specific properties of bipartite correlations to global correlation properties. We prove that strictly classical bipartite correlations may still require global entanglement and that unentangled (albeit not strictly classical) reductions may require global genuine multipartite entanglement rather than simple entanglement. On the other hand, for three qubits, the strict classicality of the bipartite reductions rules out the need for genuine multipartite entanglement. Our work sheds light on the relation between local and global properties of quantum states and on the interplay between classical and quantum properties of correlations.

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I. INTRODUCTION

Quantum correlations have a central role in quantum information processing, quantum foundations, and the physics of strongly correlated systems [1–4]. On the one hand, quantum correlations, in particular entanglement, are a resource that allows one to go beyond what is classically possible in many scenarios, from communication tasks to (measurement-based) quantum computing to quantum cryptography. On the other hand, the nonclassicality of quantum correlations, be it in the form of nonlocality, steering, entanglement, or discord, is one of the most distinctive traits of quantum mechanics and challenges our understanding of quantum mechanics itself. The interplay between local and global properties of quantum states is a key aspect in the study of quantum correlations, from both a fundamental perspective and an applicative one. For example, we may want to certify the presence of multipartite entanglement in large systems without the (often inaccessible) knowledge of the global state, using instead only the information that comes from reduced states. On the other hand, in condensed-matter physics, because of the typically local, e.g., two-body, interactions, relevant properties are dictated by the interplay between the allowed reduced states and global correlations, giving rise to phenomena such as frustration [5]. The general study of the relations between the properties of the reduced states and the properties of the global state is known as the quantum marginal problem, which has seen a growing interest over the past decade also for the reasons above [6–12].

In this work we study what can be inferred about the quality of the correlations of the global state given information about the two-body reduced states, aiming at answering the following question: What correlations need to be present globally to explain what we see locally? In [13] a characterization of multipartite entanglement in terms of even just single-party reduced states (actually, single-party spectra) was given, but under the assumption of dealing with a pure or quasipure global state. In [14–16] the possibility of dealing with global mixed states is taken into account and examples are given where two-qubit separable states are only compatible with global entanglement, intended in the sense of a lack of total separability (see Sec. II for definitions). In [16] examples are also given where genuine multipartite entanglement, a much stronger notion of global entanglement, can still be deduced from the properties of the two-body reduced states, but only when these reduced states exhibit bipartite entanglement themselves.

In this work we present several results that complement and generalize those of [14–16]. We offer a brief summary of our findings in Table I. First, we provide examples of triples of bipartite reduced states (in the simplest case, two-qubit states) that, albeit separable, are only compatible with genuine tripartite entanglement (lower right corner of Table I). This gap between the entanglement properties of the marginals and of the global state is wide. Second, we address the issue of relating the general quantumness of correlations [2] of reduced states to the quantum correlations of the global state. We find that strictly classical reduced states may still be compatible

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TABLE I. A ✓ in a cell means that there exist two-body marginal states with the quality of correlations listed in the corresponding row, which are only compatible with global states that have *at least* the property of correlations listed in the corresponding column. A ✗ means that the inference is not possible: Specifically, there are no fully classical two-qubit states that are only compatible with genuine tripartite entanglement (refer to the main text for definitions). Previous results are reported for completeness and comparison.

	Global state	
Reductions	Entangled	Genuinely multipartite entangled
Fully classical	✓	✗ (qubits)
Separable	✓[a]	✓

[a]References [14–16].

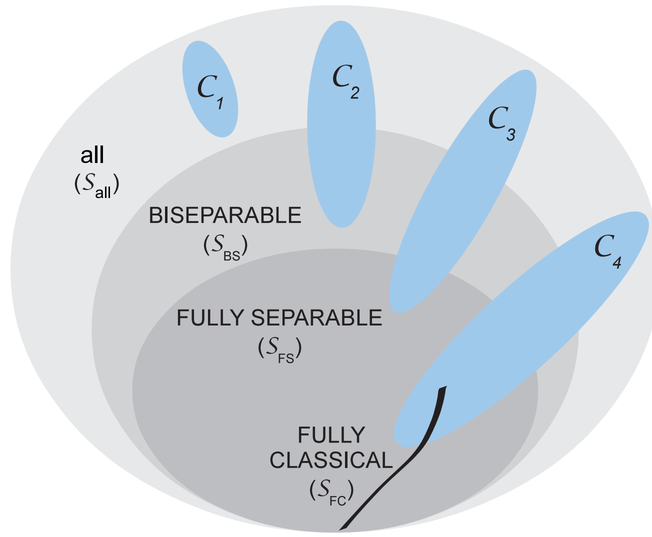

FIG. 1. (Color online) Hierarchy of correlation classes and some possible compatibility sets. The set $\mathcal{S}_{FC}$ is denoted by the line. The set $\mathcal{S}_{GME}$ of genuinely multipartite states is the complement of $\mathcal{S}_{BS}$ in the space of all states $\mathcal{S}_{all}$. The reductions corresponding to the compatibility set $\mathcal{C}_1$ are only compatible with genuine multipartite entanglement. The reductions of the states in compatibility set $\mathcal{C}_2$ are compatible with entangled biseparable states and genuinely multipartite entangled states, but not separable states. The reductions defining $\mathcal{C}_3$ are compatible with a fully separable state as well as with entangled states, but not fully classical states.

only with global entanglement (upper left corner of Table I). Third, we find that, at least for qubits, the strict classicality of the two-body correlations makes it impossible to certify genuine tripartite entanglement based on the knowledge of the reductions: Strictly classical two-qubit reduced states are always compatible with a global state that is not genuinely tripartite entangled (upper right corner of Table I).

The rest of the paper is organized as follows. In Sec. II we define the relevant notions of correlations and classicality and of compatibility of two-body reduced states in tripartite systems. In Sec. III we study the relation between the classicality of reductions and their compatibility. In Sec. IV we prove that unentangled reduced states may only be compatible with genuine multipartite entanglement at the level of the global state. We summarize in Sec. V.

II. CORRELATIONS AND COMPATIBILITY

We begin by formally defining qualitatively different types of correlations.

Definition 1. Any tripartite mixed state can be written as a mixture of an ensemble of pure states as $\rho_{ABC} = \sum_i p_i |\psi_i\rangle\langle\psi_i|_{ABC}$. We say that ρ_{ABC} is (a) fully separable if we can take each $|\psi_i\rangle_{ABC}$ to be fully factorized, e.g., $|\alpha_i\rangle_A|\beta_i\rangle_B|\gamma_i\rangle_C$; (b) biseparable if we can take each $|\psi_i\rangle_{ABC}$ to be unentangled in at least one partition, e.g., $|\alpha_i\rangle_A|\phi_i\rangle_{BC}$, $|\beta_i\rangle_B|\phi\rangle_{AC}$, or $|\gamma_i\rangle_C|\phi\rangle_{AB}$; (c) genuinely multipartite entangled if for any ensemble there is at least one $|\psi_i\rangle$ with $p_i > 0$ that is not factorized with respect to any bipartition, i.e., if ρ_{ABC} is not biseparable; or (d) fully classical if we can take each $|\psi_i\rangle_{ABC}$ to be of the form $|a_i\rangle_A|b_j\rangle_B|c_k\rangle_C$, with $\{|a_i\rangle\}$, $\{|b_j\rangle\}$, and $\{|c_k\rangle\}$ orthonormal bases on $\mathcal{H}_A$, $\mathcal{H}_B$, and $\mathcal{H}_C$, respectively, so that, overall, $\rho_{ABC} = \sum_{ijk} p_{ijk}|a_ib_jc_k\rangle\langle a_ib_jc_k|$.

Bipartite full classicality and separability are defined similarly: ρ_{AB} is fully classical if $\rho_{AB} = \sum_{ij} p_{ij}|a_ib_j\rangle\langle a_ib_j|$ for $\{|a_i\rangle\}$ and $\{|b_j\rangle\}$ orthonormal bases and separable if $\rho_{AB} = \sum_i p_i|\alpha_i\beta_i\rangle\langle\alpha_i\beta_i|$. A bipartite state is entangled if it is not separable. The notions of full separability and biseparability are redundant for bipartite states.

The set of fully classical states $\mathcal{S}_{FC}$ is a subset of the set of fully separable states $\mathcal{S}_{FS}$, which in turn is a subset of the set of biseparable states $\mathcal{S}_{BS}$; the set of genuinely multipartite entangled states $\mathcal{S}_{GME}$ is the complement of $\mathcal{S}_{BS}$ in the space of all states $\mathcal{S}_{all}$ (see Fig. 1). All the mentioned sets apart from $\mathcal{S}_{FC}$ and $\mathcal{S}_{GME}$ are convex. A biseparable state may be either separable or entangled in any bipartition, but it is by definition the convex sum of three states that are each separable in one of the bipartitions $A : BC$, $B : AC$, and $C : AB$.

We now move to define formally the notion of compatibility for reduced states.

Definition 2. Given a triple of three two-party states $\mathcal{E} = (\rho_{AB},\rho_{AC},\rho_{BC})$, its compatibility set is defined as $\mathcal{C}(\mathcal{E}) := \{\sigma_{ABC} \in \mathcal{S}_{all}|\sigma_{ij} = \rho_{ij}, ij = AB,AC,BC\}$. Any compatibility set is a convex set [10] and the property of being part of a given compatibility set defines an equivalence relation. We find it useful to denote by $\mathcal{C}(\rho_{ABC})$ the compatibility set associated with the reduced states of ρ_{ABC}, i.e., the set of all states that have the same reductions as ρ_{ABC}. A triple of two-party states $\mathcal{E}$ is said to be compatible (so that we refer to the triple as triple of reductions) if $\mathcal{C}(\mathcal{E}) \neq \emptyset$, i.e., if there is at least one global state with those reductions.

The following definition links the compatibility of reduced states to the correlation properties of global states.

Definition 3. We say that the reductions $\mathcal{E}$ are incompatible with a set $\mathcal{S}$ (or with the defining correlation property of $\mathcal{S}$) if $\mathcal{C}(\mathcal{E}) \cap \mathcal{S} = \emptyset$. We say that (a compatible) $\mathcal{E}$ is only compatible with genuine multipartite entanglement if it is incompatible with $\mathcal{S}_{BS}$.

Figure 1 illustrates the problem of deciding whether certain bipartite reductions necessarily require the presence of global correlations of a certain kind. We can always expand a generic tripartite state as

$$\rho_{ABC} = \rho_A \otimes \rho_B \otimes \rho_C + \chi_{ABC}\chi_{AB} \otimes \frac{\mathbb{1}_C}{d_C} + \chi_{AC} \otimes \frac{\mathbb{1}_B}{d_B} + \chi_{BC} \otimes \frac{\mathbb{1}_A}{d_A}, \quad (1)$$

where ρ_k is the reduced states of party k and $\mathbb{1}_k/d_k$ is the normalized identity operator on the Hilbert space $\mathcal{H}_k$. The bipartite correlation matrices χ_{kl} can be defined via $\rho_{jk} = \rho_j \otimes \rho_k + \chi_{jk}$ and satisfy $\mathrm{Tr}_k[\chi_{kl}] = \mathrm{Tr}_l[\chi_{kl}] = 0$. It is worth noting that when a bipartite marginal state ρ_{jk} is fully classical, then $[\rho_j \otimes \rho_k, \chi_{jk}] = 0$ and χ_{jk} is also diagonal in the same product basis as ρ_{jk}. The tripartite correlation matrix χ_{ABC}, which for a fixed ρ_{ABC} can be defined via (1), satisfies $\mathrm{Tr}_k[\chi_{ABC}] = 0$ for all $k \in \{A,B,C\}$. For compatible reductions $\mathcal{E} = (\rho_{AB}, \rho_{AC}, \rho_{BC})$, the compatibility set $\mathcal{C}(\mathcal{E})$ is spanned by choosing the tripartite correlation matrix χ_{ABC} so that the resulting operator in (1) is a physical state, i.e., positive semidefinite. On the other hand, to determine whether a triple of bipartite states is compatible, we first check the basic necessary condition that the single-party marginals be the same, i.e., $\mathrm{Tr}_j[\rho_{ij}] = \mathrm{Tr}_k[\rho_{ik}]$ for all $\{i,j,k\} \in \{A,B,C\}$. Next we have to search for a tripartite correlation matrix χ_{ABC} such that Eq. (1) is physical. If no such χ_{ABC} exists, the given states are not compatible.

III. CLASSICALITY OF REDUCTIONS AND GLOBAL ENTANGLEMENT

Consider the marginals from the well-known Greenberger-Horne-Zeilinger (GHZ) state $|S_{\mathrm{GHZ}}\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$: $\rho_{AB} = \rho_{BC} = \rho_{AC} = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$. These are fully classical marginals coming from a genuinely tripartite entangled state. However, these marginals are also compatible with $\frac{1}{2}(|000\rangle\langle 000| + |111\rangle\langle 111|)$, which is fully classical.

In this section we will provide an example where fully classical two-body reduced states are not compatible with a global fully classical state and actually require the presence of entanglement. On the other hand, we will prove that, in the case of three qubits, the fully classical two-body reductions are always compatible with a global states that is not genuine multipartite entangled.

A. Two-body classical states may require global quantumness of correlations

We will now derive conditions to ensure that some fully classical marginals cannot be compatible with any global fully classical state. We start with the following lemma.

Lemma 1. Suppose three states ρ_{AB}, ρ_{AC}, and ρ_{BC} are fully classical. Consider the commutators $\Delta_{ij,ik} = [\chi_{ij} \otimes \mathbb{1}_k, \chi_{ik} \otimes \mathbb{1}_j] = [\rho_{ij} \otimes \mathbb{1}_k, \rho_{ik} \otimes \mathbb{1}_j]$, where the second equality is due to the assumed classicality, i.e., $[\rho_j \otimes \rho_k, \chi_{jk}] = 0$. Then we have the following.

(i) All commutators $\Delta_{ij,ik}$ vanish if and only if there are orthonormal bases $\{|a_i\rangle\}$, $\{|b_i\rangle\}$, and $\{|c_i\rangle\}$ such that $\rho_{AB} = \sum_{ij} p_{ij}|a_i,b_j\rangle\langle a_i,b_j|$, $\rho_{BC} = \sum_{ij} q_{ij}|b_i,c_j\rangle\langle b_i,c_j|$, and $\rho_{AC} = \sum_{ij} r_{ij}|a_i,c_j\rangle\langle a_i,c_j|$.

(ii) If some commutator $\Delta_{ij,ik}$ does not vanish, then (a) at least one ρ_i is degenerate (i.e., at least two eigenvalues of some ρ_i are the same) and (b) there does not exist a tripartite fully classical state that is compatible with ρ_{AB}, ρ_{BC}, and ρ_{AC}.

Proof. (i) The "if" part is trivial. Let us prove the "only if" part. By hypothesis we may assume

$$\rho_{AB} = \sum_{ij} p_{ij}|a_i,b_j\rangle\langle a_i,b_j| = \sum_j p_j\alpha_j \otimes |b_j\rangle\langle b_j| = \sum_i p_i'|a_i\rangle\langle a_i| \otimes \beta_i, \quad (2)$$

$$\rho_{AC} = \sum_{ij} r_{ij}|a_i',c_j\rangle\langle a_i',c_j| = \sum_j r_j\alpha_j' \otimes |c_j\rangle\langle c_j| = \sum_i r_i'|a_i'\rangle\langle a_i'| \otimes \gamma_i, \quad (3)$$

$$\rho_{BC} = \sum_{ij} q_{ij}|b_i',c_j'\rangle\langle b_i',c_j'| = \sum_j q_j\beta_j' \otimes |c_j'\rangle\langle c_j'| = \sum_i q_i'|b_i'\rangle\langle b_i'| \otimes \gamma_i', \quad (4)$$

with the orthonormal bases $\{|a_i\rangle\}$ and $\{|a_i'\rangle\}$ on $\mathcal{H}_A$, $\{|b_i\rangle\}$ and $\{|b_i'\rangle\}$ on $\mathcal{H}_B$, and $\{|c_i\rangle\}$ and $\{|c_i'\rangle\}$ on $\mathcal{H}_C$ and $\alpha_j = \sum_i p_{ij}|a_i\rangle\langle a_i|/p_j$, with $p_j = \sum_i p_{ij}$ (similarly for β_j, etc.). Then $\Delta_{ij,ik} = 0$ implies

$$\begin{aligned}&[\chi_{ij} \otimes \mathbb{1}_k, \chi_{ik} \otimes \mathbb{1}_j] \\ &= [(\rho_{ij} - \rho_i \otimes \rho_j) \otimes \mathbb{1}_k, (\rho_{ik} - \rho_i \otimes \rho_k) \otimes \mathbb{1}_j] \\ &= [\rho_{ij} \otimes \mathbb{1}_k, \rho_{ik} \otimes \mathbb{1}_j] = 0,\end{aligned} \quad (5)$$

with $i,j,k \in \{A,B,C\}$. By setting $i = A$ in (5), we have $[\alpha_s, \alpha_t'] = 0 \forall s,t$. Thus the states α_s, α_t' are simultaneously diagonalizable in the orthonormal basis $\{|a_i''\rangle\}$. So we may replace the bases $\{|a_i\rangle\}$ and $\{|a_i'\rangle\}$ in (2) and (3) by $\{|a_i''\rangle\}$. This replacement may result in the change of p_i', β_i and r_i', γ_i. Since there is no confusion, we still use them in (2) and (3).

Next, by setting $i = B$ in (5), we have $[\beta_s, \beta_t'] = 0 \, \forall s,t$. Thus the states β_s, β_t' are simultaneously diagonalizable in the orthonormal basis $\{|b_i''\rangle\}$. So we may replace the bases $\{|b_i\rangle\}$ and $\{|b_i'\rangle\}$ in (2) and (4) by $\{|b_i''\rangle\}$. This replacement may result in the change of q_i', γ_i'. Since there is no confusion, we still use them in (4). Then we set $i = C$ in (5) and repeat the above argument to show that the bases $\{|c_i\rangle\}$ and $\{|c_i'\rangle\}$ in (3) and (4) can be replaced by the orthonormal basis $\{|c_i''\rangle\}$. So the assertion follows.

(ii) Suppose either condition (a) or (b) is violated. We have that either ρ_A, ρ_B, and ρ_C are all nondegenerate in the orthonormal bases $\{|a_i\rangle\}$, $\{|b_i\rangle\}$, and $\{|c_i\rangle\}$, respectively [violation of (a)], or that there exists a tripartite fully classical state $\sum_{i,j,k} f_{ijk}|a_i,b_j,c_k\rangle\langle a_i,b_j,c_k|$ that is compatible with ρ_{AB}, ρ_{BC}, and ρ_{AC} [violation of (b)]. In either case we have $\rho_{AB} = \sum_{ij} p_{ij}|a_i,b_j\rangle\langle a_i,b_j|$, $\rho_{AC} = \sum_{ij} r_{ij}|a_i,c_j\rangle\langle a_i,c_j|$, and $\rho_{BC} = \sum_{ij} q_{ij}|b_i,c_j\rangle\langle b_i,c_j|$. So (i) implies that all commutators $\Delta_{ij,kl}$ vanish and we have a contradiction. ■

An immediate consequence of Lemma 1 is the following.

Theorem 1. Let ρ_{AB}, ρ_{BC}, and ρ_{AC} be three compatible bipartite fully classical states such that (i) they all commute (all commutators $\Delta_{ij,kl}$ of Lemma 1 vanish) or (ii) all three one-body reduced states ρ_A, ρ_B, and ρ_C are nondegenerate.

Then ρ_{AB}, ρ_{BC}, and ρ_{AC} are compatible with a fully classical tripartite state.

Proof. The fact that all three single-system reductions are not degenerate implies, by Lemma 1, condition (ii), result (a), that all the commutators $\Delta_{ij,kl}$ defined in Lemma 1 vanish. By Lemma 1, condition (i) we have that ρ_A, ρ_B, and ρ_C are diagonal in the orthonormal bases $\{|a_i\rangle\}$, $\{|b_j\rangle\}$, and $\{|c_k\rangle\}$, respectively, in which ρ_{AB}, ρ_{BC}, and ρ_{AC} are explicitly classical. Most importantly, we have

$$\rho_{XY} = \sum_{ij} |x_i y_j\rangle\langle x_i y_j|\rho_{XY}|x_i y_j\rangle\langle x_i y_j|, \tag{6}$$

with $x,y \in \{a,b,c\}$ and $X,Y \in \{A,B,C\}$. Let ρ_{ABC} be any tripartite state with which the three two-body reductions are compatible. Then also the fully classical tripartite state

$$\sigma_{ABC} = \sum_{ijk} |a_i b_j c_k\rangle\langle a_i b_j c_k|\rho_{ABC}|a_i b_j c_k\rangle\langle a_i b_j c_k| \tag{7}$$

has bipartite reduced density matrices ρ_{AB}, ρ_{BC}, and ρ_{AC}. ■

Given Theorem 1, in order to construct an example where ρ_{AB}, ρ_{AC}, and ρ_{BC} are all fully classical but not compatible with any fully classical state, we first have to construct an example where ρ_{AB}, ρ_{AC}, and ρ_{BC} are classical but do not commute with each other. For this, we will need the following lemma. We recall that for a bipartite state ρ acting on the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, the partial transpose computed in the standard orthonormal basis $\{|i\rangle\}$ of system A is defined by $\rho^\Gamma = \sum_{ij} |j\rangle\langle i| \otimes \langle i|\rho|j\rangle$. One can similarly define the partial transpose Γ_B on the system B.

Lemma 2. Consider three classical-classical two-qubit states

$$\begin{aligned}\rho_{AB} &= p(|00\rangle\langle 00| + |11\rangle\langle 11|)\\ &\quad + (1/2 - p)(|01\rangle\langle 01| + |10\rangle\langle 10|), \end{aligned}\tag{8}$$

$$\begin{aligned}\rho_{BC} &= q(|b_0,0\rangle\langle b_0,0| + |b_1,1\rangle\langle b_1,1|)\\ &\quad + (1/2 - q)(|b_0,1\rangle\langle b_0,1| + |b_1,0\rangle\langle b_1,0|), \end{aligned}\tag{9}$$

$$\begin{aligned}\rho_{AC} &= r(|a_0,c_0\rangle\langle a_0,c_0| + |a_1,c_1\rangle\langle a_1,c_1|)\\ &\quad + (1/2 - r)(|a_0,c_1\rangle\langle a_0,c_1| + |a_1,c_0\rangle\langle a_1,c_0|), \end{aligned}\tag{10}$$

where $p,q,r \in (0,1/4)$ and any one of $\{|a_i\rangle\}$, $\{|b_i\rangle\}$, or $\{|c_i\rangle\}$ is a real and orthonormal basis in $\mathbf{C}^2$. Let

$$\begin{aligned}\rho_{ABC} &= -\frac{1}{4}\mathbb{1}_A \otimes \mathbb{1}_B \otimes \mathbb{1}_C\\ &\quad + \rho_{AB} \otimes \frac{\mathbb{1}_C}{2} + \rho_{AC} \otimes \frac{\mathbb{1}_B}{2} + \rho_{BC} \otimes \frac{\mathbb{1}_A}{2}. \end{aligned}\tag{11}$$

Then we have the following.

(i) If $\rho_{ABC} \geqslant 0$ then ρ_{ABC} is separable with respect to to the partition $A : BC$, $B : AC$, and $C : AB$.

(ii) Here ρ_{AB}, ρ_{BC}, and ρ_{AC} are compatible if and only if they are compatible with the biseparable state ρ_{ABC} in (11).

(iii) Suppose ρ_{AB}, ρ_{BC}, and ρ_{AC} are compatible. They are compatible with a fully separable state if and only if ρ_{ABC} is fully separable.

Proof. (i) One may directly verify that the state is invariant under partial transposition with respect to any system, i.e., $\rho^{\Gamma_X} = \rho$ for $X = A,B,C$. Since $\rho_{ABC} \geqslant 0$, the assertion follows from Theorem 2 in [17].

(ii) The "if" part is trivial; let us prove the "only if" part. Suppose the bipartite marginals ρ_{AB}, ρ_{BC}, and ρ_{AC} are compatible with a tripartite state ρ'_{ABC}. Since ρ_{AB}, ρ_{BC}, and ρ_{AC} are real, they are also compatible with the real state $(\rho'_{ABC} + \rho'^*_{ABC})/2$, so we can assume that ρ'_{ABC} is real without loss of generality. By Eq. (1), there is a Hermitian matrix χ_{ABC} such that

$$\rho'_{ABC} = \rho_{ABC} + \chi_{ABC}.$$

Since in our case both ρ and ρ' are real, also χ is real. It follows from Eqs. (8)–(10) and the fact that $\{|a_i\rangle\}$, $\{|b_i\rangle\}$, and $\{|c_i\rangle\}$ are real and orthonormal bases that ρ_{AB}, ρ_{BC}, and ρ_{AC} are invariant under the local unitary $\sigma_y \otimes \sigma_y$. So they are compatible with the state

$$\begin{aligned}&\tfrac{1}{2}[\rho'_{ABC} + (\sigma_y \otimes \sigma_y \otimes \sigma_y)\rho'_{ABC}(\sigma_y \otimes \sigma_y \otimes \sigma_y)]\\ &\quad = \tfrac{1}{2}[\rho_{ABC} + (\sigma_y \otimes \sigma_y \otimes \sigma_y)\rho_{ABC}(\sigma_y \otimes \sigma_y \otimes \sigma_y)]\\ &\qquad + \tfrac{1}{2}[\chi_{ABC} + (\sigma_y \otimes \sigma_y \otimes \sigma_y)\chi_{ABC}(\sigma_y \otimes \sigma_y \otimes \sigma_y)]\\ &\quad = \rho_{ABC} + \tfrac{1}{2}[\chi_{ABC} + (\sigma_y \otimes \sigma_y \otimes \sigma_y)\chi_{ABC}\\ &\qquad \times (\sigma_y \otimes \sigma_y \otimes \sigma_y)], \end{aligned}\tag{12}$$

where we have used that, from (11),

$$(\sigma_y \otimes \sigma_y \otimes \sigma_y)\rho_{ABC}(\sigma_y \otimes \sigma_y \otimes \sigma_y) = \rho_{ABC}.$$

We will now argue that, for a real χ,

$$\chi_{ABC} + (\sigma_y \otimes \sigma_y \otimes \sigma_y)\chi_{ABC}(\sigma_y \otimes \sigma_y \otimes \sigma_y) = 0, \tag{13}$$

so (12) proves that, for a physical state ρ', ρ is also physical, as it corresponds to the convex combination of physical states. The starting point in proving (13) is to observe that every three-qubit correlations matrix χ is by definition the linear combination of traceless Pauli matrices, i.e.,

$$\chi = \sum_{i,j,k=1}^{3} \chi_{ijk}\sigma_i \otimes \sigma_j \otimes \sigma_k. \tag{14}$$

Since χ is Hermitian, all coefficients χ_{ijk} are real. Moreover, for a real (and hence symmetric) χ only terms with an even number of $\sigma_2 = \sigma_y$ are present in the expansion, because $(\sigma_2)^T = -\sigma_2$, while $\sigma_1 = \sigma_x$ and $\sigma_3 = \sigma_z$ are symmetric. On the other hand, $\sigma_2\sigma_m\sigma_2 = -\sigma_m$ for $m = 1,3$, while, obviously, $\sigma_2\sigma_2\sigma_2 = \sigma_2$. Since each nonzero term in the expansion (14) of χ contains an even number of σ_2's, it will change sign after conjugation by $\sigma_2 \otimes \sigma_2 \otimes \sigma_2$, i.e.,

$$\begin{aligned}&(\sigma_y \otimes \sigma_y \otimes \sigma_y)\chi_{ABC}(\sigma_y \otimes \sigma_y \otimes \sigma_y)\\ &\quad = \sum_{i,j,k=1}^{3} \chi_{ijk}(\sigma_y \otimes \sigma_y \otimes \sigma_y)\sigma_i \otimes \sigma_j \otimes \sigma_k(\sigma_y \otimes \sigma_y \otimes \sigma_y)\\ &\quad = \sum_{i,j,k=1}^{3} \chi_{ijk}(\sigma_y\sigma_i\sigma_y) \otimes (\sigma_y\sigma_j\sigma_y) \otimes (\sigma_y\sigma_k\sigma_y)\\ &\quad = -\sum_{i,j,k=1}^{3} \chi_{ijk}\sigma_i \otimes \sigma_j \otimes \sigma_k\\ &\quad = -\chi_{ABC}. \end{aligned}$$

As argued, this implies $\rho_{ABC} \geqslant 0$, with biseparability following from (i).

(iii) The "if" part follows from (ii); let us prove the "only if" part. Suppose ρ_{AB}, ρ_{BC}, and ρ_{AC} are compatible with a fully separable state ρ'_{ABC}. From (12), ρ_{ABC} is the convex sum of a few fully separable states. So the assertion follows. This completes the proof. ∎

We are now ready to present our example where ρ_{AB}, ρ_{AC}, and ρ_{BC} are all fully classical but not compatible with any fully classical state.

Example 1. Consider the three-qubit state

$$\rho_{ABC}(q) = \tfrac{1}{8}(\mathbb{1}_A \otimes \mathbb{1}_B \otimes \mathbb{1}_C + q\mathbb{1}_A \otimes \sigma_1 \otimes \sigma_1 + q\sigma_2 \otimes \mathbb{1}_B \otimes \sigma_2 + q\sigma_3 \otimes \sigma_3 \otimes \mathbb{1}_C), \quad (15)$$

where σ_i, with $i = 1,2,3$, are the Pauli matrices and $\frac{-1}{\sqrt{3}} \leqslant q \leqslant \frac{1}{\sqrt{3}}$ [for q outside of this interval the matrix $\rho_{ABC}(q)$ is not positive semidefinite], $q \neq 0$. It is not hard to see that each of the bipartite marginals states is fully classical but with respect to different bases. A quick method to verify this assertion is using Theorem 1 in [18]. Moreover, the reductions of $\rho_{ABC}(q)$ do not commute with each other. Part (ii) of Lemma 1 implies that there is no tripartite fully classical state that is compatible with these bipartite marginals: $\mathcal{C}(\rho_{ABC}(q)) \cap \mathcal{S}_{\rm FS} = \emptyset$. Surprisingly, we can find a value of q for which the fully classical marginals in fact require some global entanglement. Consider the state $\omega_{ABC} = \rho_{ABC}(q = 1/\sqrt{3})$. The range of ω_{ABC} does not contain any fully factorized pure state, hence it cannot be fully separable. Now, up to a local unitary, the classical bipartite marginals ω_{AB}, ω_{AC}, and ω_{BC} and the entangled state ω_{ABC} can be written as in (8)–(11), respectively. It follows from Lemma 2, condition (iii) that ω_{AB}, ω_{AC}, and ω_{BC} cannot be compatible with any fully separable state.

In Appendix A, which focuses on the uniqueness of global states with fixed two-body reductions, we provide also an example of classical reductions compatible with a unique global state that is not fully classical, although fully separable.

B. Classical two-qubit states do not require genuine tripartite entanglement

Although, as we have just seen, compatible classical marginals may require global quantum correlations or even entanglement, it turns out that for the case of three qubits they will never require the global state to be genuinely multipartite entangled.

To see this, will need an additional lemma.

Lemma 3. Let $\{|a_i\rangle\}$ and $\{|b_i\rangle\}$ be orthonormal bases on $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Suppose the bipartite marginals ρ_{AB}, ρ_{AC}, and ρ_{BC} are compatible. They are compatible with a nongenuinely entangled tripartite state when one of the following conditions is satisfied: (i) $\rho_{AB} = \sum_i p_i |a_i\rangle\langle a_i| \otimes \rho_i$ and $\rho_{AC} = \sum_i q_j |a_i\rangle\langle a_i| \otimes \sigma_i$ or (ii) $\rho_{AB} = \sum_i r_i |a_i,b_i\rangle\langle a_i,b_i|$.

Proof. Suppose ρ_{AB}, ρ_{AC}, and ρ_{BC} are compatible with ρ_{ABC}. If hypothesis (i) is satisfied, then they are also compatible with the state $\sum_i |a_i\rangle\langle a_i|_A \otimes \mathbb{1}_{BC} \rho_{ABC} |a_i\rangle\langle a_i|_A \otimes \mathbb{1}_{BC}$, which is biseparable. On the other hand, if hypothesis (ii) is satisfied, then it follows from [19] that there is a quantum channel Λ on $\mathcal{H}_C$ such that $\rho_{ABC} = \Lambda(|\psi\rangle\langle\psi|)$ for $|\psi\rangle = \sum_i \sqrt{r_i} |a_i,b_i,i\rangle$. So we obtain $\rho_{AC} = \Lambda(\sum_i r_i |a_i,i\rangle\langle a_i,i|)$ and $\rho_{BC} = \Lambda(\sum_i r_i |b_i,i\rangle\langle b_i,i|)$. Then ρ_{AB}, ρ_{AC}, and ρ_{BC} are compatible with the fully separable state $\Lambda(\sum_i r_i |a_i,b_i,i\rangle\langle a_i,b_i,i|)$. This completes the proof. ∎

Now we are ready to give the proof of the following.

Theorem 2. Any three compatible classical-classical two-qubit states ρ_{AB}, ρ_{BC}, and ρ_{AC} are compatible with a tripartite biseparable state.

Proof. Suppose some fully classical bipartite marginals ρ_{AB}, ρ_{BC}, and ρ_{AC} are only compatible with genuinely entangled states ρ_{ABC}. Then the one-party reduced density operators ρ_A, ρ_B, and ρ_C have to be the maximally mixed states $\mathbb{1}/2$. Indeed, if, without loss of generality in the argument, ρ_A is nondegenerate, the two-party reduced states would also be compatible with a global state given by the locally (on A) dephased version of ρ_{ABC}, which would be separable in $A : BC$, leading to a contradiction. Thus, up to local unitaries, we may assume $\rho_{AB} = p|00\rangle\langle 00| + x|01\rangle\langle 01| + y|10\rangle\langle 10| + z|11\rangle\langle 11|$, where $p + x + y + z = 1$ and $0 \leqslant p \leqslant 1/4$. Since $\rho_A = \rho_B = \rho_C = \mathbb{1}/2$, we have $p + x = p + y = y + z = 1/2$. So we obtain $x = y$ and $p = z \in [0,1/4]$. Since ρ_{ABC} is genuinely entangled, the cases $p = 1/4$ and $p = 0$ are excluded by Lemma 3, hypotheses (i) and (ii), respectively. So we obtain ρ_{AB} as in Eq. (8). By similar arguments and performing suitable diagonal local unitary gates on systems A and B, the classical-classical two-qubit states ρ_{BC} and ρ_{AC} can be simplified to the forms (9) and (10), respectively. Meantime, ρ_{AB} is unchanged. Since ρ_{AB}, ρ_{BC}, and ρ_{AC} are compatible, it follows from Lemma 2, hypothesis (ii) that they are compatible with a biseparable state. This gives us a contradiction. So there are no compatible bipartite marginals that are only compatible with genuine entangled states. This completes the proof. ∎

IV. SEPARABLE REDUCTIONS CAN IMPLY GENUINE MULTIPARTITE ENTANGLEMENT

We have seen that the condition of classicality of marginals is strong enough to exclude the need for global genuine multipartite entanglement. We will now construct nonclassical separable marginals that are only compatible with global genuine multipartite entanglement, but first we need to establish some more definitions and notation.

We set $d_A = \dim \mathcal{H}_A$, $d_B = \dim \mathcal{H}_B$, and $d_C = \dim \mathcal{H}_C$. We denote by $r(M)$ and $\mathcal{R}(M)$ the rank and range of any square matrix M, respectively. A quantum state is a positive semidefinite linear operator $\rho : \mathcal{H} \to \mathcal{H}$ with $\operatorname{Tr} \rho = 1$. We say that ρ_{ABC} is an $m \times n \times l$ state, which means that the reduced density operators satisfy $r(\rho_A) = m$, $r(\rho_B) = n$, and $r(\rho_C) = l$. The ranks of the reduced density operators of ρ_{ABC} are invariant when we perform an invertible local operator (ILO) on ρ_{ABC}. That is, let $A = \bigotimes_{i=1}^{3} A_i \in {\rm GL} := {\rm GL}_{d_A}(\mathbf{C}) \times {\rm GL}_{d_B}(\mathbf{C}) \times {\rm GL}_{d_C}(\mathbf{C})$ such that $\sigma = A\rho A^\dagger$. Then $r(\rho_X) = r(\sigma_X)$, $r(\rho_{XY}) = r(\sigma_{XY})$, and $r(\rho) = r(\sigma)$, where $X,Y = A,B,C$. We also denote by $|a^*\rangle$ the vector whose components are the complex conjugates of those of $|a\rangle$. So $|a\rangle$ is real when $|a\rangle = |a^*\rangle$.

Evidently $r(\rho^\Gamma) = r(\rho^{\Gamma_B})$, where Γ, we recall, denotes partial transposition. We call the integer pair $(r(\rho), r(\rho^\Gamma))$ the birank of ρ and the two integers may be different. For such examples of two-qubit and qubit-qutrit separable states, we refer the readers to Tables I and II in [20]. Furthermore, we say that ρ is a positive under partial transposition (PPT) (non-positive under partial transposition (NPT)) state if $\rho^\Gamma \geqslant 0$ (ρ^Γ has at least one negative eigenvalue). Evidently, a separable state must be PPT. The converse is true only if $mn \leqslant 6$ [21,22].

We say a bipartite state ρ_{AB} is A finite when for any subspace $H \subset \mathcal{H}_A$, $\dim H > 1$, and any state $|x\rangle \in \mathcal{H}_B$ it holds that $H \otimes |x\rangle \not\subset \mathcal{R}(\rho_{AB})$. In other words, ρ_{AB} is not A finite when $\mathcal{R}(\rho_{AB})$ contains a two-dimensional subspace spanned by $|a_1,x\rangle,|a_2,x\rangle$ with some linearly independent states $|a_1\rangle,|a_2\rangle$. So if ρ_{AB} is not A finite, there must be infinitely many product states in $\mathcal{R}(\rho_{AB})$.

Besides these notions and notation, we will need also the following lemma.

Lemma 4. Suppose the bipartite marginals ρ_{AB}, ρ_{BC}, and ρ_{AC} are compatible with ρ_{ABC} and ρ_{AB} is A finite and B finite. Then ρ_{ABC} is either separable with respect to the partition $AB:C$ or genuinely multipartite entangled.

Proof. Suppose ρ_{ABC} is biseparable so that $\rho_{ABC} = p\alpha_{A:BC} + q\beta_{B:AC} + (1-p-q)\gamma_{C:AB}$, with $\alpha_{A:BC}$ separable in the $A:BC$ partition (similarly for $\beta_{B:AC}$ and $\gamma_{C:AB}$). We argue that $\alpha_{A:BC}$ is fully separable and a similar argument will apply to $\beta_{B:AC}$. Let $\alpha_{A:BC} = \sum_i p_i |a_i\rangle\langle a_i|_A \otimes |\psi_i\rangle\langle\psi_i|_{BC}$. Since ρ_{AB} is B finite and $|a_i\rangle \otimes \mathcal{R}(\mathrm{Tr}_C\, |\psi_i\rangle\langle\psi_i|) \subset \mathcal{R}(\rho_{AB})$, any $|\psi_i\rangle$ must be a product state. So $\alpha_{A:BC}$ is fully separable. Similarly, one can show that $\beta_{B:AC}$ is also fully separable, so ρ_{ABC} is separable with respect to $AB:C$. ■

We are now in the position to prove the following.

Theorem 3. Suppose the triple $\mathcal{E} = (\rho_{AB},\rho_{BC},\rho_{AC})$ is compatible. Then $\mathcal{C}(\mathcal{E}) \cap \mathcal{S}_{\mathrm{BS}} = \emptyset$ if all the following conditions are met: (i) For any $i,j \in \{A,B,C\}$ the state ρ_{ij} is i finite and j finite, (ii) ρ_{BC} has birank $(r(\rho_B)+1, r(\rho_B)+1)$, and (iii) ρ_{AB} has birank (r,s), $r \neq s$.

Proof. Suppose ρ_{AB}, ρ_{BC}, and ρ_{AC} are compatible with a biseparable state ρ_{ABC}. By hypothesis (i) and Lemma 4, we obtain that ρ_{ABC} is separable with respect to the partition $AB:C$. Let $\rho_{ABC} = \sum_{i=0}^{n-1} p_i \rho_i \otimes |c_i\rangle\langle c_i|$, where $p_i > 0$ and the $|c_i\rangle$ on $\mathcal{H}_C$ are pairwise linearly independent. Note that the product subspace $\mathcal{R}((\rho_i)_B) \otimes |c_i\rangle \subset \mathcal{R}(\rho_{BC}) \forall i$. This fact and (i) imply that $r((\rho_i)_B) = 1\, \forall i$. By similar arguments we have $r((\rho_i)_A) = 1$. We may assume $\rho_i = |a_i,b_i\rangle\langle a_i,b_i|$. Hence $\rho_{ABC} = \sum_{i=0}^{n-1} p_i |a_i,b_i,c_i\rangle\langle a_i,b_i,c_i|$. By (ii) we have $n \geqslant d+1$, where $d = r(\rho_B)$. Without loss of generality, we may assume that the states $|b_i\rangle$, $i = 0,\ldots,d-1$, span $\mathcal{R}(\rho_B)$. We choose a suitable ILO V such that $V|b_i\rangle \propto |i\rangle$ $(i = 0,\ldots,d-1)$, $V|b_i\rangle \propto |f_i\rangle$ $(i \geqslant d)$, and $|f_d\rangle$ is real. By performing V on the state ρ_{ABC}, we have

$$\begin{aligned}\sigma_{ABC} &= (I \otimes V \otimes I)\rho_{ABC}(I \otimes V \otimes I)^\dagger \\ &= \sum_{i=0}^{d-1} q_i |a_i,i,c_i\rangle\langle a_i,i,c_i| + \sum_{i=d}^{n-1} q_i |a_i,f_i,c_i\rangle\langle a_i,f_i,c_i|\end{aligned}$$

and $q_i > 0$ for any i. Since the operation V does not change the rank of quantum states, it follows from (ii) that σ_{BC} has birank $(d+1,d+1)$. Recall that the $|c_i\rangle$ are pairwise linearly independent. Since (i) is not changed under ILOs, it follows from (i) that $|f_d\rangle$ is not parallel to any state $|i\rangle$; otherwise ρ_{BC} would not be B finite. Since $|f_d\rangle$ is real, the two $(d+1)$-dimensional subspaces $\mathcal{R}(\sigma_{BC})$ and $\mathcal{R}(\sigma_{BC}^\Gamma)$ are equal and spanned by $|i,c_i\rangle, i = 0,\ldots,d-1$ and $|f_d,c_d\rangle$. So the states $|f_i,c_i\rangle,|f_i^*,c_i\rangle \in \mathcal{R}(\sigma_{BC})$ for any $i > d$. Then (i) implies that these $|f_i\rangle$ are real up to an overall phase. So $\sigma_{AB} = \sigma_{AB}^{\Gamma_B}$. This implies $r(\rho_{AB}) = r(\rho_{AB}^\Gamma)$, which is a contradiction with (iii). Therefore ρ_{AB}, ρ_{BC}, and ρ_{AC} are not compatible with any nongenuinely entangled state. This completes the proof. ■

We will now make use of Theorem 3 and offer an example of separable marginals only compatible with genuinely entangled tripartite states.

Example 2. Consider the family of rank-$(d+1)$ states on $\mathbb{C}^{d+1} \otimes \mathbb{C}^{d+1} \otimes \mathbb{C}^{d+1}$ given by $\rho_{ABC} = p_1\sigma_{ABC} + \sum_{m=2}^{d} p_m |mmm\rangle\langle mmm|$, with

$$\sigma_{ABC} = \tfrac{2}{3}|\xi\rangle\langle\xi| + \tfrac{1}{3}|111\rangle\langle 111|, \tag{16}$$

$|\xi\rangle = \frac{1}{2}|010\rangle + \frac{1}{2}|100\rangle + \frac{1}{\sqrt{2}}|001\rangle$, $p_1 > 0$, and $p_m \geqslant 0$. It is easy to see that the only biseparable pure state in $\mathcal{R}(\sigma_{ABC})$ is $|111\rangle$. The bipartite reduced density operators of σ_{ABC} are

$$\sigma_{AB} = \tfrac{1}{3}|\Phi^+\rangle\langle\Phi^+| + \tfrac{1}{3}|00\rangle\langle 00| + \tfrac{1}{3}|11\rangle\langle 11|, \tag{17}$$

$$\sigma_{BC} = \sigma_{AC} = \tfrac{1}{2}|\zeta\rangle\langle\zeta| + \tfrac{1}{6}|00\rangle\langle 00| + \tfrac{1}{3}|11\rangle\langle 11|, \tag{18}$$

with $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ and $|\zeta\rangle = \sqrt{\frac{2}{3}}|01\rangle + \sqrt{\frac{1}{3}}|10\rangle$. The three two-qubit marginals σ_{AB}, σ_{BC}, and σ_{AC} are PPT, so they are separable [22]. Hence, ρ_{AB}, ρ_{BC}, and ρ_{AC} are separable too; they also evidently satisfy condition (i) of Theorem 3. Furthermore, ρ_{BC} has birank $(d+2,d+2)$, while $r(\rho_B) = d+1$, and ρ_{AB} has birank $(d+2,d+3)$. So also conditions (ii) and (iii) of Theorem 3 are satisfied and we conclude that ρ_{AB}, ρ_{BC}, and ρ_{AC} are only compatible with genuinely tripartite entangled states.

The example shows that for any fixed local dimension d, there exist triples of two-qudit separable states that are only compatible with genuine multipartite entanglement. The core of our construction is the genuine multipartite entangled three-qubit state σ_{ABC} of Eq. (16). It turns out that σ_{ABC} is actually the only state compatible with its reductions. The proof of this is given in the Appendix. It is worth comparing this with the results of [23]. There it was proven that for almost all pure entangled states of three qubits $|\eta\rangle$, $\mathcal{C}(|\eta\rangle\langle\eta|) = \{|\eta\rangle\langle\eta|\}$ holds, with the exception of states of the generalized GHZ form $|g_{\mathrm{GHZ}}\rangle = \sqrt{p}|000\rangle + \sqrt{1-p}|111\rangle$ (up to local unitary transformations), which satisfy, e.g., $\{|g_{\mathrm{GHZ}}\rangle\langle g_{\mathrm{GHZ}}|, p|000\rangle\langle 000| + (1-p)|111\rangle\langle 111|\} \subset \mathcal{C}(|g_{\mathrm{GHZ}}\rangle\langle g_{\mathrm{GHZ}}|)$. Interestingly, the only three-qubit pure states that have separable reduction are of the generalized GHZ form [24]. This implies that any three-qubit state ρ such that (i) its reductions are separable and (ii) $\mathcal{C}(\rho) \cap \mathcal{S}_{\mathrm{BS}} = \emptyset$ must be mixed. Since the state σ_{ABC} has rank 2, we can think of it as the simplest possible example that satisfies (i) and (ii), with the additional property of being uniquely determined by its reductions. We generalize Example 2 in several ways, all presented in the Appendix.

Genuine multipartite entanglement from separable reductions is a robust feature

While we showed that there exist genuine multipartite states whose compatibility set contains only genuine multipartite states, it is natural to ask how common this phenomenon is, i.e., whether such states have finite volume in the set of all states. This is important also from the point of view of the potential realization of such states in the laboratory, which can never be perfect. We answer this question in the affirmative.

We introduce a parameter of compatibility of a tripartite state ρ_{ABC} with $\mathcal{E} = (\sigma_{AB},\sigma_{BC},\sigma_{AC})$ as $D(\rho_{ABC}|\mathcal{E}) := \|\rho_{AB} - \sigma_{AB}\|_2^2 + \|\rho_{BC} - \sigma_{BC}\|_2^2 + \|\rho_{AC} - \sigma_{AC}\|_2^2$, where we have used the Hilbert-Schmidt norm $\|X\|_2 = \sqrt{\mathrm{Tr}(X^\dagger X)}$.[1] We further define $D_{\mathrm{BS}}(\mathcal{E}) := \min_{\rho\in\mathcal{S}_{\mathrm{BS}}} D(\rho_{ABC}|\mathcal{E})$. We have $D_{\mathrm{BS}}(\mathcal{E}) > 0$ for any triple $\mathcal{E}$ such that $\mathcal{C}(\mathcal{E}) \cap \mathcal{S}_{\mathrm{BS}} = \emptyset$, even if the triple of reduced states is compatible, as in Example 2. Finally, given a tripartite state σ_{ABC}, we define $D(\rho_{ABC}|\sigma_{ABC}) := D(\rho_{ABC}|(\sigma_{AB},\sigma_{BC},\sigma_{AC}))$ and $D_{\mathrm{BS}}(\sigma_{ABC}) := \min_{\rho\in\mathcal{S}_{\mathrm{BS}}} D(\rho_{ABC}|\sigma_{ABC})$.

Now consider a genuinely entangled multipartite state $\bar{\sigma}_{ABC}$ with separable reductions such that $D_{\mathrm{BS}}(\bar{\sigma}_{ABC}) > 0$ and the convex combination of $\bar{\sigma}_{ABC}$ with an arbitrary fully separable state ρ^{FS}: $\tau_p(\bar{\sigma}_{ABC},\rho^{\mathrm{FS}}) := (1-p)\bar{\sigma}_{ABC} + p\rho^{\mathrm{FS}}$ for $0 \leqslant p \leqslant 1$. Since the set of biseparable states is closed, there exists $\bar{p} > 0$ such that $\tau_p(\bar{\sigma}_{ABC},\rho^{\mathrm{FS}})$ is genuinely multipartite entangled for all ρ^{FS} and all $0 \leqslant p < \bar{p}$. Since ρ^{FS} is fully separable, so are the two-party reduced states of $\tau_p(\bar{\sigma}_{ABC},\rho^{\mathrm{FS}})$. Furthermore, since D is continuous, there exists $\bar{p}_D > 0$ such that $D_{\mathrm{BS}}(\tau_p(\bar{\sigma}_{ABC},\rho^{\mathrm{FS}})) > 0$ for all ρ^{FS} and all $0 \leqslant p < \bar{p}_D$.

For any local finite dimensions, the set of fully separable states has nonzero volume among all states, because there exists a ball of fully separable states around the maximally mixed state [25]. Thus, the argument above proves that also the set of tripartite states whose two-party marginals are separable but only compatible with genuine multipartite entanglement has nonzero volume.

V. CONCLUSION

We analyzed the relation between the character of correlations of tripartite states and the ones exhibited by their bipartite reductions, i.e., a version of the quantum marginal problem that focuses on the compatibility of bipartite reductions with certain global properties. We constructed examples where separable reductions are only compatible with genuine multipartite entanglement. This separation between the character of correlations of bipartite reductions and what can be inferred about the quality of correlations of the global state, based only on the knowledge of the reductions, is large. On the other hand, at least for qubits we were able to prove that compatible reductions that are fully classical can always originate from a biseparable global state. Nonetheless, bipartite reductions that are fully classical may still require the presence of some entanglement in the global state. Our results show that the relation between global and local correlations is far from trivial. Notably, the notion of fully classical correlations is strong enough to overcome the need for genuine multipartite entanglement, but not the potential need for global entanglement altogether. An interesting open question is whether compatible classical-classical marginals in high dimension are always compatible with a biseparable tripartite state. Another question is how to quantitatively bound the certifiable genuine multipartite entanglement in terms of the nonclassicality of the two-body reductions, at least in the three-qubit case. The latter problem is reminiscent of the case of entanglement distribution, where the nonclassicality of correlations, rather than the entanglement, present between a quantum carrier and distant laboratories constitutes a bound on the entanglement that can be generated between the laboratories by exchanging the carrier [26,27].

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APPENDIX A: UNIQUENESS OF GLOBAL STATES COMPATIBLE WITH GIVEN REDUCTIONS

We first prove that σ_{ABC} in Eq. (16) in the paper is the only state compatible with its reductions, a fact of interest in its own.

Proposition 1. For σ_{ABC} in Eq. (16) $\mathcal{C}(\sigma_{ABC}) = \{\sigma_{ABC}\}$ holds.

Proof. Suppose $\rho = \rho_{ABC}$ has the same reductions as σ_{ABC}, i.e., $\rho \in \mathcal{C}(\sigma_{ABC})$. We can always write its spectral decomposition as $\rho = \sum_{i=0}^{7} p_i|\psi_i\rangle\langle\psi_i|$, where $|\psi_i\rangle = \sqrt{q_i}|0,\alpha_i\rangle + \sqrt{1-q_i}|1,\varphi_i\rangle$, with $|\alpha_i\rangle,|\varphi_i\rangle \in \mathcal{H}_B \otimes \mathcal{H}_C \forall i$. We have $\sigma_{BC} = \sum_i p_i(q_i|\alpha_i\rangle\langle\alpha_i| + (1-q_i)|\varphi_i\rangle\langle\varphi_i|)$. It follows from Eq. (5) that $r(\sigma_{BC}) = 3$. So any four states of $|\alpha_i\rangle, i = 0,\ldots,7$ are linearly dependent. Using the freedom in the choice of the pure-state ensemble representation of a mixed states [28], we can choose a suitable linear combination of $|\psi_i\rangle$, $i = 0,1,2,3$, such that it is equal to $|1\rangle_A|\varphi_3'\rangle_{BC}$. So the state can be written as $\rho = \sum_{i=0}^{3} r_i|\psi_i'\rangle\langle\psi_i'| + \sum_{i=4}^{7} p_i|\psi_i\rangle\langle\psi_i|$, where $|\psi_3'\rangle = |1\rangle_A|\varphi_3'\rangle_{BC}$. By applying this procedure to another four states $|\psi_0'\rangle,|\psi_1'\rangle,|\psi_2'\rangle,|\psi_j\rangle$ with $j = 4,5,6,7$, respectively, we can realize $|\psi_j\rangle = |1\rangle_A|\varphi_j'\rangle_{BC}$.

By relabeling the states, we can write $\rho = \sum_{i=0}^{2} p_i'|\psi_i\rangle\langle\psi_i| + p_3'|1\rangle\langle 1| \otimes \rho_0$ with ρ_0 on $\mathcal{H}_B \otimes \mathcal{H}_C$. We have $\mathcal{R}(|1\rangle\langle 1| \otimes (\rho_0)_B) \subset \mathcal{R}(\sigma_{AB})$ and $|11\rangle \in \mathcal{R}(\sigma_{AB})$ by Eq. (17). Since σ_{AB} is X finite for $X = A,B$, we have $(\rho_0)_B = |1\rangle\langle 1|$. By a similar argument we can show $(\rho_0)_C = |1\rangle\langle 1|$. So we have $\rho = \sum_{i=0}^{2} p_i'|\psi_i\rangle\langle\psi_i| + p_3'|111\rangle\langle 111|$.

Let $|\psi_i\rangle = \sum_{j,k,l=0}^{1} c_{i,m}|jkl\rangle$, where $i = 0,1,2$ and $m = 4j + 2k + l$. By Eq. (17) we have $c_{i2} = c_{i4}$ and $c_{i3} = c_{i5}$. By Eq. (18) we have $c_{i2} = c_{i1}/\sqrt{2}$ and $c_{i6} = c_{i5}/\sqrt{2}$. These

[1] We make this choice for the sake of concreteness, but our argument is only based on continuity of D in its arguments and the fact that $D(\rho_{ABC}|\mathcal{E})$ is positive and vanishes if an only if $\rho_{ABC} \in \mathcal{C}(\mathcal{E})$.

equations imply, for $i=0,1,2$, that

$$|\psi_i\rangle = c_{i0}|000\rangle + c_{i1}\left(|001\rangle + \frac{1}{\sqrt{2}}|010\rangle + \frac{1}{\sqrt{2}}|100\rangle\right) + c_{i3}\left(|011\rangle + |101\rangle + \frac{1}{\sqrt{2}}|110\rangle\right) + c_{i7}|111\rangle. \quad (A1)$$

The coefficients of $|11\rangle\langle 11|$ in both σ_{AB} and σ_{AC} are 1/3, so $c_{03}=c_{13}=c_{23}=0$. By replacing $|\psi_i\rangle$, $i=0,1,2$, by a suitable linear combination of them, we may assume $c_{11}=c_{21}=c_{20}=0$. So the tripartite state can be rewritten as $\rho=\sum_{i=0}^{2} p_i''|\psi_i\rangle\langle\psi_i|$, where

$$\begin{aligned}|\psi_0\rangle &= c'_{00}|000\rangle + c'_{01}\left(|001\rangle + \frac{1}{\sqrt{2}}|010\rangle + \frac{1}{\sqrt{2}}|100\rangle\right) + c'_{07}|111\rangle,\\ |\psi_1\rangle &= c'_{10}|000\rangle + c'_{17}|111\rangle, \quad (A2)\\ |\psi_2\rangle &= |111\rangle.\end{aligned}$$

Since $r(\sigma_{BC})=3$, we have $c'_{01}\neq 0$. By Eq. (17) we have $c'_{01}c'_{00}=c'_{01}c'_{07}=0$. So $c'_{00}=c'_{07}=0$. By Eq. (17) again we have $p''_0=\frac{2}{3}$, $|c'_{01}|=\frac{1}{\sqrt{2}}$, and $c'_{10}=0$. Now we see $\rho=\sigma_{ABC}$ in Eq. (16). This completes the proof. ■

We further derive (and later use in Appendix B) the following lemma.

Lemma 5. Suppose ρ_{AB}, ρ_{BC}, and ρ_{AC} are only compatible with a tripartite state ρ_{ABC} and σ_{AB}, σ_{BC}, and σ_{AC} are compatible with another tripartite state σ_{ABC}. If $\mathcal{R}(\sigma_{ABC})\subseteq\mathcal{R}(\rho_{ABC})$, then σ_{ABC} is the only state with which σ_{AB}, σ_{BC}, and σ_{AC} are compatible.

Proof. Suppose σ_{AB}, σ_{BC}, and σ_{AC} are compatible with another state $\sigma'_{ABC}\neq\sigma_{ABC}$. Since $\mathcal{R}(\sigma_{ABC})\subseteq\mathcal{R}(\rho_{ABC})$, we may find a small enough $p>0$ and a tripartite state α_{ABC} such that $\rho_{ABC}=p\sigma_{ABC}+(1-p)\alpha_{ABC}$. So the bipartite reductions ρ_{AB}, ρ_{BC}, and ρ_{AC} are compatible with the state $p\sigma'_{ABC}+(1-p)\alpha_{ABC}$, which is different from ρ_{ABC}. This gives us a contradiction. ■

We conclude this section by presenting separable marginals that are only compatible with a unique quantum correlated (unentangled) state.

Proposition 2. The separable states $\rho_{AB}=\rho_{BC}=\rho_{AC}=p|00\rangle\langle 00|+(1-p)|a,a\rangle\langle a,a|$, where $|a\rangle=\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$, are only compatible with the separable state $\rho_{ABC}=p|000\rangle\langle 000|+(1-p)|a,a,a\rangle\langle a,a,a|$.

Proof. We will use the following observation in the proof and it is easy to verify. For any $X,Y\in\{A,B,C\}$ there are only two product states $|00\rangle,|a,a\rangle\in\mathcal{R}(\rho_{XY})$, which also span the space $\mathcal{R}(\rho_{XY})$. That is, any state in $\mathcal{R}(\rho_{XY})$ is the linear combination of $|00\rangle$ and $|a,a\rangle$.

It is clear that $\rho_{AB},\rho_{BC},\rho_{AC}$ are compatible with ρ_{ABC}. Suppose they are compatible with another three-qubit state $\sigma_{ABC}=\sum_i p_i|\psi_i\rangle\langle\psi_i|$. By applying the observation to the system B,C we have $|\psi_i\rangle=f_i|\alpha_i,00\rangle+g_i|\beta_i,a,a\rangle$ with some complex numbers f_i,g_i. By applying the observation to system A,B we have $g_i|\beta_i\rangle\propto|a\rangle$ and hence $f_i|\alpha_i\rangle\propto|0\rangle$. So we may assume $|\psi_i\rangle=f'_i|000\rangle+g'_i|a,a,a\rangle$. As a result, the range of the state σ_{ABC} is spanned by the product states $|000\rangle,|a,a,a\rangle$. By simple algebra one can see that the only feasible σ_{ABC} compatible with ρ_{AB}, ρ_{BC}, and ρ_{AC} is the convex sum of $|000\rangle\langle 000|$ and $|a,a,a\rangle\langle a,a,a|$. By using the condition $\rho_{XY}=\sigma_{XY}$ we obtain $\sigma_{ABC}=\rho_{ABC}$. This completes the proof. ■

APPENDIX B: GENERALIZATIONS OF EXAMPLE 2

We provide here some further examples of states with separable reductions that are only compatible with genuine multipartite entanglement, also making use of Proposition 1 and Lemma 5.

Note that $|111\rangle\in\mathcal{R}(\sigma_{ABC})$ for σ_{ABC} in Eq. (16). It follows from Lemma 5 that for any $p\in(0,1)$, the separable states $p\sigma_{AB}+(1-p)|11\rangle\langle 11|$, $p\sigma_{AC}+(1-p)|11\rangle\langle 11|$, and $p\sigma_{BC}+(1-p)|11\rangle\langle 11|$ are uniquely compatible with the state $p\sigma_{ABC}+(1-p)|111\rangle\langle 111|$. So we have generated a family of separable bipartite marginals that are uniquely compatible with a genuinely entangled state, extending Example 2.

We now generalize Example 2 to a different family of states that satisfy the conditions in Theorem 3. Let σ_{ABC} be as in Eq. (16) and the product state $|a,b\rangle\in\mathcal{R}(\sigma_{BC})\cap\mathcal{R}(\sigma_{BC}^{\Gamma})$. Such a product state always exists because $r(\sigma_{BC})=r(\sigma_{BC}^{\Gamma})=3$ and there is a product state in any two-dimensional two-qubit subspace. For example, we can choose $|a,b\rangle=(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})\otimes(\sqrt{\frac{1}{3}},\sqrt{\frac{2}{3}})$. We have the following corollary now.

Corollary 1. Let $\vec{p}=(p_1,\ldots,p_n)$, $\sum_{i=1}^{n}p_i=1$, $p_1>0$, $p_i\geqslant 0$. For $i>1$ suppose the product states $|a_i,b_i\rangle\in\mathcal{R}(\sigma_{BC})\cap\mathcal{R}(\sigma_{BC}^{\Gamma})$, where $|a_i\rangle$ is real and σ_{BC} is the reduced density operator of the state σ_{ABC} in Eq. (16). The three reduced density operators of the three-qubit state $\sigma_{\vec{p}}=p_1\sigma_{ABC}+\sum_{i=2}^{n}p_i|a_i,a_i,b_i\rangle\langle a_i,a_i,b_i|$ are only compatible with genuinely entangled states.

Proof. It is sufficient to show that the three reduced density operators $(\sigma_{\vec{p}})_{AB}$, $(\sigma_{\vec{p}})_{AC}$, and $(\sigma_{\vec{p}})_{BC}$ satisfy the three conditions (i), (ii), and (iii) in Theorem 3. Recall that σ_{ABC} satisfies these conditions. Since $|a_i,a_i\rangle\in\mathcal{R}(\sigma_{AB})$ and $|a_i,b_i\rangle\in\mathcal{R}(\sigma_{BC})=\mathcal{R}(\sigma_{AC})$, we have $\mathcal{R}((\sigma_{\vec{p}})_{XY})=\mathcal{R}(\sigma_{XY})$ for any $X,Y\in\{A,B,C\}$. So condition (i) is satisfied. Next, the same argument shows that $(\sigma_{\vec{p}})_{AB}$ has birank (3,4), which is exactly condition (iii). Third, the hypothesis $|a_i,b_i\rangle\in\mathcal{R}(\sigma_{BC})\cap\mathcal{R}(\sigma_{BC}^{\Gamma})$ and $|a_i\rangle$ is real imply that the birank of $(\sigma_{\vec{p}})_{BC}$ is (3,3). So condition (ii) is also satisfied. ■

[1] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).

[2] K. Modi, A. Brodutch, H. Cable, T. Paterek, and V. Vedral, Rev. Mod. Phys. **84**, 1655 (2012).

[3] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, Rev. Mod. Phys. **86**, 419 (2014).

[4] J. Eisert, M. Cramer, and M. B. Plenio, Rev. Mod. Phys. **82**, 277 (2010).

[5] K. Binder and A. P. Young, Rev. Mod. Phys. **58**, 801 (1986); *Frustrated Spin Systems*, edited by H. T. Diep (World Scientific, Singapore, 2005); C. M. Dawson and M. A. Nielsen, Phys. Rev. A **69**, 052316 (2004).
[6] S. Bravyi, Quantum Inf. Comput. **4**, 12 (2004).
[7] A. Klyachko, arXiv:quant-ph/0409113.
[8] M. Christandl and G. Mitchison, Commun. Math. Phys. **3**, 261 (2006).
[9] M. Christandl, M. B. Sahinoglu, and M. Walter, arXiv:1210.0463.
[10] J. Chen, Z. Ji, D. Kribs, Z. Wei, and B. Zeng, J. Math. Phys. **53**, 102201 (2012).
[11] P. D. Johnson and L. Viola, Phys. Rev. A **88**, 032323 (2013).
[12] J. Chen, Z. Ji, D. Kribs, N. Lütkenhaus, and B. Zeng, Phys. Rev. A **90**, 032318 (2014).
[13] M. Walter, B. Doran, D. Gross, and M. Christandl, Science **340**, 1205 (2013).
[14] G. Toth, C. Knapp, O. Gühne, and H. J. Briegel, Phys. Rev. Lett. **99**, 250405 (2007).
[15] G. Toth, C. Knapp, O. Gühne, and H. J. Briegel, Phys. Rev. A **79**, 042334 (2009).
[16] L. E. Wurflinger, J.-D. Bancal, A. Acin, N. Gisin, and T. Vertesi, Phys. Rev. A **86**, 032117 (2012).
[17] B. Kraus, J. I. Cirac, S. Karnas, and M. Lewenstein, Phys. Rev. A **61**, 062302 (2000).
[18] L. Chen, E. Chitambar, K. Modi, and G. Vacanti, Phys. Rev. A **83**, 020101(R) (2011).
[19] M. Christandl and A. Winter, J. Math. Phys. **45**, 829 (2004).
[20] L. Chen and D. Ž. Đoković, Phys. Rev. A **86**, 062332 (2012).
[21] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).
[22] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996).
[23] N. Linden, S. Popescu, and W. K. Wootters, Phys. Rev. Lett. **89**, 207901 (2002).
[24] A. V. Thapliyal, Phys. Rev. A **59**, 3336 (1999).
[25] L. Gurvits and H. Barnum, Phys. Rev. A **68**, 042312 (2003).
[26] T. K. Chuan, J. Maillard, K. Modi, T. Paterek, M. Paternostro, and M. Piani, Phys. Rev. Lett. **109**, 070501 (2012).
[27] A. Streltsov, H. Kampermann, and D. Bruss, Phys. Rev. Lett. **108**, 250501 (2012).
[28] L. P. Hughston, R. Josza, and W. K. Wootters, Phys. Lett. A **183**, 14 (1993).