Self-interactions as predicted by the Dirac-Maxwell equations

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We solve the Maxwell-Dirac equations to study the dynamics of a spatially localized charged particle in one spatial dimension. While the coupling of the Maxwell equations to the Dirac equation predicts correctly the attractive or repulsive interaction between different particles, it also reveals an unphysical interaction of a single electron or positron with itself leading to an enhanced spatial spreading of the wave packet. Using a comparison with a relativistic ensemble of mutually interacting classical quasiparticles, we suggest that this quantum mechanical self-repulsion can be understood in terms of relativistic classical mechanics. We show that due to the simple form of the Coulomb law in one spatial dimension it is possible to find analytical expressions of the time-dependent spatial width for the interacting classical ensemble. A better understanding of the dynamical impact of this unavoidable self-repulsion effect is relevant for recent studies of the field-induced pair creation process from the vacuum.

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While the creation dynamics of electron-positron pairs in supercritical electromagnetic fields in the quantum vacuum has been widely studied [1–3], only recently has the contribution of the mutual electron-positron attraction been investigated [4–10]. A fundamentally accurate and space-time resolved theory that would involve the force-intermediating photons as dynamically coupled quantum particles is presently out of the question for computational and also conceptual reasons. To obtain some first qualitative ideas about the impact of these forces, one can approximate the photons by self-consistent but classical electromagnetic fields whose dynamics are governed by the Maxwell equations, which contain the charge and current densities of the created particles as source terms. However, due to the absence of any quantum field theoretical estimates it is presently rather difficult to evaluate the accuracy of such an approximation for the pair-creation process. For example, as neither the source terms in the Maxwell equation nor the electromagnetic fields are described as field theoretical operators, these equations predict that even portions within a wave packet of a single particle can repel each other.

One could try to interpret this self-repulsion based on Born's statistical interpretation of a wave packet. A probability density represents a temporal average of infinitely many measurements of the same single particle. In an approach where the Maxwell field is not second quantized, however, the entire spatial density acts as a source term in the Maxwell equations, which then produces a field with which all portions of the wave function act simultaneously. In other words, different portions of the *same* particle wave packet could then interact with each other like particles in a classical ensemble of many particles or in a charge cloud. If the field is second-quantized, the electron is treated more like a real inseparable particle, which cannot be divided into different parts and thus the self-interaction should be absent.

We point out that at present it is not clear if this effect is intrinsically quantum mechanical at all or how accurately it could be modeled in terms of simple classical mechanics. A better understanding of the self-repulsion mechanism would also help us to distinguish it from the true (multiparticle) repulsion between two equally charged particles. With respect to the pair creation dynamics it is therefore important to obtain a better estimate of the magnitude (and therefore dynamical relevance) of this unphysical self-repulsion effect.

In this Brief Report we compare the effect on the enhancement of wave packet spreading with the predictions from a classical ensemble of fully interacting charged quasiparticles [11,12] in the relativistic and nonrelativistic regime. The quantum mechanical description for a particle with charge Q and mass M interacting with a scalar and vector potential is given by the coupled Maxwell-Dirac equations in one spatial dimension [10]

$$i\partial_t\psi(z,t) = [c\sigma_1[p - QA(z,t)/c] + \sigma_3 Mc^2 + QV(z,t)]\psi(z,t), \quad (1)$$

$$(\partial_{ct}^2 - \partial_z^2)V(z,t) = 4\pi q(z,t), \quad (2a)$$

$$\left(\partial_{ct}^2 - \partial_z^2\right)A(z,t) = 4\pi c^{-1} j(z,t). \quad (2b)$$

Here we have used the Lorenz gauge $\partial_z A = -\partial_{ct} V$ and atomic and cgs units, where the four fundamental constants [amount of the charge of the electron, its mass, and Coulomb's and Planck's constants $1/(4\pi\varepsilon_0)$ and $\hbar$] are all unity by definition. The corresponding electric field is given as $E = -\partial_z V - \partial_{ct} A$. Also, $c = 137.036$ a.u. is the speed of light and σ_1 and σ_3 are the two Pauli matrices. For a single particle, the charge and current densities are obtained from the two-component wave function ψ, as $q(z,t) \equiv Q\psi(z,t)^\dagger\psi(z,t)$ and $j(z,t)^\dagger \equiv cQ\psi(z,t)^\dagger\sigma_1\psi(z,t)$. We have introduced the (standard minimal coupling) parameter Q (which in one dimension has the same units as the square root of the force) to control the strength of the back reaction of the particle onto the internal fields.

Next we discuss the initial conditions. In order to model a spatially localized particle that is initially centered at $z = 0$ and has a spatial width of Δz_0, we use the state

$$\psi(z, t=0) = \left(2\Delta z_0^2/\pi\right)^{1/4} \int dp\, \exp\left[-p^2\, \Delta z_0^2\right] \times W_p(u;z), \quad (3)$$

where the two-component $W_p(u;z)$ denotes the energy eigenstate of the force-free Hamiltonian $c\sigma_1 p + \sigma_3 Mc^2$ with momentum p and positive energy $w_p \equiv [M^2c^4 + c^2p^2]^{1/2}$. The energy eigenstates for positive (up) and negative (down) energies are given in their spatial representation by

$$W_p(u;z) \equiv \chi[1, cp/(Mc^2 + w_p)]\exp[ipz], \tag{4a}$$

$$W_p(d;z) \equiv \chi[-cp/(Mc^2 + w_p), 1]\exp[ipz], \tag{4b}$$

where $\chi \equiv (2\pi)^{-1/2}[1 + c^2p^2/(w_p + Mc^2)^2]^{-1/2}$ denotes the normalization factor. If the initial width exceeds the Compton wavelength, the associated initial charge density can be approximated by a Gaussian distribution $q(z, t = 0) \approx Q(2\pi)^{-1/2}\Delta z_0^{-1}\exp[-(z/\Delta z_0)^2/2]$. Using Gauss' law $[\partial_z^2 V(z) = -4\pi q(z, t = 0)]$ we further approximate the initial conditions for the Maxwell equations as $V(z, t = 0) = -4\pi\int_{-\infty}^{z} dz' \int_{-\infty}^{z'} dz'' q(z'', t = 0)$ and $A(z, t = 0) = \partial_t A(z, t = 0) = \partial_t V(z, t = 0) = 0$. In order to have a potential that satisfies initially the periodic boundaries at $z = \pm L/2$ of our numerical grid (of total length L) $V(z = -L/2, t = 0) = V(z = L/2, t = 0) = 0$, we have determined the coefficients α and β such that the new potential $V(z) + \alpha z + \beta$ can vanish at the boundaries. In order to solve the coupled Maxwell-Dirac equations, we have used an FFT-based split-operator technique [13–15] for the Dirac and Maxwell equations. The algorithmic details are presented in Ref. [10].

The corresponding relativistic but classical ensemble of N mutually (and instantaneously) interacting quasiparticles (of effective charge $q_{\rm eff} = Q/N$ and effective mass $m_{\rm eff} = M/N$ each) is modeled here by the Hamilton function

$$H = \Sigma_{i=1}^{N}\left[\left(m_{\rm eff}^2c^4 + c^2p_i^2\right)^{1/2} + q_{\rm eff}\Sigma_{j=1}^{N}V(z_i, z_j)\right], \tag{5}$$

where $z_i(t)$ and $p_i(t)$ are the position and momentum of the ith quasiparticle. Here we use a gauge where A is zero. The quantum mechanical wave packet has the same total charge Q (given by the minimum coupling constant in the underlying QED Hamiltonian [16]), as the corresponding classical ensemble of N particles. Due to our choice of $m_{\rm eff}$ both systems have also the same total mass M, which we chose to be unity in our numerical simulations.

The form of the pairwise interaction potential $V(z_i, z_j)$ can be obtained from the Maxwell equations in one spatial dimension [Eqs. (2)]. For a single (positive) unit charge at rest that is localized at $z = 0$ [i.e., $q(z) = \delta(z)$] Gauss' law predicts a steady state potential $V(z) = -2\pi|z|$, corresponding to an electric field $E(z) = 2\pi z/|z|$ that is spatially constant to the left and right of the charge. We therefore use $V(z_i, z_j) = -2\pi q_{\rm eff}|z_i - z_j|$ in Eq. (5). The resulting Hamilton equations of motion for the ith particle are

$$\partial_t z_i(t) = \partial H/\partial p_i = c^2 p_i\left[m_{\rm eff}^2c^4 + c^2p_i^2\right]^{-1/2}, \tag{6a}$$

$$\partial_t p_i(t) = -\partial H/\partial z_i = q_{\rm eff}^2 2\pi\,\Sigma_{j=1}^{N}(z_i - z_j)/|z_i - z_j|. \tag{6b}$$

To be consistent with the quantum initial conditions, we choose independently Gaussian distributed initial positions and momenta, centered at zero and with initial widths Δz_0 and $\Delta z_0^{-1}/2$, respectively.

As a consequence of the scaling of each particle's effective charge $q_{\rm eff} = Q/N$, not only the ensemble's total charge but also the total interaction energy $\Sigma_{i=1}^{N} q_{\rm eff}\Sigma_{j=1}^{N}V(z_i, z_j)$ is finite. If for simplicity we assume that the particles' spatial distribution is uniform and extended from 0 to Δz_0, i.e., $z_i = i\Delta z_0/N$ (where $i = 1, 2, \ldots, N$), we can estimate the total interaction energy as $-2\pi q_{\rm eff}^2\Delta z_0/N\,\Sigma_{i=1}^{N}\,\Sigma_{i=1}^{N}|i - j| = -(2\pi q_{\rm eff}^2\Delta z_0/N)N(N^2 - 1)/3 \to -2\pi/3\,Q^2\Delta z_0$ for the large N limit. As we have assumed a mass $m_{\rm eff}$ for each particle, also the total kinetic energy $\Sigma_{i=1}^{N}[m_{\rm eff}^2c^4 + c^2p_i^2]^{1/2}$ is finite. The average energy per quasiparticle, however, is inversely proportional to N.

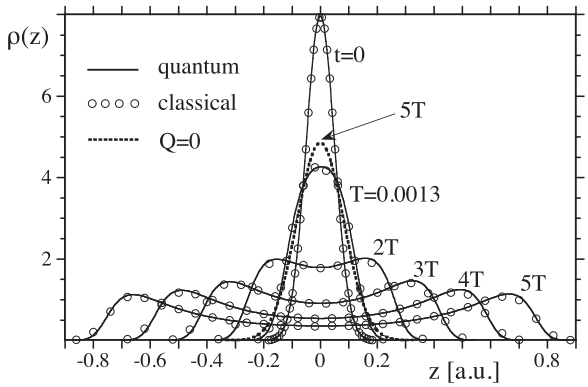

FIG. 1. The spatial probability density at various instants of time $(n - 1)$ 0.0013 a.u. for the quantum (solid) and classical mechanical (open circles) system. (Parameters are $L = 6.4$ a.u., $N_z = 2048$ spatial grid points, initial width $\Delta z_0 = 0.05$ a.u. The number of particles in the classical ensemble was 10 000, $Q = -100$, $M = 1$.)

In Fig. 1 we present six snapshots of the spatial probability density $\rho(z, t) \equiv \psi(z, t)^\dagger\psi(z, t)$. To illustrate the self-repulsion for comparison we have also shown by the dashed line the corresponding final density in the absence of any interaction ($Q = 0$), reflecting simply the usual quantum mechanical spreading associated with the nonvanishing variance in the momentum $\Delta p_0 = 1/(2\Delta z_0)$. The open circles are the spatial distribution associated with the fully interacting classical ensemble. The agreement between the quantum and classical systems is superb.

If the initial conditions for the Maxwell equations had been chosen as $V(z, t = 0) = A(z, t = 0) = \partial_t A(z, t = 0) = \partial_t V(z, t = 0) = 0$, the resulting long-time spatial distribution would be unchanged. In this case the charge density would first induce the corresponding electric field. After a time delay, however, the self-spreading would evolve similarly as shown in Fig. 1.

Below we will examine the time dependence of the spatial width $\Delta z(t)$ more directly. It turns out that due to the special properties of Coulomb's law in one spatial dimension, the temporal growth of the spatial width of the entire classical ensemble $\Delta z(t)$ can be obtained analytically if we assume that the initial velocity of each particle is negligible. As the amount of the electric field between the particles is independent of the interparticle distance, the particles can maintain their relative ordering of their positions in time. As a result each particle experiences a different but temporally constant force due to the other particles. The trajectory for the ith particle can therefore be obtained from the Hamilton equations (6) of motion as $z_i(t) = z_{0,i} + [m_{\rm eff}^2c^4 f_i^{-2} + c^2t^2]^{1/2} - m_{\rm eff}c^2/f_i$, where the

force f_i for each particle does not change in time and is therefore determined from all initial positions $z_j(t=0) \equiv z_{0,j}$ as

$$f_i = q_{\rm eff}^2 2\pi \Sigma_{i=1}^{N} (z_{0,i} - z_{0,j})/|z_{0,i} - z_{0,j}|. \quad (7)$$

The nonrelativistic limit ($c \to \infty$) of each trajectory is, of course, $z_i(t) = z_{0,i} + a_i t^2/2$, where $a_i \equiv (f_i/m_{\rm eff})$ is the (N-independent) acceleration of each particle.

We will use this particular solution for the ith particle to compute the time-dependent standard deviation of the position, whose square is defined as $\Delta z(t)^2 \equiv \langle z(t)^2\rangle - \langle z(t)\rangle^2$. From now on $\langle\cdots\rangle$ denotes the average over all particle positions and momenta in phase space. Using that $\langle a\rangle = 0$ we obtain

$$\Delta z(t)^2 = \Delta z_0^2 + \langle z_0 (c^4 a^{-2} + c^2 t^2)^{1/2}\rangle - c^2\langle z_0/a\rangle + \langle [(c^4 a^{-2} + c^2 t^2)^{1/2} - c^2/a]^2\rangle. \quad (8)$$

The corresponding nonrelativistic limit can be expressed directly in the terms of the variance of the initial accelerations Δa as

$$\Delta z(t)^2 = \Delta z_0^2 + \Delta a^2 t^4/4 + \langle z_0\, a\rangle t^2. \quad (9)$$

While the usual (velocity variance-induced) nonrelativistic spreading (for $Q = 0$)

$$\Delta z(t)^2 = \Delta z_0^2 + \Delta v_0^2 t^2 \quad (10)$$

grows asymptotically linearly in time, $\Delta z(t) \to \Delta v_0 t$, the spreading induced by an acceleration variance grows quadratically, $\Delta z(t) \to \Delta a\, t^2/2$.

The self-spreading dominates the long-time behavior. However, the analytical solutions can be used to estimate for which parameters the self-repulsion can be neglected for short times in the nonrelativistic regime. The ratio of the linear terms in Eqs. (9) and (10) is given by the dimensionless parameter $\Xi \equiv \langle z_0 a\rangle/\Delta v_0^2$. Only if $\Xi \ll 1$ the self repulsion can be neglected relative to the natural spreading. If we assume that the particle positions are uniformly distributed in space, we can compute the average $\langle z_0 a\rangle = (1/N)\Sigma_{i=1}^{N} z_{0,i} q_{\rm eff}^2/m_{\rm eff} 2\pi \Sigma_{j=1}^{N} (z_{0,i} - z_{0,j})/|z_{0,i} - z_{0,j}|$. This expression can be simplified to $\langle z_0 a\rangle = (1/N)^3 Q^2/M\, 2\pi\ \Delta z_0 \Sigma_{i=1}^{N} \Sigma_{j=1}^{N} i(i-j)/|i-j|$. In the large N limit we obtain $\langle z_0\, a\rangle = Q^2\pi\Delta z_0/(3M)$. If we further assume that the initial quantum state minimizes the Heisenberg uncertainty relationship, i.e, $\Delta v_0 = 1/(2M\Delta z_0)$ the parameter Ξ becomes $\Xi = (4\pi/3)\ Q^2 M\ \Delta {z_0}^3$. For example, for the data displayed in Figs. 1 and 2 ($Q = -100$, $M = 1$, $\Delta z_0 = 0.05$) we obtain $\Xi \approx 5.2$, which explains why the analytical formula (for which we had to assume $\Delta v_0 = 0$) worked so accurately. This also means, if the initial particle distribution is sufficiently narrow, $\Delta z_0 \ll [(4\pi/3) Q^2 M]^{-1/3}$, the self-spreading can be neglected compared to the natural spreading.

In Fig. 2 we compare the time evolution of the spatial width obtained from various models. The first graph is the quantum mechanical width $\Delta z(t)$ obtained from the coupled Dirac-Maxwell equations. This graph is indistinguishable from the one obtained from the classical ensemble given by Eq. (8). The agreement is again rather good suggesting that the initial width in the momentum distribution Δp_0 (which we neglected in our analytical derivation) is not so important here. To illustrate the importance of the interparticle forces, for comparison we have

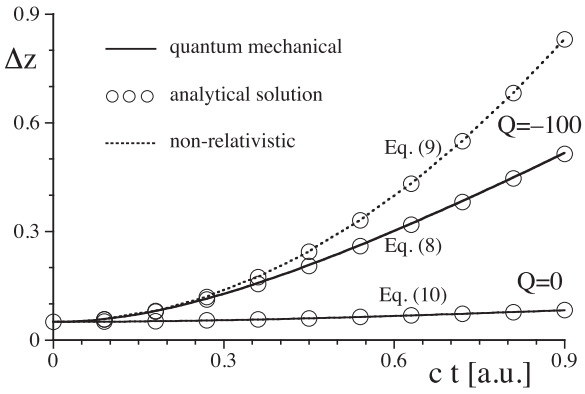

FIG. 2. The spatial width $\Delta z(t)$ of the charged particle as a function of time. The lowest curve/circles are the quantum/classical data for $Q = 0$. For $Q = -100$, the quantum calculation is compared with the analytical formula. (Parameters are $L = 6.4$ a.u., $N_z = 2048$ spatial grid points, initial width $\Delta z_0 = 0.05$ a.u. 10 000 particles in the classical ensemble.)

also shown the spreading behavior for a particle that is not coupled to the Maxwell equation, coresponding to $Q = 0$, and given by Eq. (10). We see that at the final time the natural spreading has only increased the width by a factor of 1.65, while the self-interaction increased the width by a factor of 10.3 during the same time interval.

Finally, by comparing with the nonrelativistic limit Eq. (9), we see that the dynamics is highly relativistic, especially at long interaction times when all particles (except the one exactly in the middle of the distribution) approach the speed of light c. While for early times when the particles have speeds significantly less than c, the agreement is rather good as expected, however, at longer times the width grows quadratically in time leading to a width that can exceed the relativistic limit $\Delta z(t) \sim ct$.

In summary, we have shown that the unavoidable self-repulsion of a charged quantum particle when described by the coupled Maxwell-Dirac equation can be modeled very accurately and therefore understood in terms of a classical mechanical ensemble of mutually interacting quasiparticles, for which analytical solutions are available. This agreement permits us now to estimate the impact of this effect, which is unavoidable unless the photon field is second quantized or the source terms in the Maxwell equations are not chosen *ab initio* but modified phenomenologically, such as done for example in the Vlasov equation [4–7].

This finding suggests that in a focal region where the electron's and positron's probability density overlap the magnitude of the (unphysical) electron self-repulsion is equal to the amount of the (physical) attractive force between an electron and a positron. This is supported by the fact that in the present formalism two parts of the same electron contribute to the total charge and current densities in the same way in the Maxwell equation as two different electrons or an electron and a positron would contribute. In other words, studies that investigate *ab initio* the effect of the electron-positron interaction on the pair creation process without the required second quantization of the photon field have to be rather carefully examined [8–10]. The predictions of the pair creation process might be rather

different, when the internally generated field is described with the required second quantization.

While these conclusions are based on the spatially restricted Coulomb law, we would expect that the impact of the self-repulsion in *ab initio* approaches will also be a serious issue in higher dimensions. While the availability of a fully analytical solution for the spreading behavior relied on the spatial restriction of our system, the agreement between the quantum and classical dynamics should also prevail in higher dimensions, for which only numerical solutions might be available.

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