Maximal incompatibility of locally classical behavior and global causal order in multiparty scenarios

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Quantum theory in a global spacetime gives rise to nonlocal correlations, which cannot be explained causally in a satisfactory way; this motivates the study of theories with reduced global assumptions. Oreshkov, Costa, and Brukner [Nat. Commun. **3**, 1092 (2012)] proposed a framework in which quantum theory is valid locally but where, at the same time, no global spacetime, i.e., predefined causal order, is assumed beyond the absence of logical paradoxes. It was shown for the two-party case, however, that a global causal order always emerges in the classical limit. Quite naturally, it has been conjectured that the same also holds in the multiparty setting. We show that, counter to this belief, classical correlations locally compatible with classical probability theory exist that allow for deterministic signaling between three or more parties incompatible with any predefined causal order.

I. MOTIVATION AND MAIN RESULT

According to Bell [1], correlations cry out for explanation. In such a spirit, already Einstein, Podolsky, and Rosen (EPR) [2] had asked for an extension of quantum theory that incorporates a *causal explanation* [3,4] of the correlations arising when two parts of an entangled quantum state are measured. Such an explanation can describe the emergence of the correlations either through *preshared information* or through *influences*. Because of relativity, EPR argued further, the latter cannot be the cause of such correlations. Later, (finite-speed) influences were ruled out by theory [5,6] and experiments [7–13]. Therefore, still according to EPR's reasoning, physical quantities need to be *predefined*. This, however, had been rejected by Bell [14] under the assumption that spatially separated settings can be chosen (at least partially [15–17]) freely and independently; such correlations are called *nonlocal*. Remarkably, this means that there are *not predefined yet correlated* physical quantities emerging in a spacelike separated way. However, although the EPR program as such may have failed, it seems natural to continue to ask for a causal explanation of nonlocal correlations. A possible approach is to refrain from considering spacetime as fundamental, treating it as emerging (potentially along with other macroscopic quantities) from a deeper fundament [18–21]—comparably to temperature. A step in this direction was taken by Hardy [22,23] with his program of merging general relativity with quantum theory, in which he proposes to extend the latter to *dynamical causal orders*, a feature of relativity (see Ref. [24] for a recent review on quantum theory and causality). Chiribella, D'Ariano, and Perinotti [25,26] studied quantum supermaps called "quantum combs" that allow for superpositions of causal orders. Based on Hardy's idea, Oreshkov, Costa, and Brukner [27] developed a framework for quantum correlations without predefined causal order by dropping the assumption of a *global background time* while keeping the assumptions that *locally*, nature is described by quantum theory and that *no logical paradoxes* arise. Some causal structures emerging from this framework cannot be predefined [27–29]—just like physical quantities exhibiting nonlocal correlations [14,30–32]. If, in the *two-party* case, we consider the classical limit of the quantum

systems, i.e., enforce both parties' physics to be described by *classical probability theory* (instead of quantum theory), then a predefined causal order *always emerges* [27]. This is in accordance with our experience and, hence, natural and unsurprising; it strongly indicates that the same may hold in the *multiparty* case [33]. This, however, fails to be true, as we show in the present work.

II. INPUT-OUTPUT SYSTEMS AND CAUSAL ORDER

By definition, measurement settings and outcomes are classical, i.e., perfectly distinguishable. Therefore, we think of physical systems as black boxes which we probe with classical inputs and that respond with classical outputs. When taking this perspective, we describe all *physical quantities*, i.e., *outputs*, as functions of *inputs*. The party S is described by a set of *inputs* $V(S) = \{A_i\}_I$ chosen *freely* by S, and by a set of *outputs* $Q(S) = \{X_j\}_J$ the party can access (instantiations of the inputs and outputs are denoted by the same letters but in lowercase). Since we refrain from assuming global *spacetime* as given *a priori*, we cannot define *free randomness* based on such a causal structure, as done elsewhere [34,35]. Instead, we take the concept of *free randomness* as fundamental—in accordance with a recent trend to derive properties of quantum theory from information-theoretic principles [36–43]—and *postulate* inputs as being *free*.

Outputs are functions of inputs. Based on this relationship, we define *causal order*. If an output X_j is a function of A_i, we say that X_j *depends causally* on A_i and is in the *causal future* of A_i or, equivalently, that A_i is in the *causal past* of X_j, denoted by $X_j \succeq A_i$ or $A_i \preceq X_j$. The negations of these relations are denoted by $\not\succeq$ and $\not\preceq$. This definition neither induces a causal order between outputs nor between inputs nor between any output and the input it does not depend on.

Let us introduce a second party T described by the set of inputs $V(T) = \{B_k\}_K$ and with access to the outputs $Q(T) = \{Y_\ell\}_L$. Outputs can depend on inputs of both parties. If party S has an output that depends on an inputs of T, then we say that T can *signal* to S (see Fig. 1). In the following, we will assume *unidirectional* signaling: If S can signal to T, then T cannot signal to S. This enables us to causally order parties. If at least one output of S depends on an input of T,
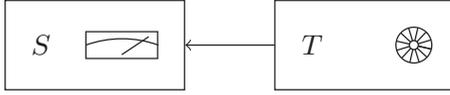
FIG. 1. If party T can freely choose an input (here, visualized by a knob), and party S can read off an output that depends on T's input, then T can *signal* to S, which implies that S is in the *causal future* of T ($S \succeq T$).

but no output of T depends on any input of S (which is the condition for unidirectional signaling), then S is in the causal future of T. Formally, if there exist $X \in Q(S)$ and $B \in V(T)$ fulfilling $X \succeq B$ and if for all $Y \in Q(T)$ and for all $A \in V(S)$, the relation $Y \not\succeq A$ holds, then we have $S \succeq T$.

Consider a two-party scenario with parties S, T, each having a single input A, B, a single output X, Y, respectively, and a shared random variable Λ. We call a theory compatible with *predefined causal order* if all achievable probability distributions $P(x,y|a,b)$ can be written as a convex mixture of possible causal orders, i.e.,

$$
\begin{aligned}
P(x,y|a,b) = {}& \Pr(\alpha) \sum_\lambda \Pr(\lambda|\alpha) \Pr(x|a,\lambda,\alpha) \Pr(y|a,b,\lambda,\alpha) \\
& + \Pr(\neg\alpha) \sum_\lambda \Pr(\lambda|\neg\alpha) \Pr(x|a,b,\lambda,\neg\alpha) \\
& \times \Pr(y|b,\lambda,\neg\alpha),
\end{aligned}
$$

where α is the event $S \preceq T$, and λ is an instantiation of Λ that depends on a input not in either of the sets $V(S)$ or $V(T)$. For more than two parties, the definition of predefined causal order becomes more subtle. Suppose we have three parties S, T, and U, where S is in the causal past of both T and U. We call a causal order *predefined* even if S is free to choose the causal order between T and U [44,45]. In general, in a predefined causal order, a party is allowed to determine the causal order between all parties in her causal future. Hence, a theory with the parties $S_0,\ldots,S_{n-1}$, inputs $A_0,\ldots,A_{n-1}$ (shorthand $\vec{A}$), and outputs $X_0,\ldots,X_{n-1}$ ($\vec{X}$), respectively, is compatible with *predefined causal order* if all achievable probability distributions $P(\vec{x}|\vec{a})$ can be written as

$$
\begin{aligned}
P(\vec{x}|\vec{a}) = {}& \sum_{i=0}^{n-1} \Pr(\alpha_i \wedge \neg\alpha_0 \wedge \cdots \wedge \neg\alpha_{i-1}) \\
& \times \Pr(\vec{x}|\vec{a}, S_i \text{ is first}),
\end{aligned}
$$

where α_i is the event that each party $S_{j(\neq i)}$ either is in the causal future of S_i ($S_i \preceq S_j$) or has no causal relation with S_i ($S_i \not\preceq S_j$ and $S_i \not\succeq S_j$). The term $\Pr(\vec{x}|\vec{a}, S_i \text{ is first})$ is a convex mixture of distributions compatible with the causal structures in which S_i is first and chooses the causal order between the remaining parties.

III. GAME

The following multiparty game cannot be won in a scenario with predefined causal order. Denote by $S_0,\ldots,S_{n-1}$ the parties that participate in the game. Each party S_i has a uniformly distributed binary input A_i as well as a binary output X_i and access to the shared random variable M uniformly distributed in the range $\{0,\ldots,n-1\}$. The random

variable M belongs to a dummy party (we need her as a source of shared randomness). For given $M = m$, the game is won whenever S_m's output X_m is the parity of the inputs to all other parties, i.e., $X_m = \bigoplus_{i \neq m} A_i$. Therefore, the success probability for winning the game is

$$
p_{\text{succ}} = \frac{1}{n} \sum_{m=0}^{n-1} \Pr\left(X_m = \bigoplus_{i \neq m} A_i \,\middle|\, M = m \right). \tag{1}
$$

In a setup with predefined causal order, this success probability is upper bounded by $1 - 1/(2n)$. To see this, note that if, without loss of generality, S_0 is first, she will remain first. For $n > 2$, the last party can be specified by S_0. Thus, all the terms of the sum in expression (1) are 1 except for the first summand, reflecting the fact that S_0 herself has to guess the parity of the other's inputs, which is $1/2$. By repeating the experiment $\omega(n)$ times, one can bring the winning probability arbitrarily closely to 0.

IV. FRAMEWORK FOR CLASSICAL CORRELATIONS WITHOUT CAUSAL ORDER

Instead of assuming that locally, nature is described by *quantum theory* [27], we take the *classical limit* of the systems and thus assume that locally, nature is described by *classical probability theory*. In addition to this, we require the probabilities of the outcomes to be nonnegative and to sum up to 1; this excludes logical paradoxes [26,27]. We suppose that each party has a closed laboratory that can be opened once—which is when the only interaction with the environment happens. When a laboratory is opened, the party receives, manipulates, and outputs a state. Thus, in the setting of local validity of classical probability theory, such a laboratory is described by a conditional probability distribution $P_{O|I}$, where I is the input to and O is the output from the laboratory.

Let us consider the parties as described in the game. We denote the input to S_i by I_i and the output from S_i by O_i. Therefore, the ith local laboratory is described by the distribution $P_{X_i,O_i|A_i,I_i}$. Because we do not make global assumptions other than that the overall picture should describe a probability distribution, we describe everything outside the laboratories by the distribution W (see Fig. 2) satisfying the
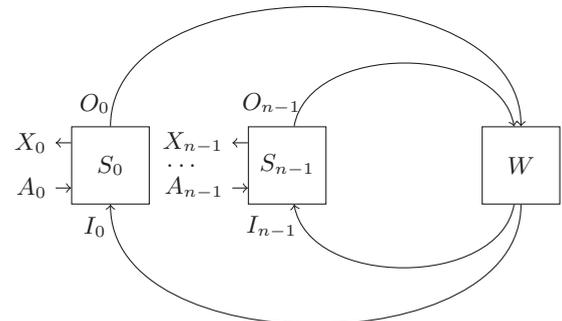

FIG. 2. Party S_i is described by $P_{X_i,O_i|A_i,I_i}$. Her output O_i is fed to W, which describes *everything* outside the laboratories. Therefore, W also provides the input I_i and is described by $W = P_{I_0,\ldots,I_{n-1}|O_0,\ldots,O_{n-1}}$.

condition that, for any choice of $a_0, \ldots, a_{n-1}$, i.e., for any probability distribution $P_{X_0,O_0|I_0}, \ldots, P_{X_{n-1},O_{n-1}|I_{n-1}}$, the values of the product with W, i.e., the values of

$$P_{X_0,O_0|I_0} \cdot \cdots \cdot P_{X_{n-1},O_{n-1}|I_{n-1}} \cdot P_{I_0,\ldots,I_{n-1}|O_0,\ldots,O_{n-1}},$$

and in general of

$$P_{O_0|I_0} \cdot \cdots \cdot P_{O_{n-1}|I_{n-1}} \cdot P_{I_0,\ldots,I_{n-1}|O_0,\ldots,O_{n-1}},$$

are non-negative and sum up to 1. For tackling this condition formally, we represent a probability distribution P_X as a real positive diagonal matrix $\hat{P}_X$ with trace 1 and diagonal entries $P_X(x)$. A conditional probability distribution $P_{X|Y}$ is a collection of (unconditional) probability distributions $P_{X|Y=y}$ for each value of y. Thus, we represent $P_{X|Y}$ similarly, yet with trace $|\mathcal{Y}|$, where $\mathcal{Y}$ is the set of values y can take, and we use the symbol $\hat{P}_{X|Y}$. The values $P_{X|Y=y}(x)$ are ordered with respect to the ordering of the subscripts of $P_{X|Y}$, e.g., for binary X and Y the matrix $\hat{P}_{X|Y}$ is

$$\begin{pmatrix} P_{X|Y=0}(0) & 0 & 0 & 0 \\ 0 & P_{X|Y=1}(0) & 0 & 0 \\ 0 & 0 & P_{X|Y=0}(1) & 0 \\ 0 & 0 & 0 & P_{X|Y=1}(1) \end{pmatrix}.$$

The condition that the probabilities $P_{X|Y=y}(x)$ sum up to 1 for fixed y is reflected by the condition that if we trace out X from the matrix $\hat{P}_{X|Y}$ (denoted by $\mathrm{Tr}_X \hat{P}_{X|Y}$), we are left with the identity. The product of two distributions $\hat{P}_X$ and $\hat{P}_Y$ in the matrix representation corresponds to the tensor product denoted by $\hat{P}_X \otimes \hat{P}_Y$. To obtain the marginal distribution from a joint distribution, we use the partial trace. This implies that the output state of a laboratory $P_{O_i|I_i}$, given the input state P_{I_i}, is $\mathrm{Tr}_{I_i}(\hat{P}_{O_i|I_i} \cdot (\mathbb{1}_{O_i} \otimes \hat{P}_{I_i}))$, where $\mathbb{1}_{O_i}$ is the identity matrix with the same dimension as $\hat{P}_{O_i}$. This allows us to use the framework of Oreshkov, Costa, and Brukner [27], where we restrict ourselves to diagonal matrices, i.e., all objects (W and local operations) are simultaneously diagonalizable in the computational basis and can, hence, be expressed using the identity $\mathbb{1}$ and the Pauli matrix σ_z. We know from their framework [27] that if we express $P_{I_0,\ldots,I_{n-1}|O_0,\ldots,O_{n-1}}$ as a matrix $W = c \sum_i g_i$, where c is a normalization constant and $g_i = R_{i,0} \otimes \cdots \otimes R_{i,n-1} \otimes T_{i,0} \otimes \cdots \otimes T_{i,n-1}$. For every i, the summand g_i represents a channel from all S_j with $\mathrm{Tr} T_{i,j} = 0$ to all S_k with $\mathrm{Tr} R_{i,k} = 0$. In order to avoid logical paradoxes, g_i must describe a channel where at least one party is a recipient without being a sender [27]. In other words, g_i must either be the identity or there exists j such that $T_{i,j} = \mathbb{1}$ and $\mathrm{Tr} R_{i,j} = 0$.

V. WINNING THE GAME PERFECTLY

To win the game using this framework, we need to provide the distribution $P_{I_0,\ldots,I_{n-1}|O_0,\ldots,O_{n-1}}$ and all distributions describing the laboratories. For that purpose, we use the fact that, if a set $\{g_i\}_I$ of matrices with all eigenvalues in $\{-1,1\}$ forms an Abelian group with respect to matrix multiplication, then $\sum_{i \in I} g_i$ is a positive semidefinite matrix. To prove this, take the eigenvector $\vec{v}$ which has the smallest

eigenvalue $\lambda_{\min}$, i.e.,

$$\sum_{i \in I} g_i \vec{v} = \sum_{i \in I} \lambda_i^{\vec{v}} \vec{v} = \lambda_{\min} \vec{v},$$

where $\lambda_i^{\vec{v}}$ is the eigenvalue of g_i with respect to the eigenvector $\vec{v}$. Let g_{i_0} be an element contributing negatively to $\lambda_{\min}$, i.e., $g_{i_0} \vec{v} = -\vec{v}$. As the set forms a group, for every j there exists a $k \neq j$ such that $g_{i_0} \cdot g_j = g_k$. This implies $-\lambda_j^{\vec{v}} = \lambda_k^{\vec{v}}$ and $\sum_{i \in I} \lambda_i^{\vec{v}} = 0$.

A. Construction of W_n for odd n

We construct the distribution $P_{I_0,\ldots,I_{n-1}|O_0,\ldots,O_{n-1}}$ for odd $n > 2$. Let $\{g_i\}_I$ be the set of matrices that can be written as $g_i = g_{i,1} \otimes g_{i,2} \otimes \cdots \otimes g_{i,n}$, with the objects $g_{i,k} \in \{\mathbb{1}, \sigma_z\}$, and with an even number of σ_z for each $i \in I$. We use the notation $g_{i,j:k}$ to denote the matrix $g_{i,j} \otimes g_{i,j+1} \otimes \cdots \otimes g_{j,k}$. The fact $\sigma_z^2 = \mathbb{1}$ implies that the product $g_i \cdot g_j$ for every $i,j \in I$ is a tensor product of $\mathbb{1}$ and σ_z with an even number of σ_z and is thus an element of $\{g_i\}_I$. Furthermore, all elements mutually commute, have all eigenvalues in $\{-1,1\}$ and, hence, each element is an involution. Therefore, their sum is a positive semidefinite matrix. The distribution $P_{I_0,\ldots,I_{n-1}|O_0,\ldots,O_{n-1}}$ as a matrix W_n is built by taking the sum over all group elements, where the matrix $g_{i,k}$ of the group element g_i contributes to the input I_k of party S_k, and to the output $O_{k-1 \bmod n}$ of the party labeled by $(k - 1 \bmod n)$,

$$W_n = \hat{P}_{I_0,\ldots,I_{n-1}|O_0,\ldots,O_{n-1}} = \frac{1}{2^n} \sum_{i \in I} g_i \otimes g_{i,2:n} \otimes g_{i,1}.$$

By construction, W_n is positive semidefinite, i.e., all probabilities are positive. Because n is odd, there exists for each group element g_i ($\neq \mathbb{1}$) at least one position k such that $g_{i,k} \otimes g_{i,k+1 \bmod n} = \sigma_z \otimes \mathbb{1}$, which excludes logical paradoxes. Furthermore, for every $i \in \{0,1,\ldots,n-1\}$ the object W_n contains the channel from all parties $S_{j(\neq i)}$ to S_i—a condition to perfectly win the game.

B. Example: W_3

For illustration, we construct W_3. The group from which W_3 is constructed is $\{g_0, g_1, g_2, g_3\}$ with the group elements

$$g_0 = \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1},$$
$$g_1 = \mathbb{1} \otimes \sigma_z \otimes \sigma_z,$$
$$g_2 = \sigma_z \otimes \mathbb{1} \otimes \sigma_z,$$
$$g_3 = \sigma_z \otimes \sigma_z \otimes \mathbb{1}.$$

The matrix W_3 is thus

$$W_3 = \frac{1}{8}(\mathbb{1}^{\otimes 6} + \mathbb{1} \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \mathbb{1}$$
$$+ \sigma_z \otimes \mathbb{1} \otimes \sigma_z \otimes \mathbb{1} \otimes \sigma_z \otimes \sigma_z$$
$$+ \sigma_z \otimes \sigma_z \otimes \mathbb{1} \otimes \sigma_z \otimes \mathbb{1} \otimes \sigma_z).$$

The second summand of W_3 represents a channel from S_0, S_1 to S_2, the third summand represents a channel from S_1, S_2 to S_0, and finally, the last summand represents a channel from S_0, S_2 to S_1.

It can easily be verified that if the three parties S_0, S_1, and S_2 input $P_{O_0,O_1,O_2}(o_0,o_1,o_2) = 1$ into W_3, then W_3 outputs
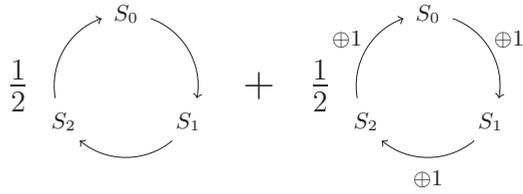
FIG. 3. The conditional probability distribution W_3 is a mixture of a circular identity channel with a circular bit-flip channel.

the distribution

$$P_{I_0,I_1,I_2}(i_0,i_1,i_2) = \begin{cases} \frac{1}{2}, & i_0 = o_2, \quad i_1 = o_0, \quad i_2 = o_1 \\ \frac{1}{2}, & i_0 = \bar{o}_2, \quad i_1 = \bar{o}_0, \quad i_2 = \bar{o}_1 \\ 0, & \text{otherwise}, \end{cases}$$

where $\bar{o}_i = o_i \oplus 1$. Therefore, W_3 implements a uniform mixture of the loops where the input of party $S_{i \bmod 3}$ is sent to party $S_{i+1 \bmod 3}$, and where the input of party $S_{i \bmod 3}$ is flipped and sent to $S_{i+1 \bmod 3}$ (see Fig. 3) [46]. It is evident from Fig. 3 that logical paradoxes are not possible. If all intermediate parties forward what they receive (by applying any reversible transformation), both loops (see Fig. 3) cancel each other out, i.e., the correlations interfere destructively. Then again, if one intermediate party does not forward what she receives, the loop is broken. Conversely, any party can signal to her predecessor in the loop, because then an even number of bit-flips are applied, and thus the correlations interfere constructively. The same reasoning holds for any W_n for odd $n > 2$.

C. Construction of W_n for even n

The above construction works for odd $n > 2$. For even n, the group contains the element $\sigma_z^{\otimes n}$, which leads to a logical paradox since all inputs are correlated to all outputs [27]. This

can also be seen in Fig. 3 where, for even n, the sum of both channels leads to a logical paradox. In this case (n even), we double the dimensions of the output of the second-to-last party and of the input of the last party and construct the distribution based on the group of matrices for the case of $n-1$. Let $\{g_i\}_I$ be the group used to construct W_{n-1}. The set for n parties is the Abelian subgroup $\{g_i \otimes g'_i\}_I \cup \{\bar{g}_i \otimes g'_i\}_I$, where $g'_i = g_{i,1} \otimes g_{i,2}$ and $\bar{g}_i = g_i \cdot \sigma_z^{\otimes(n-1)}$.

The distribution $P_{I_0,\ldots,I_{n-1}|O_0,\ldots,O_{n-1}}$ as a matrix W_n is constructed as before, with the exception that g'_i is considered a *single* submatrix,

$$W_n = \frac{1}{2^{n+1}} \sum_{i \in I} (g_i \otimes g'_i \otimes g_{i,2:n-1} \otimes g'_i \otimes g_{i,1} + \bar{g}_i \otimes g'_i \otimes \bar{g}_{i,2:n-1} \otimes g'_i \otimes \bar{g}_{i,1}).$$

Again, by construction, W_n fulfills all requirements and contains all channels required to perfectly win the game.

D. Example: W_4

As an example, we construct the matrix W_4. The group $\{h_0,h_1,h_2,h_3,h_4,h_5,h_6,h_7\}$ for W_4 is constructed from the group $\{g_0,g_1,g_2,g_3\}$ and has the elements

$$h_0 = g_0 \otimes g'_0 = (\mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}) \otimes (\mathbb{1} \otimes \mathbb{1}),$$
$$h_1 = g_1 \otimes g'_1 = (\mathbb{1} \otimes \sigma_z \otimes \sigma_z) \otimes (\mathbb{1} \otimes \sigma_z),$$
$$h_2 = g_2 \otimes g'_2 = (\sigma_z \otimes \mathbb{1} \otimes \sigma_z) \otimes (\sigma_z \otimes \mathbb{1}),$$
$$h_3 = g_3 \otimes g'_3 = (\sigma_z \otimes \sigma_z \otimes \mathbb{1}) \otimes (\sigma_z \otimes \sigma_z),$$
$$h_4 = \bar{g}_0 \otimes g'_0 = (\sigma_z \otimes \sigma_z \otimes \sigma_z) \otimes (\mathbb{1} \otimes \mathbb{1}),$$
$$h_5 = \bar{g}_1 \otimes g'_1 = (\sigma_z \otimes \mathbb{1} \otimes \mathbb{1}) \otimes (\mathbb{1} \otimes \sigma_z),$$
$$h_6 = \bar{g}_2 \otimes g'_2 = (\mathbb{1} \otimes \sigma_z \otimes \mathbb{1}) \otimes (\sigma_z \otimes \mathbb{1}),$$
$$h_7 = \bar{g}_3 \otimes g'_3 = (\mathbb{1} \otimes \mathbb{1} \otimes \sigma_z) \otimes (\sigma_z \otimes \sigma_z).$$

The matrix W_4 is thus

$$W_4 = \frac{1}{32}(\mathbb{1}^{\otimes 10} + \mathbb{1} \otimes \sigma_z \otimes \sigma_z \otimes \mathbb{1} \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \mathbb{1} \otimes \sigma_z \otimes \mathbb{1} + \sigma_z \otimes \mathbb{1} \otimes \sigma_z \otimes \sigma_z \otimes \mathbb{1} \otimes \mathbb{1} \otimes \sigma_z \otimes \sigma_z \otimes \mathbb{1} \otimes \sigma_z$$

$$+ \sigma_z \otimes \sigma_z \otimes \mathbb{1} \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \mathbb{1} \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z + \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \mathbb{1} \otimes \mathbb{1} \otimes \sigma_z \otimes \sigma_z \otimes \mathbb{1} \otimes \mathbb{1} \otimes \sigma_z$$

$$+ \sigma_z \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \sigma_z \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \sigma_z \otimes \sigma_z + \mathbb{1} \otimes \sigma_z \otimes \mathbb{1} \otimes \sigma_z \otimes \mathbb{1} \otimes \sigma_z \otimes \mathbb{1} \otimes \sigma_z \otimes \mathbb{1} \otimes \mathbb{1}$$

$$+ \mathbb{1} \otimes \mathbb{1} \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \mathbb{1} \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \mathbb{1}).$$

The second to the fifth summands represent the channels that are used to perfectly win the game.

The conditional probability distribution W_4 responds to input $P_{O_0,O_1,O_2,O_3}(o_0,o_1,o_{2,1},o_{2,2},o_3) = 1$ with the following output

$$P_{I_0,I_1,I_2,I_3}(i_0,i_1,i_2,i_{3,1},i_{3,2}) = \begin{cases} \frac{1}{4}, & i_0 = o_3, \quad i_1 = o_0, \quad i_2 = o_1, \quad i_{3,1} = o_{2,1}, \quad i_{3,2} = o_{2,2} \\ \frac{1}{4}, & i_0 = \bar{o}_3, \quad i_1 = o_0, \quad i_2 = \bar{o}_1, \quad i_{3,1} = o_{2,1}, \quad i_{3,2} = \bar{o}_{2,2} \\ \frac{1}{4}, & i_0 = o_3, \quad i_1 = \bar{o}_0, \quad i_2 = \bar{o}_1, \quad i_{3,1} = \bar{o}_{2,1}, \quad i_{3,2} = o_{2,2} \\ \frac{1}{4}, & i_0 = \bar{o}_3, \quad i_1 = \bar{o}_0, \quad i_2 = o_1, \quad i_{3,1} = \bar{o}_{2,1}, \quad i_{3,2} = \bar{o}_{2,2} \\ 0, & \text{otherwise}, \end{cases}$$

where $o_{2,1}, o_{2,2}$ are both bits of the random variable O_2; $i_{3,1}, i_{3,2}$ are both bits of the random variable I_3, and where $\bar{o}_i = o_i \oplus 1$. Therefore, W_4 implements a uniform distribution of four circular channels (see Fig. 4).

By construction of W_4, no logical paradox arises. More intuitively, in any strategy that does not break any of the four circular channels of Fig. 4 (i.e., every party's output depends on its input), parties S_2 and S_3 use the first bit, the second,
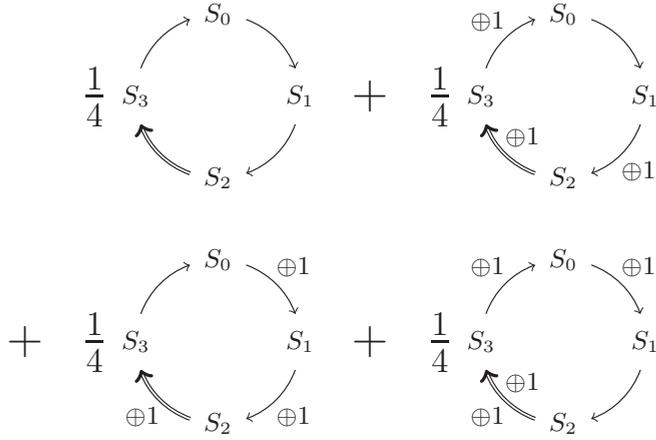
FIG. 4. The conditional probability distribution W_4 is a mixture of four circular channels. The channel from S_2 to S_3 is a two-bit channel (double line). If the $\oplus 1$ operation for the channel from S_2 to S_3 is outside the circle, then the $\oplus 1$ operation is applied to the first channel, i.e., the first bit is flipped, if it is inside the circle, then the $\oplus 1$ operation is applied to the second channel, i.e., the second bit is flipped.

or both bits to communicate. If they use the first bit, then the correlations arising from the first two loops and the last two loops of Fig. 4 interfere constructively. Both pairs together, however, break the loop. If they use the second bit, then the first and the third loop, and the second with the last loop yield the same output in every cycle. In total, all loops cancel each other out. For the last case as well, where both bits are used for communication, the correlations from the first and the last loop interfere constructively, and so do the second and the third. Ultimately, again, all loops cancel each other out, and no logical paradox can be created.

For larger even n, the conditional probability distribution W_n as well is constructed out of four loops, as in Fig. 4, that cancel each other out when one tries to build a logical paradox. For $n = 2$, the same construction does not work, because the two-bit channel cannot be used to signal from its source to its destination—it can only be used when combined with other channels. In a two-party scenario, however, in order to win the game, each party needs to signal to the other.

E. Winning strategy

For odd n, the strategy $Q_i^m = \hat{P}_{X_i = x_i, O_i | A_i = a_i, I_i}$ for party S_i to win the game is

$$Q_i^m = Q_{i,O}^m \otimes Q_{i,I}^m$$
$$= \left(\frac{\mathbb{1} + (-1)^{a_i'} \sigma_z}{2} \right) \otimes \left(\frac{\mathbb{1} + (-1)^{x_i} \sigma_z}{2} \right),$$

where $a_i' = a_i$ for $i \equiv m + 1 \pmod{n}$, and $a_i' = a_i + x_i$ otherwise. The strategies for even n are equivalent to the strategies for odd n, except that S_{n-2} has a two-bit output and S_{n-1} has a two-bit input. Depending on M, they use the first, second, or both bit(s) to receive or send the desired bit. All local operations are classical since they are diagonal, i.e., consist only of measuring and preparing states in the σ_z basis.

The distribution $P(x_m | a_0, \ldots, a_{n-1}, M = m)$ is

$$\sum_{\substack{x_i \in \{0,1\} \\ i \neq m}} P(\vec{x} | \vec{a}, M = m)$$

$$= \sum_{\substack{x_i \in \{0,1\} \\ i \neq m}} \mathrm{Tr}\Big(\big(Q_{0,I}^m \otimes \cdots \otimes Q_{n-1,I}^m$$

$$\otimes Q_{0,O}^m \otimes \cdots \otimes Q_{n-1,O}^m \big) \cdot W_n \Big)$$

$$= \frac{1}{2} (1 + (-1)^{X_m + \sum_{i \neq m} A_i}),$$

where we rearranged the submatrices of Q_i^m in the trace expression such that the ordering of the conditional probabilities in W_n match. This result is obtained because, after taking the trace, each term except 1 and $(-1)^{X_m + \sum_{i \neq m} A_i}$ is either zero or depends on a variable $X_{i(\neq m)}$ which, in the process of marginalization over $X_{i(\neq m)}$, cancels out. For each m, the winning probability is

$$\Pr \left(X_m = \bigoplus_{i \neq m} A_i \,\middle|\, M = m \right) = 1.$$

Therefore, the game is won with certainty.

F. Example: $n = 3$

The probability of obtaining x_0 in the case $M = 0$ is

$$P(x_0 | a_0, a_1, a_2, M = 0)$$

$$= \sum_{x_1, x_2 \in \{0,1\}} \mathrm{Tr}\Big(\big(Q_{0,I}^0 \otimes Q_{1,I}^0 \otimes Q_{2,I}^0$$

$$\otimes Q_{0,O}^0 \otimes Q_{1,O}^0 \otimes Q_{2,O}^0 \big) \cdot W_3 \Big)$$

$$= \frac{1}{8} \frac{\mathrm{Tr}(\mathbb{1}^{\otimes 6})}{2^6} \sum_{x_1, x_2 \in \{0,1\}} (1 + (-1)^{x_0 + x_1 + x_2 + a_0 + a_1}$$

$$+ (-1)^{x_0 + a_1 + a_2} + (-1)^{x_1 + x_2 + a_0 + a_2})$$

$$= \frac{1}{2} (1 + (-1)^{x_0 + a_1 + a_2}).$$

Therefore, the probability of the event $X_0 = A_1 \oplus A_2$ is 1. The distribution of X_1 in the case $M = 1$ is

$$P(x_1 | a_0, a_1, a_2, M = 1)$$

$$= \frac{1}{8} \sum_{x_0, x_2 \in \{0,1\}} (1 + (-1)^{x_0 + x_2 + a_0 + a_1}$$

$$+ (-1)^{x_0 + x_1 + x_2 + a_1 + a_2} + (-1)^{x_1 + a_0 + a_2})$$

$$= \frac{1}{2} (1 + (-1)^{x_1 + a_0 + a_2}),$$

and, finally, the distribution of X_2 in the case $M = 2$ is

$$P(x_2 | a_0, a_1, a_2, M = 2)$$

$$= \frac{1}{8} \sum_{x_0, x_1 \in \{0,1\}} (1 + (-1)^{x_2 + a_0 + a_1}$$

$$+ (-1)^{x_0 + x_1 + a_1 + a_2} + (-1)^{x_0 + x_1 + x_2 + a_0 + a_2})$$

$$= \frac{1}{2} (1 + (-1)^{x_2 + a_0 + a_1}).$$

The probabilities of the events $X_1 = A_0 \oplus A_2$ and $X_2 = A_0 \oplus A_1$ are both 1. Therefore, the game is won with certainty.

Intuitively, in the case $M = m$, party $S_{m+1 \bmod 3}$ sends $O_{m+1 \bmod 3} = a_{m+1 \bmod 3}$ on both circular channels of Fig. 3. Thus, party $S_{m+2 \bmod 3}$ receives the uniform mixture of $I_{m+2 \bmod 3} = a_{m+1 \bmod 3}$ (left channel of Fig. 3) and $I_{m+2 \bmod 3} = a_{m+1 \bmod 3} \oplus 1$ (right channel of Fig. 3). Party $S_{m+2 \bmod 3}$ thereafter sends $O_{m+2 \bmod 3} = I_{m+2 \bmod 3} \oplus a_{m+2 \bmod 3}$, i.e., the uniform mixture of $O_{m+2 \bmod 3} = a_{m+1 \bmod 3} \oplus a_{m+2 \bmod 3}$ and $O_{m+2 \bmod 3} = a_{m+1 \bmod 3} \oplus a_{m+2 \bmod 3} \oplus 1$, on both circular channels, yielding the deterministic input $I_m = a_{m+1 \bmod 3} \oplus a_{m+2 \bmod 3}$ to party S_m.

G. Example: $n = 4$

In the example $n = 4$, we explicitly write the local operations for the third and fourth parties, as the third party has a two-bit output, and the fourth party has a two-bit input. The local operations for the third party (S_2) are

$$Q_2^{m=0} = \left(\frac{\mathbb{1} \otimes \mathbb{1} + (-1)^{a_2 + x_2} \sigma_z \otimes \mathbb{1}}{4} \right) \otimes Q_2',$$

$$Q_2^{m=1} = \left(\frac{\mathbb{1} \otimes \mathbb{1} + (-1)^{a_2} \sigma_z \otimes \sigma_z}{4} \right) \otimes Q_2',$$

$$Q_2^{m=2} = \left(\frac{\mathbb{1} \otimes \mathbb{1}}{4} \right) \otimes Q_2',$$

$$Q_2^{m=3} = \left(\frac{\mathbb{1} \otimes \mathbb{1} + (-1)^{a_2 + x_2} \mathbb{1} \otimes \sigma_z}{4} \right) \otimes Q_2',$$

with

$$Q_2' = \left(\frac{\mathbb{1} + (-1)^{x_2} \sigma_z}{2} \right).$$

Party 4 (S_3) uses

$$Q_3^{m=0} = Q_3' \otimes \left(\frac{\mathbb{1} \otimes \mathbb{1} + (-1)^{x_3} \sigma_z \otimes \mathbb{1}}{2} \right),$$

$$Q_3^{m=1} = Q_3' \otimes \left(\frac{\mathbb{1} \otimes \mathbb{1} + (-1)^{x_3} \sigma_z \otimes \sigma_z}{2} \right),$$

$$Q_3^{m=2} = \left(\frac{\mathbb{1} + (-1)^{a_3} \sigma_z}{2} \right) \otimes \left(\frac{\mathbb{1} \otimes \mathbb{1}}{2} \right),$$

$$Q_3^{m=3} = Q_3' \otimes \left(\frac{\mathbb{1} \otimes \mathbb{1} + (-1)^{x_3} \mathbb{1} \otimes \sigma_z}{2} \right),$$

where we use shorthand Q_3' for

$$Q_3' = \left(\frac{\mathbb{1} + (-1)^{a_3 + x_3} \sigma_z}{2} \right).$$

The distributions of X_0, X_1, X_2, X_3, under the condition $M = 0$, $M = 1$, $M = 2$, $M = 3$, respectively, are

$$P(x_0 | a_0, a_1, a_2, a_3, M = 0) = \tfrac{1}{2}(1 + (-1)^{x_0 + a_1 + a_2 + a_3}),$$

$$P(x_1 | a_0, a_1, a_2, a_3, M = 1) = \tfrac{1}{2}(1 + (-1)^{x_1 + a_0 + a_2 + a_3}),$$

$$P(x_2 | a_0, a_1, a_2, a_3, M = 2) = \tfrac{1}{2}(1 + (-1)^{x_2 + a_0 + a_1 + a_3}),$$

$$P(x_3 | a_0, a_1, a_2, a_3, M = 3) = \tfrac{1}{2}(1 + (-1)^{x_3 + a_0 + a_1 + a_2}).$$

Therefore, the event $X_0 = A_1 \oplus A_2 \oplus A_3$ given $M = 0$, the event $X_1 = A_0 \oplus A_2 \oplus A_3$ in the case $M = 1$, the event $X_2 = A_0 \oplus A_1 \oplus A_3$ if $M = 2$, and the event $X_3 = A_0 \oplus A_1 \oplus A_2$ in the case $M = 3$ have probability 1. Which implies that the game is won with certainty.

By consulting Fig. 4, we can describe the strategy in the following way. If $M = m$, then party $S_{m+1 \bmod 4}$ sends $a_{m+1 \bmod 4}$ to the next party by using all four channels of Fig. 4. Each of the next two parties in clockwise orientation, i.e., party $S_{m+2 \bmod 4}$ and party $S_{m+3 \bmod 4}$, sends the parity of what she receives from the previous party and her input ($a_{m+2 \bmod 4}$, $a_{m+3 \bmod 4}$, respectively). Depending on M, parties S_2 and S_3 use the first, the second, or both single-bit channels. In particular, if $M = 0$, then S_2 uses the first channel to communicate to S_3—the second cannel is ignored. For $M = 1$ they use both channels, i.e., the parity of the inputs to both channels is equal to the bit S_2 sends. For $M = 2$, the two-bit channel between S_2 and S_3 is ignored. Finally, for $M = 3$ they use the second channel. By doing so, S_m obtains $a_{m+1 \bmod 4} + a_{m+2 \bmod 4} + a_{m+3 \bmod 4}$, as the introduced bit-flips from the four channels (see Fig. 4) cancel each other out.

VI. CONCLUSION

In an attempt to construct a theory that combines aspects of general relativity and quantum theory, Oreshkov, Costa, and Brukner [27] proposed a framework for quantum correlations without causal order. They proved that some correlations are incompatible with any *a priori* causal order and, therefore, are *not compatible with predefined causal order* although they satisfy quantum theory *locally*. We consider the classical limit of this framework and show that in sharp contrast to the two-party scenario [27], *classical and logically consistent multiparty correlations can be incompatible with any predefined causal order*. To show this, we propose a game that cannot be won in a scenario with predefined causal order, but is won with certainty when no causal order is fixed.

Recently, the ideas of indefinite causal order and of superpositions of causal orders were applied to quantum computation [26,47–51]. Furthermore, Aaronson and Watrous [52] showed that closed timelike curves render classical and quantum computing equivalent. Our result is similar in the sense that the winning probability of the game is the same for the quantum and for the classical framework. Since the W object in Fig. 2 can be thought of as a *channel back in time*, closed timelike curves can be interpreted as being part of the framework. Closed timelike curves *per se* are consistent with general relativity [53]. However, Aaronson and Watrous take Deutsch's approach [54] to closed timelike curves which, as opposed to the framework studied here, is a *nonlinear* extension of quantum theory—such extensions are known to allow for communication faster than at the speed of light [55].

[1] J. S. Bell, J. Phys., Colloq. **42**, 41 (1981).

[2] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).

[3] H. Reichenbach, in *The Direction of Time*, edited by M. Reichenbach (University of California Press, Berkeley, 1956), Chap. 19, pp. 157–167.

[4] H. M. Wiseman, Special Issue of J. Phys. A: Math. Theor. (to be published), arXiv:1402.0351.

[5] C. J. Wood and R. W. Spekkens, arXiv:1208.4119.

[6] T. J. Barnea, J.-D. Bancal, Y.-C. Liang, and N. Gisin, Phys. Rev. A **88**, 022123 (2013).

[7] A. Aspect, J. Dalibard, and G. Roger, Phys. Rev. Lett. **49**, 1804 (1982).

[8] A. Suarez and V. Scarani, Phys. Lett. A **232**, 9 (1997).

[9] G. Weihs, T. Jennewein, C. Simon, H. Weinfurter, and A. Zeilinger, Phys. Rev. Lett. **81**, 5039 (1998).

[10] A. Aspect, Nature (London) **398**, 189 (1999).

[11] M. A. Rowe, D. Kielpinski, V. Meyer, C. A. Sackett, W. M. Itano, C. Monroe, and D. J. Wineland, Nature (London) **409**, 791 (2001).

[12] D. Salart, A. Baas, J. A. W. van Houwelingen, N. Gisin, and H. Zbinden, Phys. Rev. Lett. **100**, 220404 (2008).

[13] M. Giustina, A. Mech, S. Ramelow, B. Wittmann, J. Kofler, J. Beyer, A. Lita, B. Calkins, T. Gerrits, S. W. Nam, R. Ursin, and A. Zeilinger, Nature (London) **497**, 227 (2013).

[14] J. S. Bell, Physics **1**, 195 (1964).

[15] M. J. W. Hall, Phys. Rev. A **84**, 022102 (2011).

[16] J. Barrett and N. Gisin, Phys. Rev. Lett. **106**, 100406 (2011).

[17] G. Pütz, D. Rosset, T. J. Barnea, Y.-C. Liang, and N. Gisin, arXiv:1407.5634.

[18] Parmenides, in *Parmenides of Elea: A Verse Translation with Interpretative Essays and Commentary to the Text*, edited and translated by M. J. Henn (Praeger Publishers, Westport, Connecticut, 2003), Chap. 2, pp. 24–29.

[19] W. K. Wootters, Int. J. Theor. Phys. **23**, 701 (1984).

[20] L. Bombelli, J. Lee, D. Meyer, and R. D. Sorkin, Phys. Rev. Lett. **59**, 521 (1987).

[21] G. M. D'Ariano, AIP Conf. Proc. **1363**, 63 (2011).

[22] L. Hardy, arXiv:gr-qc/0509120.

[23] L. Hardy, J. Phys. A: Math. Theor. **40**, 3081 (2007).

[24] Č. Brukner, Nat. Phys. **10**, 259 (2014).

[25] G. Chiribella, G. M. D'Ariano, and P. Perinotti, Phys. Rev. A **80**, 022339 (2009).

[26] G. Chiribella, G. M. D'Ariano, P. Perinotti, and B. Valiron, Phys. Rev. A **88**, 022318 (2013).

[27] O. Oreshkov, F. Costa, and Č. Brukner, Nat. Commun. **3**, 1092 (2012).

[28] Ä. Baumeler and S. Wolf, in *Proceedings of International Symposium on Information Theory* (IEEE, Piscataway, NJ, 2014), pp. 526–530.

[29] Č. Brukner, arXiv:1404.0721.

[30] B. S. Cirel'son, Lett. Math. Phys. **4**, 93 (1980).

[31] D. M. Greenberger, M. A. Horne, and A. Zeilinger, in *Bell's Theorem, Quantum Theory, and Conceptions of the Universe*, edited by M. Kafatos (Kluwer Academic Publishers, Dordrecht, 1989), pp. 69–72.

[32] D. M. Greenberger, M. A. Horne, A. Shimony, and A. Zeilinger, Am. J. Phys. **58**, 1131 (1990).

[33] F. Costa, Ph.D. thesis, Universität Wien, 2013 (unpublished), http://othes.univie.ac.at/29237/.

[34] R. Colbeck and R. Renner, Nat. Commun. **2**, 411 (2011).

[35] G. Ghirardi and R. Romano, Found. Phys. **43**, 881 (2013).

[36] C. M. Caves, C. A. Fuchs, and R. Schack, Phys. Rev. A **65**, 022305 (2002).

[37] Č. Brukner and A. Zeilinger, in *Time, Quantum and Information*, edited by L. Castell and O. Ischebeck (Springer-Verlag, Berlin, Heidelberg, 2003), pp. 323–354.

[38] R. Clifton, J. Bub, and H. Halvorson, Found. Phys. **33**, 1561 (2003).

[39] G. Brassard, Nat. Phys. **1**, 2 (2005).

[40] M. Pawłowski, T. Paterek, D. Kaszlikowski, V. Scarani, A. Winter, and M. Zukowski, Nature (London) **461**, 1101 (2009).

[41] G. Chiribella, G. M. D'Ariano, and P. Perinotti, Phys. Rev. A **84**, 012311 (2011).

[42] M. P. Müller and L. Masanes, New J. Phys. **15**, 053040 (2013).

[43] C. Pfister and S. Wehner, Nat. Commun. **4**, 1851 (2013).

[44] F. Costa (private communication).

[45] C. Giarmatzi (private communication).

[46] Č. Brukner (private communication).

[47] L. Hardy, in *Quantum Reality, Relativistic Causality, and Closing the Epistemic Circle*, The Western Ontario Series in Philosophy of Science, Vol. 73 (Springer, Netherlands, 2009), pp. 379–401.

[48] G. Chiribella, Phys. Rev. A **86**, 040301 (2012).

[49] T. Colnaghi, G. M. D'Ariano, S. Facchini, and P. Peri-notti, Phys. Lett. A **376**, 2940 (2012).

[50] M. Araújo, F. Costa, and Č. Brukner, arXiv:1401.8127.

[51] T. Morimae, Phys. Rev. A **90**, 010101 (2014).

[52] S. Aaronson and J. Watrous, Proc. R. Soc. London, Ser. A **465**, 631 (2009).

[53] K. Gödel, Rev. Mod. Phys. **21**, 447 (1949).

[54] D. Deutsch, Phys. Rev. D **44**, 3197 (1991).

[55] N. Gisin, Phys. Lett. A **143**, 1 (1990).