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Abraham and Minkowski momenta in the optically induced motion of fluids

Ulf Leonhardt
Department of Physics of Complex Systems, Weizmann Institute of Science, Rehovot 76100, Israel
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Does the Abraham or the Minkowski momentum describe the momentum transport of light in media? Here we show that this is a question of fluid dynamics. In momentum transport, neither the Abraham nor the Minkowski momentum is fundamental, but they emerge depending on the fluid-mechanical response of the medium on the light. If the fluid is not brought into motion, the Minkowski momentum emerges, if it moves the Abraham momentum appears.

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I. INTRODUCTION

According to McIntyre [1],

> Controvercies over the "momentum" of waves have repeatedly wasted the time of physicists for over ... a century. The persistence of the controvercies is surprising, since regardless of whether classical or quantum dynamics is used the facts of the matter are simple and unequivocal, are well checked by laboratory experiment, are clearly explained in several published papers, and on the theoretical side can easily be verified by straightforward calculations.

It is indeed surprising [2] that the momentum of waves, and here, in particular, the momentum of light, has been the subject of controversy since 1909 [3,4]. "The argument has not, it is true, been carried on at high volume, but the list of disputants is very distinguished" [5].

Are the facts of the matter simple and unequivocal? Consensus has been reached on the meaning of the two principal contenders for the momentum of light in media: The Minkowski momentum is the canonical, and the Abraham momentum the kinetic, momentum [6]. The Minkowski momentum corresponds to the wave; the Abraham momentum, to the particle aspects of light [7]. Or, from a geometrical perspective [8], the Minkowski momentum is the covariant, and the Abraham momentum the contravariant, momentum with respect to the geometry of light in media [9]. However, as Brillouin [10] wrote in 1925, "It is not ultimately the density of momentum which matters, but rather the *flux of momentum*." In this paper we show how subtle the question of the momentum flux of light is.

The physical situation considered here can hardly be simpler: light is propagating through two homogeneous media with a planar interface; part of the light is transmitted, the rest is reflected. Depending on the balance between the incoming and the outgoing momentum flux, the interface experiences a pressure difference that may lead to measurable physical effects that ought to be "well checked by laboratory experiment" [1]. But are they? Three experiments [11–13], the first by Ashkin and Dziedzic [11], seem to show that the momentum transported across the interface is Minkowski's; one experiment [14] indicates quantitatively that it is Abraham's. This paper is inspired by the latter experiment; it asks, and hopefully answers, the question how such seemingly similar experiments on the momentum transport of light can show such different modes of behavior.

The calculations required to solve this puzzle cannot "be easily verified by straightforward calculations" [1], as they involve fluid mechanics. The paper derives an analytical solution of the linearized stationary Navier-Stokes equation [15,16]. The solution describes the velocity profile of a viscous fluid in response to an incident light beam that, as it turns out, facilitates the transport of the Abraham momentum. The transport of the Minkowski momentum emerges trivially if the light is not able to put the fluid in motion. However, the paper cannot discriminate between the precise circumstances in which one or the other mode of momentum transfer appears; this is left to future research.

II. MOMENTUM BALANCE

Let us begin at the beginning—the fact that a dielectric medium consists of individual building blocks, molecules and atoms—and consider the mechanical force of light on them. Suppose that the light is sufficiently off resonance such that we can safely ignore absorption and dissipation. Note that in this case we implicitly neglect the dissipative force of light [17] that leads to the radiation pressure on atoms used, e.g., in laser cooling. Nevertheless, as we see in the next section, the remaining reactive force accounts for the radiation pressure on bulk media.

Consider a single molecule or atom interacting with light. The particle should be small enough so that we can regard it as an induced electric dipole of polarizability α and mass m moving at velocity $\boldsymbol{v}$. We could also consider the contribution coming from the magnetic dipole, but the results will be completely analogous to the purely electric case. The electric dipole experiences the electric field as the potential [18]

$$V = -\frac{\alpha}{2} E'^2. \tag{1}$$

The resulting force is proportional to the gradient of the intensity of light and is known as the optical gradient force or optical dipole force. Note that the dipole responds to the electric field $\boldsymbol{E}'$ in its own rest frame, i.e., in a frame comoving with the particle, which differs from the electric field $\boldsymbol{E}$ in the laboratory frame. From special relativity we get to lowest order in v/c [18],

$$E'^2 = E^2 - 2\boldsymbol{v}\cdot(\boldsymbol{E}\times\boldsymbol{B}), \tag{2}$$

where $\boldsymbol{B}$ denotes the magnetic induction. Throughout this paper we use SI units. As usual, c denotes the speed of light in vacuum. Note that the distinction between laboratory and comoving frame becomes significant in the momentum of the particle [8]. Indeed, we obtain from the Lagrangian

$$L = \frac{m}{2} v^2 - V \tag{3}$$

the canonical momentum

$$\boldsymbol{p} = \frac{\partial L}{\partial \boldsymbol{v}} = m\boldsymbol{v} - \alpha \boldsymbol{E} \times \boldsymbol{B}. \tag{4}$$

We see that the canonical momentum $\boldsymbol{p}$ differs from the kinetic momentum $m\boldsymbol{v}$ by $\alpha \boldsymbol{E} \times \boldsymbol{B}$ (which is known as the Röntgen interaction [19,20]). This difference is one reason for the Abraham-Minkowski controversy [3,4,21], although not the only one, as we are going to see.

Let us now proceed from the individual particles to bulk media. Imagine that many of such point dipoles constitute a fluid of mass density ρ and flow velocity $\boldsymbol{u}$ described by the continuity equation

$$\partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0. \tag{5}$$

Suppose, for simplicity, that the canonical momentum $\boldsymbol{p}$ is given by the gradient of the velocity potential φ times the mass m such that

$$\boldsymbol{u} = \nabla \varphi + \frac{\alpha}{m} \boldsymbol{E} \times \boldsymbol{B}. \tag{6}$$

For Bose-Einstein condensates [22] φ is proportional to the phase of the mean-field wave function. For more general fluids, the assumption expressed in Eq. (6) has the advantage that we can easily write down the Lagrangian density $\mathscr{L}_{\mathrm{M}}$ that generates the equations of motions. From the Lagrangian we can deduce the constitutive equations of the medium and, from them, calculate the momentum balance. As the resulting equations of motion can also be deduced from general thermodynamical arguments [23,24], we expect that our results remain the same irrespective whether or not $\boldsymbol{p}$ can be expressed as a gradient. In the case of a gradient $\boldsymbol{p}$, the Lagrangian density $\mathscr{L}_{\mathrm{M}}$ should generate the Bernoulli equation [15,16]

$$\partial_t \varphi + \frac{u^2}{2} - \frac{\alpha}{2m} E^2 + W = 0 \tag{7}$$

in addition to the continuity equation, (5). Here W denotes the enthalpy that, for isentropic motion [16], is directly related to the pressure p by

$$\nabla W = \frac{\nabla p}{\rho}. \tag{8}$$

We write down the Lagrangian density

$$\mathscr{L}_{\mathrm{M}} = -\rho \left(\partial_t \varphi + \frac{u^2}{2} - \frac{\alpha}{2m} E^2 + W \right). \tag{9}$$

The Euler-Lagrange equation for ρ trivially gives the Bernoulli equation, (7), while the Euler-Lagrange equation for φ generates the continuity equation, (5), as required. The negative sign in front of ρ was chosen such that the action is minimal for the field-free case, where we also obtain for the energy density $\rho u^2/2 + p$. These features uniquely determine the Lagrangian density $\mathscr{L}_{\mathrm{M}}$.

Let us now add to $\mathscr{L}_{\mathrm{M}}$ the Lagrangian density $\mathscr{L}_{\mathrm{F}}$ of the free electromagnetic field, with

$$\mathscr{L}_{\mathrm{F}} = \frac{\varepsilon_0}{2}(E^2 - c^2 B^2), \tag{10}$$

where ε_0 denotes the electric permittivity of the vacuum. We thus have, for the total Lagrangian density,

$$\mathscr{L} = \mathscr{L}_{\mathrm{M}} + \mathscr{L}_{\mathrm{F}}. \tag{11}$$

Writing the electric field and the magnetic induction in terms of the vector potential $\boldsymbol{A}$ in Coulomb gauge,

$$\boldsymbol{E} = -\partial_t \boldsymbol{A}, \quad \boldsymbol{B} = \nabla \times \boldsymbol{A}, \tag{12}$$

we obtain [20] from the total Lagrangian, (11), Maxwell's equations

$$\begin{aligned} \nabla \cdot \boldsymbol{D} = 0, \quad \nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B}, \\ \nabla \cdot \boldsymbol{B} = 0, \quad \nabla \times \boldsymbol{H} = \partial_t \boldsymbol{D}, \end{aligned} \tag{13}$$

with the constitutive equations

$$\begin{aligned} \boldsymbol{D} &= \varepsilon_0[(1+\chi)\boldsymbol{E} + \chi \boldsymbol{u} \times \boldsymbol{B}], \\ \boldsymbol{H} &= \varepsilon_0(c^2 \boldsymbol{B} + \chi \boldsymbol{u} \times \boldsymbol{E}) \end{aligned} \tag{14}$$

and the susceptibility

$$\chi = \frac{\alpha \rho}{\epsilon_0 m}. \tag{15}$$

These are the constitutive equations of a moving medium [23] to lowest order in u/c. From them we deduce the momentum balance. In the Appendix we derive, from the equation of continuity, (5), the Bernoulli equation, (7), the isentropic relation, (8), Maxwell's equations, (13), and the constitutive equations, (14) and (15),

$$\begin{aligned} &\partial_t(\rho \nabla \varphi + \boldsymbol{D} \times \boldsymbol{B}) + \nabla \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u}) \\ &= \nabla \cdot \left(\sigma + \frac{\varepsilon_0}{2} \chi E'^2 \mathbb{1} - p \mathbb{1} \right). \end{aligned} \tag{16}$$

Here E'^2 denotes the electric-field intensity in a locally comoving frame, as in Eq. (2) with $\boldsymbol{v} = \boldsymbol{u}$, and σ is Abraham's stress tensor,

$$\begin{aligned} \sigma = \boldsymbol{D} \otimes \boldsymbol{E} + \boldsymbol{B} \otimes \boldsymbol{H} + \chi \boldsymbol{u} \otimes \varepsilon_0 \boldsymbol{E} \times \boldsymbol{B} \\ - \tfrac{1}{2}(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{B} \cdot \boldsymbol{H}). \end{aligned} \tag{17}$$

The Abraham tensor consists of Maxwell's stress tensor [18] plus the *Ruhstrahl* contribution $\chi \boldsymbol{u} \otimes \varepsilon_0 \boldsymbol{E} \times \boldsymbol{B}$ due to Abraham [4]. One verifies by direct calculation that Abrahams's tensor, as given by Eq. (17), is symmetric for moving media with the constitutive equations, (14), whereas Maxwell's stress tensor is not symmetric in this case. Formula (16) holds for dilute media where the particles constituting the media do not directly influence their electromagnetic response such that the susceptibility χ is proportional to the density ρ, as in Eq. (15). Otherwise, the explicit χ in formula (16) needs to be replaced by $\rho\,(\partial \chi / \partial \rho)$ [23,24].

Let us now remove the scaffolding of our derivation and write down the essential results. Although we have borrowed a few elements of relativity, most of them are not relevant in practice (at least for the existing experimental tests of the momentum of light). For electromagnetic radiation the magnetic induction B scales like E/c in SI units, so we ignore the contributions of motion in the constitutive equations:

$$\boldsymbol{D} = \varepsilon_0(1+\chi)\boldsymbol{E}, \quad \boldsymbol{H} = \varepsilon_0 c^2 \boldsymbol{B}. \tag{18}$$

We define the quantity

$$\boldsymbol{g} = \rho\nabla\varphi + \boldsymbol{D}\times\boldsymbol{B}, \tag{19}$$

for which we obtain, from Eq. (16) in the nonrelativistic limit,

$$\partial_t \boldsymbol{g} + \nabla\cdot(\rho\boldsymbol{u}\otimes\boldsymbol{u}) = \nabla\cdot\left(\sigma + \frac{\varepsilon_0}{2}\chi E^2 \mathbb{1} - p\mathbb{1}\right) \tag{20}$$

with Maxwell's stress tensor

$$\sigma = \boldsymbol{D}\otimes\boldsymbol{E} + \boldsymbol{B}\otimes\boldsymbol{H} - \tfrac{\mathbb{1}}{2}(\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{B}\cdot\boldsymbol{H}). \tag{21}$$

Equation (20) describes a conservation law. As $\rho\nabla\varphi$ accounts for the canonical momentum of the fluid, we interpret the conserved density, (19), as the total momentum density of light and matter. The term $\boldsymbol{D}\times\boldsymbol{B}$ denotes the Minkowski momentum density of light [3]. We thus conclude that temporal changes in the Minkowski momentum are mirrored by temporal changes in the canonical momentum. For Bose-Einstein condensates, φ corresponds to the phase of the wave function, so the Minkowski momentum is imprinted on the phase and appears in matter-wave interferometry, in agreement with experiment [25]. However, with the help of relationship (6), we may also express the total momentum density of Eq. (19) in terms of the kinetic momentum density $\rho\boldsymbol{u}$. All we have to do is subtract $\chi\varepsilon_0\boldsymbol{E}\times\boldsymbol{B}$ from Minkowski's $\boldsymbol{D}\times\boldsymbol{B}$ and use the constitutive equations, (18). We obtain

$$\boldsymbol{g} = \rho\boldsymbol{u} + \frac{\boldsymbol{E}\times\boldsymbol{H}}{c^2}, \tag{22}$$

where $\boldsymbol{E}\times\boldsymbol{H}$ is the Abraham momentum density [4]. Therefore, temporal changes in the Abraham momentum result in mechanical motion, again in agreement with experiment [26].

However, other experimental tests of the momentum of light in media are based on changes in space and not in time. For example, at the interface between two homogeneous media, where light is partially reflected and may exert mechanical forces, the media vary in space and not in time. What matters here is the *momentum transport* in space, and not the momentum conservation in time. The momentum transport is described by the right-hand side of the conservation law of Eq. (20), which reveals a much more complex picture than the ambiguity between Abraham and Minkowski momentum in Eqs. (19) and (22). Three terms contribute to the momentum transport: the stress tensor σ of the electromagnetic field, the dipole potential of Eq. (1), and the internal pressure p of the medium. Suppose that the pressure balances the energy density of the induced dipoles,

$$p = \frac{\varepsilon_0}{2}\chi E^2, \tag{23}$$

apart from an overall constant. In this case, the divergence of Maxwell's stress tensor σ, where σ is given by Eq. (21), describes the force density on the dielectric medium. What is the momentum transfer at the interface between two media?

III. PLANAR INTERFACE

Consider the propagation of light between two uniform media with a planar interface. The media shall have the refractive indices n_1 and n_2 (Fig. 1), where the refractive index is related to the susceptibility by

$$n_\nu^2 = 1 + \chi_\nu. \tag{24}$$

In each medium, light propagates with the local wave number k_ν, which depends on n_ν and the angular frequency ω or the free-space wavelength λ as

$$k_\nu = n_\nu\frac{\omega}{c} = n_\nu\frac{2\pi}{\lambda}. \tag{25}$$

Suppose that the light propagates predominantly perpendicular to the dielectric interface, in the z direction (Fig. 1). In this case, the vector potential $\boldsymbol{A}_\nu$ is given by the expression

$$\boldsymbol{A}_\nu = \frac{\boldsymbol{\alpha}\sqrt{P}\,e^{-i\omega t}}{\sqrt{2\varepsilon_0 n_1 c}\,\omega}\left(T_\nu u_{+\nu}(\boldsymbol{r})\,e^{ik_\nu z} + R_\nu u_{-\nu}(\boldsymbol{r})\,e^{-ik_\nu z}\right) + \text{c.c.}, \tag{26}$$

where c.c. denotes the complex conjugate. The vector $\boldsymbol{\alpha}$ describes the polarization direction, a unity vector orthogonal to the z direction, and $u_{\pm\nu}(\boldsymbol{r})$ are the profiles of the incident and reflected beams with the transmission and reflection coefficients T_ν and R_ν. The constant P describes the power of the light field. For light coming from medium 1 (Fig. 1) we have

$$T_1 = 1, \quad R_2 = 0. \tag{27}$$

As a consequence of the field equations, (12) and (13), and the constitutive equations, (18), the vector potential in the

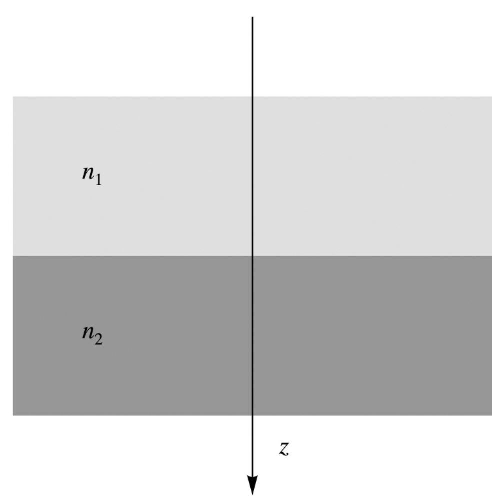

FIG. 1. Light shall propagate in the z direction perpendicularly to the planar interface between two infinitely extended media with uniform refractive indices n_1 and n_2.

Coulomb gauge [18] obeys the Helmholtz equation in each homogeneous region,

$$\left(\nabla^2 + k_\nu^2\right) \boldsymbol{A}_\nu = 0, \tag{28}$$

and hence $(\partial_x^2 + \partial_y^2 + \partial_z^2 + k_\nu^2) u_{\pm\nu} \exp(\pm i k_\nu z) = 0$. In the paraxial approximation [27] we assume that the beam profile does not vary much in the propagation direction and hence we may ignore the second z derivative of $u_{\pm\nu}$. We obtain the optical Schrödinger equation:

$$\left(\pm 2 i k_\nu \, \partial_z + \partial_x^2 + \partial_y^2\right) u_{\pm\nu} = 0. \tag{29}$$

At the interface we put $u_{-\nu} = u_{+\nu}$; the reflection and transmission are then completely described by R_ν and T_ν. As the Schrödinger equation, (29), conserves the norm, we may require

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |u_{\pm\nu}(\boldsymbol{r})|^2 \, dx \, dy = 1. \tag{30}$$

The intensity of the incident light is given by

$$I(\boldsymbol{r}) = P |u_{+1}(\boldsymbol{r})|^2. \tag{31}$$

We see this from a calculation of the Poynting vector $\boldsymbol{E} \times \boldsymbol{H}$ averaged over an optical cycle that describes the energy flux and hence the intensity. We consider the incident light, i.e., put $R_1 = 0$, ignore the evolution of $u(\boldsymbol{r})$, and obtain the z component of the Poynting vector, Eq. (31), as required. Note that from the normalization, Eq. (30), it follows that P is indeed the incident light power.

Consider the electromagnetic stress at the interface, described by the stress tensor of Eq. (21). Without loss of generality we put the polarization vector $\boldsymbol{\alpha}$ in the x direction. We obtain from the field equations, (12), and the constitutive equations, (18), for the vector potential specified by Eq. (26) with Eqs. (24), (25), and (31), the field correlations

$$\begin{aligned} \langle D_x E_x \rangle &= \frac{n_\nu^2 I}{n_1 c} (|T_\nu|^2 + |R_\nu|^2 + 2|T_\nu||R_\nu| \cos \Phi_\nu), \\ \langle B_y H_y \rangle &= \frac{n_\nu^2 I}{n_1 c} (|T_\nu|^2 + |R_\nu|^2 - 2|T_\nu||R_\nu| \cos \Phi_\nu), \end{aligned} \tag{32}$$

with $\Phi_\nu = k_\nu z + \phi_\nu$. The ϕ_ν denote the phases of $T_\nu R_\nu^*$ and the angle brackets indicate that we have averaged over an optical cycle. We have also neglected derivatives with respect to x and y. All other correlation functions constituting the stress tensor of Eq. (21) vanish, so σ is diagonal. Since σ depends essentially only on z, the momentum transfer $\nabla \cdot \sigma$ is thus solely given by σ_{zz}, for which we obtain

$$\sigma_{zz}^{(\nu)} = -\frac{n_\nu^2 I}{n_1 c} (|T_\nu|^2 + |R_\nu|^2). \tag{33}$$

To check whether this result makes physical sense, suppose that the incident light propagates in empty space ($n_1 = 1$) and is totally reflected ($R_1 = 1$). In this case we have $\sigma_{zz}^{(1)} = -2I/c$ and $\sigma_{zz}^{(2)} \sim 0$, so the total momentum transfer amounts to twice the incoming momentum flux I/c, as one would expect. Light exerts a pushing force on the total reflection, which we may easily understand if the medium is a metal with negative $\varepsilon = 1 + \chi_2$. In this case, n_2 is purely imaginary, so the field and hence $\sigma_{zz}^{(2)}$ exponentially decays in the medium, and χ_2 is negative, which corresponds to a repulsive dipole potential according to Eq. (1). Less intuitive is the pushing force if the reflection is created by a Bragg mirror [27] made of periodic layers with real refractive indices. In this case, the field also decays after each period and the incident light is completely reflected, so the momentum transfer is positive as well and amounts to $2I/c$, although χ is always positive. Presumably, the pressure balance of Eq. (23) in the medium creates an overall pushing force.

Consider now the case of two homogeneous media with $n_1 < n_2$ and a sharp planar interface. The reflection and transmission coefficients are given by the Fresnel coefficients [18]

$$R_1 = \frac{n_1 - n_2}{n_1 + n_2}, \quad T_2 = \frac{2n_1}{n_1 + n_2}. \tag{34}$$

We obtain from expression (33) the force density

$$f_z = (\nabla \cdot \sigma)_z = 2n_1 \frac{n_1 - n_2}{n_1 + n_2} \frac{I}{c} \, \delta(z), \tag{35}$$

which, for $n_1 < n_2$, pulls medium 2 back to the direction the light comes from. The δ function in Eq. (35) indicates that the force density is concentrated at the interface. We can understand f_z as the gradient $\partial_z p_{\mathrm{Mrad}}$ of the pressure

$$p_{\mathrm{Mrad}} = \begin{cases} \left(1 + R_1^2\right) n_1 I/c: & z < 0. \\ \left(1 - R_1^2\right) n_2 I/c: & z > 0. \end{cases} \tag{36}$$

We may interpret p_{Mrad} as the radiation pressure *à la* Minkowski, because it consists of the Minkowski momentum $n_\nu I/c$ with the appropriate prefactors describing reflection and transmission: in the region of incidence, $z < 0$, the prefactor is the sum of the ingoing and the reflected part; and in the region with $z > 0$ the prefactor accounts for the remaining transmitted light—there it is the difference between the ingoing and the reflected part. We thus get the impression that the momentum transfer is governed by the Minkowski momentum: the Minkowski momentum emerges as the effective optomechanical momentum. We may also read the force density f_z as the gradient of the difference in fluid pressure, where the pressure is given by Eq. (23), because

$$p_2 - p_1 = \frac{\varepsilon_0}{2} \left(n_2^2 - n_1^2\right) \langle E^2 \rangle = 2n_1 \frac{n_2 - n_1}{n_2 + n_1} \frac{I}{c}. \tag{37}$$

This pressure difference must be counterbalanced by the surface tension [15] causing an outward bulge of the fluid with a higher index, in agreement with experiments [11–13]. Yet She *et al.* [14] have observed the exact opposite, an indentation of the fluid surface by the same magnitude. How is this possible?

IV. FLUID DYNAMICS

The pulling force of the light is a consequence of the balance between the fluid pressure and the optical dipole force: along the beam the light puts a lateral force on the fluid that the pressure compensates, but as the pressure is isotropic, it also acts on the interface between the two media, bulging it out. Therefore, a pushing force can only appear if the light potential is not balanced by the pressure. In this case, local forces are acting in the bulk of the medium, causing it to flow. Consequently, to explain the experiment of She *et al.* [14], we must consider the fluid mechanics driven by the dipole force of light.

Suppose that medium 1 is simply air (empty space) and medium 2 is an incompressible fluid with some viscosity η such as the water or oil in the experiment [14]. To keep the notation simple, we drop the reference to the medium, the index 2, in the quantities referring to it. Incompressibility implies that

$$\rho = \text{const} \quad \text{and} \quad \text{hence} \quad \chi = \text{const}. \tag{38}$$

From the continuity equation, (5), it follows that

$$\nabla \cdot \boldsymbol{u} = 0. \tag{39}$$

We assume the electromagnetic field to be stationary such that

$$\partial_t \langle \boldsymbol{E} \times \boldsymbol{H} \rangle = \boldsymbol{0}, \tag{40}$$

where, as in the previous section, angle brackets indicate averaging over an optical cycle. In addition, we obtain from Eqs. (A6), (A7), and (A9) in the limit $u/c \to 0$ and the incompressibility, Eq. (38),

$$\boldsymbol{0} = \partial_t \langle \boldsymbol{D} \times \boldsymbol{B} \rangle = \nabla \cdot \langle \sigma \rangle + (\nabla \chi) \frac{\varepsilon_0}{2} \langle E^2 \rangle = \nabla \cdot \langle \sigma \rangle. \tag{41}$$

Inserting Eqs. (40) and (41) in the conservation law of Eq. (20) for the Abraham momentum of Eq. (22) we obtain the Navier-Stokes equation [15,16],

$$\rho \left[\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla) \boldsymbol{u} \right] = \nabla \left(\frac{\varepsilon_0}{2} \chi \langle E^2 \rangle - p \right) + \eta \nabla^2 \boldsymbol{u}, \tag{42}$$

where we have used the incompressibility relations, (38) and (39), and included the viscosity of the fluid by adding the term $\eta \nabla^2 \boldsymbol{u}$ to the force density (which is the only source of viscosity for incompressible fluids [16]). As the fluid evolves on much longer time scales than the optical oscillations, we are justified in averaging the dynamics over an optical cycle.

The Navier-Stokes equation, (42), describes the fluid dynamics. We see that $\boldsymbol{u} = \boldsymbol{0}$ together with the pressure balance, (23), establishes an obvious solution of the Navier-Stokes equation, a solution that, as we have seen, conforms to the Minkowski pressure of light. Suppose now that the light has brought the fluid into a steady flow with $\boldsymbol{u} \neq \boldsymbol{0}$ and

$$\partial_t \boldsymbol{u} = \boldsymbol{0}. \tag{43}$$

Solving the Navier-Stokes equation in all but trivial situations is notoriously difficult. We were only able to deduce an approximate solution for the simple model we describe below, but we argue that the main features of this solution are sufficiently general. Our mathematical method is inspired by the optical analog [28] of acoustic streaming [29], but the physics we consider and hence our results are different. In optical streaming, an incident light beam is gradually absorbed due to scattering, whereas in our case the light is not absorbed and exerts the nondissipative dipole force on the fluid.

Imagine that the light beam in the fluid is much thinner than the typical scales of the fluid flow (which are of the order of a centimeter). We idealize the beam to be infinitely thin. We also linearize the Navier-Stokes equation, (42), for the steady flow, Eq. (43), outside the line where the light has a direct effect and get there

$$\eta \nabla^2 \boldsymbol{u} = \nabla p. \tag{44}$$

It is useful to cast this equation in a different form. For incompressible fluids we have $\nabla \times (\nabla \times \boldsymbol{u}) = \nabla(\nabla \cdot \boldsymbol{u}) - \nabla^2 \boldsymbol{u} = -\nabla^2 \boldsymbol{u}$ and hence

$$\eta \nabla \times (\nabla \times \boldsymbol{u}) + \nabla p = \boldsymbol{0}. \tag{45}$$

This is the form of the linearized, stationary Navier-Stokes equation we are going to use. We express $\boldsymbol{u}$ in terms of the vector stream function [16]

$$\boldsymbol{u} = \nabla \times \boldsymbol{\psi}, \tag{46}$$

which automatically takes care of the incompressibility condition, Eq. (39). The curl of the linearized Navier-Stokes equation, (45), gives

$$\nabla \times \{ \nabla \times [\nabla \times (\nabla \times \boldsymbol{\psi})] \} = 0 \tag{47}$$

within the linear regime of the viscous fluid dynamics. We seek a cylindrically symmetric solution for $\boldsymbol{\psi}$ that is regular at the light beam, decays exponentially far away from it, and satisfies no-slip boundary conditions at the surface of the fluid and the bottom of the container with depth d. We may assume that the fluid surface is planar, as in the experiment by She *et al.* [14] the surface deformation if of the order of 10 nm, the beam is about 1 mm wide, and the container a few centimeters deep.

Let us employ cylindrical coordinates $\{r,\phi,z\}$ with metric $dl^2 = dr^2 + r^2 d\phi^2 + dz^2$, the light beam being concentrated in the line at $r = 0$ and the surface lying at $z = 0$. We assume that the vector stream function depends only on r and z and points in the ϕ direction,

$$\psi_i = (0, \psi(r,z), 0), \tag{48}$$

where ψ_i is a one-form (a covariant vector) carrying a lower index [9]. We obtain, from the curl in cylindrical coordinates [9],

$$u^i = \begin{pmatrix} u_r \\ 0 \\ u_z \end{pmatrix} \quad \text{with} \quad u_r = -\frac{1}{r} \partial_z \psi, \quad u_z = \frac{1}{r} \partial_r \psi. \tag{49}$$

The velocity is a vector and hence carries a superscript index. Note, however, that the cylindrical coordinates u_r and u_z are the same for both one-forms and vectors. Consider a stream line where $dr = u_r dt$ and $dz = u_z dt$. We see from Eq. (49) that here $d\psi = 0$: the stream lines are the contour lines of the stream function $\psi(r,z)$ (Fig. 2). Next we calculate the curl of $\boldsymbol{u}$; we get 0 for its r and z components, and

$$(\nabla \times \boldsymbol{u})_\phi = r \left(\partial_z u_r - \partial_r u_z \right) = -D^2 \psi, \tag{50}$$

where D^2 is defined as

$$D^2 = \partial_r^2 - \frac{1}{r} \partial_r + \partial_z^2. \tag{51}$$

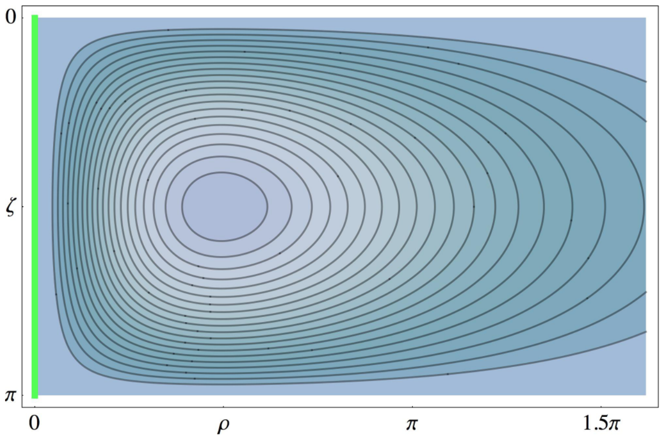

FIG. 2. (Color online) The light beam [vertical (green) bar at left] puts the fluid into motion, forming a vortex ring around it. The picture shows the stream lines on a cut through one side of the vortex ring. The stream lines are the contour lines of the stream function, Eq. (59), in terms of the scaled coordinates of Eq. (55).

To find a geometrical meaning for D^2 we note that for our r- and z-dependent $\boldsymbol{\psi}$ pointing in the ϕ direction, we have $\nabla\cdot\boldsymbol{\psi}=0$. Hence

$$\nabla^2\boldsymbol{\psi} = -\nabla\times(\nabla\times\boldsymbol{\psi}) = -\nabla\times\boldsymbol{u} = D^2\boldsymbol{\psi}. \tag{52}$$

Consequently, D^2 describes the Laplacian on one-forms of the type of Eq. (48). As the Laplacian depends on the geometric nature of the object it is acting on [9], D^2 differs from the familiar Laplacian $\partial_r^2 + r^{-1}\partial_r + \partial_z^2$ of a scalar in cylindrical coordinates. Applying the curl two more times gives, according to Eq. (47),

$$D^4\psi = 0, \tag{53}$$

the equation we are going to solve. We write ψ as

$$\psi = \psi_0(\varrho)\sin\zeta \tag{54}$$

in terms of the dimensionless variables

$$\zeta = \frac{\pi}{d}z, \quad \varrho = \frac{\pi}{d}r. \tag{55}$$

With this ansatz, the flow of Eq. (49) obeys the no-slip boundary conditions

$$u_z(0) = 0, \quad u_z(d) = 0 \tag{56}$$

at the surface of the fluid and the bottom of the container (note that z increases with increasing depth; see Fig. 1). For the Laplacian of Eq. (51) acting on the function defined in Eq. (54) with the variables of Eq. (55) we have

$$D^2 = \frac{\pi^2}{d^2}D_\varrho^2, \quad D_\varrho^2 = \partial_\varrho^2 - \frac{1}{\varrho}\partial_\varrho - 1. \tag{57}$$

We get, for the modified Bessel functions [30] from their recurrence relations and differential formulas,

$$D_\varrho^2\frac{\varrho^2}{2}K_0(\varrho) = -\varrho K_1(\varrho), \quad D_\varrho^2\frac{\varrho^2}{2}I_0(\varrho) = -\varrho I_1(\varrho),$$
$$D_\varrho^2\,\varrho K_1(\varrho) = 0, \quad D_\varrho^2\,\varrho I_1(\varrho) = 0. \tag{58}$$

Hence the functions $\{\varrho K_1,\varrho I_1,\varrho^2K_0,\varrho^2I_0\}$ form a fundamental system of the fourth-order differential equation, (53), with the ansatz of Eq. (54). The only solution that is decaying for $r\to\infty$ and regular at $r=0$ is

$$\psi = \mathcal{U}\frac{r^2}{2}K_0(\varrho)\sin\zeta, \tag{59}$$

with the constant parameter $\mathcal{U}$. To calculate the velocity profile we use the relation

$$\partial_\varrho K_0(\varrho) = -K_1(\varrho), \tag{60}$$

which also follows from the recurrence relations and differential formulas of the modified Bessel functions [30]. We obtain, for the velocity components of Eq. (49),

$$u_r = -\mathcal{U}\frac{\varrho}{2}K_0(\varrho)\cos\zeta,$$
$$u_z = \mathcal{U}\left(K_0(\varrho) - \frac{\varrho}{2}K_1(\varrho)\right)\sin\zeta. \tag{61}$$

We have thus found a solution of the fluid dynamics in the linear regime with the correct boundary and regularity conditions. It corresponds to a vortex ring around the light beam (Fig. 2). Our solution depends on the parameter $\mathcal{U}$, which carries the dimensions of a velocity.

In order to relate the velocity parameter $\mathcal{U}$ to the intensity of the light beam, we need to calculate the pressure from the linearized Navier-Stokes equation, (45). We obtain from Eqs. (50), (57), and (58)

$$(\nabla\times\boldsymbol{u})_\phi = \mathcal{U}\varrho K_1(\varrho)\sin\zeta, \tag{62}$$

and consequently,

$$[\nabla\times(\nabla\times\boldsymbol{u})]_r = -\frac{1}{r}\partial_z(\nabla\times\boldsymbol{u})_\phi = -\frac{\pi^2}{d^2}\mathcal{U}K_1(\varrho)\cos\zeta, \tag{63}$$

$$[\nabla\times(\nabla\times\boldsymbol{u})]_z = \frac{1}{r}\partial_r(\nabla\times\boldsymbol{u})_\phi$$
$$= \frac{\pi^2}{d^2}\mathcal{U}\frac{1}{\varrho}\partial_\varrho\,\varrho K_1(\varrho)\sin\zeta. \tag{64}$$

From relation (60) of the modified Bessel functions and [30]

$$\frac{1}{\varrho}\partial_\varrho\,\varrho K_1(\varrho) = -K_0(\varrho), \tag{65}$$

we see that $\eta\nabla\times(\nabla\times\boldsymbol{u})$ is the gradient of a scalar, the pressure

$$p_{\mathrm{B}} = -\eta\,\mathcal{U}\frac{\pi}{d}K_0(\varrho)\cos\zeta, \tag{66}$$

as it should be according to the linearized Navier-Stokes equation, (45). Note, however, that the z component of the flow velocity of Eq. (61) diverges logarithmically for $r\to0$, as $K_0(\varrho)\sim-\gamma-\ln(\varrho/2)$, where γ is Euler's constant and $K_1(\varrho)\sim\varrho^{-1}$ [30]. Consequently, near the light beam the flow must enter the nonlinear regime of the full Navier-Stokes equation, (42). Moreover, our solution of the linear equation, (45), suggests the existence of an additional, localized contribution to the pressure. To see this, integrate $[\nabla\times(\nabla\times\boldsymbol{u})]_z$

over an infinitesimally small disk around $r = 0$. Read the right-hand side of Eq. (64) as the two-dimensional divergence in polar coordinates [9] of a vector field pointing in the ϱ direction and having the amplitude $(\pi/d)\mathcal{U}K_1(\varrho)\sin\zeta$. From Gauss' theorem it follows that the area integral is given by the boundary integral $2\pi\varrho\,\mathcal{U}\,K_1(\varrho)\sin\zeta \sim 2\pi\,\mathcal{U}\,\sin\zeta$, as $K_1(\varrho) \sim \varrho^{-1}$ [30]. On the other hand, the area integral of the pressure given by Eq. (66) vanishes over an infinitesimally small disk around $r = 0$. Consequently, the total pressure of the fluid consists of two contributions,

$$p = p_{\rm B} + p_{\rm L}, \tag{67}$$

the bulk pressure given by Eq. (66) driving the fluid flow against the viscosity and a localized pressure $p_{\rm L}$ in direct response to the light. We obtain

$$p_{\rm L} = 2d\,\eta\,\mathcal{U}\,\delta^{(2)}(\boldsymbol{r})\cos\zeta, \tag{68}$$

where $\delta^{(2)}(\boldsymbol{r})$ denotes the two-dimensional δ function [31]. We see that the pressure, given by Eq. (68), changes sign between the bottom of the container and the surface of the fluid. At the bottom, the fluid must form a viscous layer of zero velocity [15,16], which implies that the optical potential of the light and the localized pressure balance each other there:

$$\frac{\varepsilon_0}{2}\chi\langle E^2\rangle = p_{\rm L}(d) = -2d\,\eta\,\mathcal{U}\,\delta^{(2)}(\boldsymbol{r}). \tag{69}$$

Due to the sign change between bottom and surface pressure, we obtain at the surface

$$p_{\rm L}(0) = -\frac{\varepsilon_0}{2}\chi\langle E^2\rangle = -2\,\frac{n-1}{n+1}\,\frac{I}{c}, \tag{70}$$

the exact opposite of the Minkowski pressure of Eq. (37) for $n_1 = 1$ and $n_2 = n$: Abraham's pressure! Depending on whether or not the flow pattern, (61), is generated, the Abraham or the Minkowski momentum appears to cause pressure on the surface of the fluid; the ground state carries the Minkowski pressure, Eq. (37), and the "first excited state" produces the Abraham pressure, Eq. (70).

We can easily calculate the energy of the fluid flow given by Eq. (61). The integral over the pressure of Eq. (67) from 0 to d vanishes, because of the $\cos\zeta$ term in both pressure contributions, Eqs. (66) and (68). Hence the energy is entirely given by the kinetic energy,

$$\begin{aligned} E_{\rm kin} &= \int_0^\infty \int_0^d \left(u_r^2 + u_z^2\right)\frac{\rho}{2}\,dz\,2\pi\,r\,dr \\ &= \int_0^\infty \int_0^d ((\partial_z\psi)^2 + (\partial_r\psi)^2)\frac{\pi\rho}{r}\,dz\,dr \\ &= -\int_0^\infty \int_0^d \psi(D^2\psi)\frac{\pi\rho}{r}\,dz\,dr, \end{aligned} \tag{71}$$

where we used Eq. (49) in the second line and partial integration and formula (51) for the Laplacian D^2 in the third line. We insert our solution, Eq. (59), and use one of the relations, (58), to obtain

$$\begin{aligned} E_{\rm kin} &= \frac{\rho\,d^3\,\mathcal{U}^2}{2\pi^2}\int_0^\infty K_0(\varrho)\,K_1(\varrho)\,\varrho^2 d\varrho \int_0^\pi (\sin\zeta)^2\,d\zeta \\ &= \frac{\rho\,d^3\,\mathcal{U}^2}{8\pi}. \end{aligned} \tag{72}$$

In the integral over the modified Bessel functions we utilized relation (65) for $K_0(\varrho)$, which reduces the integral to $[\varrho K_1(\varrho)]^2/2$ evaluated at $\varrho = 0$, which gives $1/2$ [30].

Let us estimate the order of magnitude for the flow velocity. Integrating our result, Eq. (70), for the localized pressure, Eq. (68), over the surface area and using Eq. (31) for the light intensity gives

$$\mathcal{U} = -\frac{n-1}{n+1}\frac{P}{\eta\,dc}. \tag{73}$$

For water, $(n-1)/(n+1) \approx 0.14$ and $\eta = 10^{-3}$ kg/ms. Hence we get, for 1 W of power and a vessel of a few centimeters' depth, a flow velocity of $|\mathcal{U}| \approx 10\ \mu$m/s. For such a velocity and gradients $\boldsymbol{\nabla}$ of the order of an inverse centimeter the nonlinear contribution to the Navier-Stokes equation, (42), is $\boldsymbol{u}\cdot\boldsymbol{\nabla} \approx 10^{-3}$ Hz. This is smaller than the viscosity term $(\eta/\varrho)\nabla^2 \approx 10^{-2}$ Hz, which justifies the linear approximation made. At the light beam, however, the velocity varies over a scale of less than a millimeter. There we can no longer ignore the nonlinear dynamics of the fluid. Nevertheless, we were able to use the linear regime of the fluid mechanics to derive a characteristic flow pattern (Fig. 2) that leads to a surface deformation in agreement with our experiment.

V. CONCLUSIONS

We have studied theoretically the momentum transport of light in fluids, where, according to Brillouin [10], "it is not ultimately the density of momentum which matters, but rather the *flux of momentum*." For light propagating through a planar interface between two homogeneous media, we have identified two distinct regimes: (i) If the incident light is not able to bring the fluid into motion, the resulting lateral optical force creates pressure in the fluid, which bulges the surface out. This pressure is exactly the same as one would obtain from a simple momentum balance using the Minkowski momentum of light in the fluid. (ii) If the light beam brings the fluid into motion, the resulting pressure on the fluid changes sign, causing the exact opposite: an indentation of the fluid surface, as one would obtain from the Abraham momentum. Therefore, in experiments [11–14] probing the momentum transfer of light in fluids, the momentum of light is not a fundamental quantity; it emerges as the result of the interplay between optics and fluid mechanics, and may appear as Minkowski's [3] or Abraham's [4].

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APPENDIX

In this Appendix we deduce the conservation law of the total momentum for light and matter, Eq. (16). This conservation law has been derived in the fully relativistic regime using a geometric method [8] based on the idea [32] that light establishes a space-time geometry for matter. Here we show that the conservation law, Eq. (16), also follows in a more elementary way, at the lowest order of u/c, from the fluid mechanics of polarizable dipoles and Maxwell's equations in moving media.

We begin by calculating the time derivative of the canonical momentum density of matter $\rho\nabla\varphi$. We obtain from the continuity equation, (5), the velocity of Eq. (6) and the susceptibility of Eq. (15),

$$\partial_t(\rho\nabla\varphi) = \rho\nabla\partial_t\varphi - (\nabla\cdot\rho\boldsymbol{u})\,\boldsymbol{u} + (\nabla\cdot\chi\boldsymbol{u})\,\boldsymbol{P}, \tag{A1}$$

where

$$\boldsymbol{P} = \varepsilon_0\boldsymbol{E}\times\boldsymbol{B}. \tag{A2}$$

For $\partial_t\varphi$ we apply the Bernoulli equation, (7), which contains $u^2/2$, for which we note

$$\rho\nabla\frac{u^2}{2} + \rho\,(\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} = \rho[\boldsymbol{u}\times(\nabla\times\boldsymbol{u})] = \nabla(\boldsymbol{u}\cdot\chi\underline{\boldsymbol{P}}) - \chi\,(\boldsymbol{u}\cdot\nabla)\,\boldsymbol{P}, \tag{A3}$$

where we have used Eq. (6) for the velocity and the fact that $\nabla\times\nabla\varphi = \boldsymbol{0}$. Here and in the following we indicate with underlining that only the underlined expression should be differentiated after applying a differential operator from the left. We use the identity

$$\nabla\cdot(\rho\boldsymbol{u}\otimes\boldsymbol{u}) = (\nabla\cdot\rho\boldsymbol{u})\,\boldsymbol{u} + \rho\,(\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} \tag{A4}$$

and obtain from relation (A1), the Bernoulli equation, (7), with the isentropic relation, (8), and our previous expressions, (A3) and (A4):

$$\partial_t(\rho\nabla\varphi) + \nabla\cdot(\rho\boldsymbol{u}\otimes\boldsymbol{u}) - \nabla\cdot(\chi\boldsymbol{u}\otimes\boldsymbol{P}) = -\nabla(\boldsymbol{u}\cdot\chi\underline{\boldsymbol{P}}) + \chi\nabla\varepsilon_0\frac{E^2}{2} - \nabla p. \tag{A5}$$

Let us turn to the time derivative of the Minkowski momentum density $\boldsymbol{D}\times\boldsymbol{B}$. We obtain from Maxwell's equations, (13),

$$\begin{aligned}\partial_t(\boldsymbol{D}\times\boldsymbol{B}) &= -\boldsymbol{D}\times(\nabla\times\boldsymbol{E}) - \boldsymbol{B}\times(\nabla\times\boldsymbol{H})\\ &= (\boldsymbol{D}\cdot\nabla)\boldsymbol{E} - \nabla(\boldsymbol{D}\cdot\underline{\boldsymbol{E}}) + (\boldsymbol{B}\cdot\nabla)\boldsymbol{H} - \nabla(\boldsymbol{B}\cdot\underline{\boldsymbol{H}})\\ &= \nabla\cdot(\boldsymbol{D}\otimes\boldsymbol{E} + \boldsymbol{B}\otimes\boldsymbol{H}) - \nabla\Sigma,\end{aligned} \tag{A6}$$

where $\nabla\Sigma$ stands for $\nabla(\boldsymbol{D}\cdot\underline{\boldsymbol{E}}) + \nabla(\boldsymbol{B}\cdot\underline{\boldsymbol{H}})$. We obtain for Σ, from the constitutive equations, (14),

$$\begin{aligned}\Sigma &= \varepsilon_0(1+\chi)\boldsymbol{E}\cdot\underline{\boldsymbol{E}} + \epsilon_0\chi(\boldsymbol{u}\times\boldsymbol{B})\cdot\underline{\boldsymbol{E}} + \frac{1}{\varepsilon_0 c^2}\boldsymbol{H}\cdot\underline{\boldsymbol{H}}\\ &\quad - \frac{\chi}{c^2}(\boldsymbol{u}\times\boldsymbol{E})\cdot\underline{\boldsymbol{H}}\\ &\approx \varepsilon_0(1+\chi)\boldsymbol{E}\cdot\underline{\boldsymbol{E}} + \frac{1}{\varepsilon_0 c^2}\boldsymbol{H}\cdot\underline{\boldsymbol{H}} - \chi\boldsymbol{u}\cdot\underline{\boldsymbol{P}},\end{aligned} \tag{A7}$$

in lowest relativistic order. Putting Eqs. (A5)–(A7) together we get, for the total momentum density of Eq. (19),

$$\partial_t\boldsymbol{g} + \nabla\cdot(\rho\boldsymbol{u}\otimes\boldsymbol{u}) - \nabla\cdot(\chi\boldsymbol{u}\otimes\boldsymbol{P}) = \nabla\cdot(\boldsymbol{D}\otimes\boldsymbol{E} + \boldsymbol{B}\otimes\boldsymbol{H}) - \nabla\left(\varepsilon_0\frac{E^2}{2} + \frac{H^2}{2\varepsilon_0 c^2} + p\right). \tag{A8}$$

From the constitutive equations, (14) we obtain

$$\begin{aligned}\boldsymbol{D}\cdot\boldsymbol{E} + \boldsymbol{B}\cdot\boldsymbol{H} &= \varepsilon_0(1+\chi)E^2 + \varepsilon_0 c^2 B^2\\ &= \varepsilon_0(1+\chi)E^2 + \frac{H^2}{\varepsilon_0 c^2} - 2\chi\boldsymbol{u}\cdot\boldsymbol{P}\\ &\approx \varepsilon_0 E^2 + \frac{H^2}{\varepsilon_0 c^2} + \varepsilon_0\chi E'^2,\end{aligned} \tag{A9}$$

in lowest relativistic order and with E'^2 being defined in Eq. (2). From Eqs. (A8) and (A9) follows the conservation law of Eq. (16) with Abraham's stress tensor of Eq. (17).

[1] M. E. McIntyre, J. Fluid Mech. **106**, 331 (1981).
[2] R. Peierls, *More Surprises in Theoretical Physics* (Princeton University Press, Princeton, NJ, 1991).
[3] H. Minkowski, Nachr. Ges. Wiss. Göttn. Math.-Phys. Kl. **53**, 472 (1908).
[4] M. Abraham, Rend. Circ. Matem. Palermo **28**, 1 (1909).
[5] E. I. Blount, quoted in Ref. [24].
[6] S. M. Barnett, Phys. Rev. Lett. **104**, 070401 (2010).
[7] U. Leonhardt, Nature **444**, 823 (2006).
[8] U. Leonhardt, Phys. Rev. A **73**, 032108 (2006).
[9] U. Leonhardt and T. G. Philbin, *Geometry and Light: The Science of Invisibility* (Dover, Mineola, NY, 2010).
[10] L. Brillouin, Ann. Phys. **4**, 528 (1925); translated in Ref. [1].
[11] A. Ashkin and J. M. Dziedzic, Phys. Rev. Lett. **30**, 139 (1973).
[12] A. Casner and J.-P. Delville, Phys. Rev. Lett. **87**, 054503 (2001).
[13] N. G. C. Astrath, L. C. Malacarne, M. L. Baesso, G. V. B. Lukasievicz, and S. E. Bialkowski, Nat. Commun. **5**, 4363 (2014).
[14] W. She, L. Zhang, N. Peng, and U. Leonhardt (unpublished).
[15] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon, Oxford, UK, 1987).
[16] G. Falkovich, *Fluid Mechanics* (Cambridge University Press, Cambridge, 2011).
[17] C. Cohen-Tannoudji and D. Guery-Odelin, *Advances in Atomic Physics* (World Scientific, Singapore, 2011).
[18] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1998).
[19] W. C. Röntgen, Ann. Phys. Chem. **271**, 264 (1888).
[20] U. Leonhardt and P. Piwnicki, Phys. Rev. Lett. **82**, 2426 (1999).
[21] S. M. Barnett and R. Loudon, Phil. Trans. R. Soc. A **368**, 927 (2010).
[22] L. Pitaevskii and S. Stringari, *Bose-Einstein Condensation* (Clarendon Press, Oxford, UK, 2003).
[23] L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media* (Pergamon, Oxford, UK, 1984).
[24] J. P. Gordon, Phys. Rev. A **8**, 14 (1973).
[25] G. K. Campbell, A. E. Leanhardt, J. Mun, M. Boyd, E. W. Streed, W. Ketterle, and D. E. Pritchard, Phys. Rev. Lett. **94**, 170403 (2005).

[26] G. B. Walker and D. G. Lahoz, Nature **253**, 339 (1975).
[27] M. Born and E. Wolf, *Principles of Optics* (Cambridge University Press, Cambridge, 1999).
[28] H. Chraibi, R. Wunenburger, D. Lasseux, J. Petit, and J.-P. Delville, J. Fluid Mech. **688**, 195 (2011).
[29] C. Eckart, Phys. Rev. **73**, 68 (1948).
[30] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions* (McGraw–Hill, New York, 1981).
[31] Note that we cannot obtain this result from integrating the r component of the pressure gradient—from $[\nabla \times (\nabla \times \boldsymbol{u})]_r$ of Eq. (63)—as the integral over the derivative of a δ function vanishes.
[32] U. Leonhardt, Phys. Rev. A **62**, 012111 (2000).