Universal scaling in the statistics and thermodynamics of a Bose-Einstein condensation of an ideal gas in an arbitrary trap

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We analytically calculate the critical phenomena of a Bose-Einstein condensation of an ideal gas in an arbitrary trap with any mesoscopic or macroscopic number of particles and find all universality classes of the system's statistics and thermodynamics. In particular, we find analytically the universal fine structure of the famous discontinuity in the value or/and derivative of the specific heat in the critical region around the λ point.

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I. A UNIVERSAL DEPENDENCE OF THE CRITICAL PHENOMENA ON THE TRAP'S FORM AND BOUNDARIES

The properties of the mesoscopic systems with a large, but finite number of particles are strongly affected by the well-known finite size effects. Do the latter effects survive and remain imprinted in the properties of the macroscopic systems in the bulk limit? In the present paper we answer "yes" to that question. Namely, we find a universal, self-similar dependence of the statistical and thermodynamic properties of an ideal gas on the trap's form and boundaries. That universal dependence determines a fine structure of the critical region in the Bose-Einstein condensation (BEC) and remains the same even in the bulk limit.

Note that usually a negative answer to the above question is assumed [1–8]. However, it does not take into account the anomalously large fluctuations and critical phenomena near the critical point of a second-order phase transition to a phase with a long-range order. They occur in the critical region, defined as a region of parameters, where an order parameter of the phase transition is on the order of the dispersion of its fluctuations. (That definition is similar to a thermodynamic one given by a Ginzburg-Levanyuk criterion [9,10].) The finite size effects disappear in the bulk limit only if a system is far from the critical point.

A fundamental reason for that effect lies in the very nature of the long-range second-order phase transitions associated with the singular behavior of the excitations, possessing the longest possible wavelengths. The properties of those excitations are directly controlled by the boundary conditions and predetermine the thermodynamics and statistics in the critical region. In other words, the effect is due to a singular (infinitely large) contribution of the excitations within an infinitesimally narrow spectral interval in the infrared limit of the energy spectrum. More generally, a fine structure of the critical region, which looks nonanalytical or singular in the thermodynamic limit, is determined by the solution of the corresponding mesoscopic problem with an explicit account for the details of the lower-energy states and boundary conditions.

In order to disclose that remarkable phenomenon it is necessary to explicitly solve the problem on the influence of the form and boundary conditions of the trap on the statistics and thermodynamics of the second-order phase transition for a system containing a finite, mesoscopic number of particles N. Then, taking the thermodynamic limit, one can find the above-mentioned singular contribution and reveal the universal structure of the critical region. Namely, the solution of that problem is given in the present paper for the case of the BEC in an ideal gas.

The BEC statistics in the systems with a mesoscopic number of the trapped atoms N attracted great interest in the recent years due to numerous experiments, where usually $N \sim 10^2$–10^7 [2,3,11–14]. It is also closely related to still missing microscopic theory of BEC and other second-order phase transitions [14–18]. The problem of an influence of the boundaries and form of the trap on the BEC phase transition in the mesoscopic systems was discussed by many authors [6,11,19–38], but remained unsolved until recently even for the case of an ideal gas.

A well-known phenomenological renormalization-group approach focuses on the first few coefficients in the Taylor series for the thermodynamic quantities in the close vicinity of the critical point, in particular on the critical exponents, and does not solve the problem [8,39–42]. A Landau mean-field theory works only far outside the critical region, where fluctuations of the order parameter are not essential at all [1,4,5,7]. The usually employed grand-canonical-ensemble approximation fails to describe correctly the BEC critical fluctuations since it allows for the unphysical fluctuations of the number of particles in the trap and fixes only the mean number of particles [2,3,6,43–46]. Thus, it is necessary to solve essentially more complicated, but correctly formulated problem of the BEC in the canonical ensemble.

Many authors addressed the problem of the BEC in the canonical ensemble (see, for example, [5,6,43,44]) and, in particular, the problem of the BEC of an ideal gas in mesoscopic traps (see, for example, [25,33–38,46–52]). However, the analysis was mainly restricted to numerical simulations for some particular values of the mesoscopic system's parameters out of a countless number of possible choices.

II. THE MAIN RESULTS OF THE PAPER

The universal scaling of the BEC statistics and thermodynamics for the ideal gas was first found in [17,18], where it was described in detail for the case of a box trap with the periodic boundary conditions. In the present paper we essentially extend that analysis to arbitrary traps. We give a full picture and explicit description of the influence of the boundary conditions and form of the trap on the universal scaling of the main statistical and thermodynamic quantities.

We find that all trapped BECs fall into two universality classes, Gaussian and anomalous, which demonstrate essentially different scalings. The main results include a proof of that universality for any trap (Sec. IV), the explicit analytical solutions for the statistics in both universality classes (Sec. VI), and the analytical expressions for the main thermodynamic and statistical quantities in the whole critical region of BEC for various traps (Sec. VII). Also, we find a canonical solution that describes the mesoscopic effects beyond the universality for any trap via the confluent hypergeometric distribution (Sec. VIII). We specify and plot these analytical solutions for various particular traps, listed in Sec. V. At the same time, we stress that the universal behavior of the BEC statistics and thermodynamics in the critical region described in this paper is valid for arbitrary traps. It does not rely on the trapping potential's separability or power-law asymptotics of the trap's spectrum.

In the Appendices we adopt a powerful Mellin-transform technique, developed in [26–29], to the calculation of the characteristic function, needed for the present analysis of BEC universality. We start with a partition function, associated, in the present case, with a one-particle energy spectrum of the trap. We then introduce a regularized trap function [see Eqs. (B3) and (A5)], which is found by means of the Mellin transformation from the trap partition function, Eq. (C3), and yields a remarkable exact solution in Eq. (51) for the logarithm of the characteristic function of the universal distribution of the total unconstrained noncondensate occupation. The latter distribution determines the major statistical and thermodynamic quantities of the mesoscopic system via its simple cutoff in Eq. (7). It is calculated as the Fourier transform of the characteristic function. Thus, employing the canonical Fourier and Mellin transforms, we calculate analytically the critical BEC phenomena.

III. EXACT SOLUTION FOR A MESOSCOPIC SYSTEM

We consider N noninteracting Bose atoms trapped in a potential $U(\mathbf{r})$ in an equilibrium state with a temperature T. The single-particle eigenstates ψ_q satisfy a Schrödinger equation with the discrete energies ϵ_q,

$$\left[-\frac{\hbar^2}{2m}\Delta + U(\mathbf{r})\right]\psi_q(\mathbf{r}) = \varepsilon_q\psi_q(\mathbf{r}), \quad \epsilon_q = \varepsilon_q - \varepsilon_0, \quad (1)$$

counted from a nondegenerate ground state $q = 0$. An integer q orders all eigenstates in increasing energies $0 < \epsilon_1 \leqslant \epsilon_2 \leqslant \cdots$. A dimensionless spectrum λ_q,

$$\frac{\epsilon_q}{T} = \alpha\lambda_q, \quad \alpha = \frac{\epsilon_1}{T}, \quad \lambda_0 = 0, \quad q = 0,1,2,\ldots, \quad (2)$$

is defined by the parameter α, which is determined by the energy ϵ_1 of the first excited level. The Hamiltonian of the system

$$\hat{H}_0 = \sum_q \varepsilon_q \hat{n}_q, \quad \sum_q \hat{n}_q = N, \quad (3)$$

is accompanied by the canonical-ensemble particle-number constraint. Thus, the occupation operators $\hat{n}_q$ of the different eigenstates are not independent.

The particle-number constraint in Eq. (3) is fully responsible for the BEC phenomenon via the so-called constraint-cutoff mechanism. Namely, it strongly constrains the many-body Hilbert space by making the Fock ground-state occupation, $n_0 = N - \sum_{q\neq 0} n_q$, determined by the occupations of the excited Fock states. Those occupations are governed by the following Hamiltonian $\hat{H}$ on the restricted Fock space and are themselves constrained by a cutoff boundary:

$$\hat{H} = \sum_{q\neq 0}\epsilon_q\hat{n}_q, \quad \sum_{q\neq 0} n_q \leqslant N. \quad (4)$$

The latter can be expressed as a step-function $\theta(N-\hat{n})$ cutoff via the operator of the total occupation of the excited states $\hat{n} = \sum_{q\neq 0}\hat{n}_q$, that is, the noncondensate occupation. Thus, an equilibrium density matrix of that system is given by the Gibbs distribution

$$\hat{\rho} = \frac{e^{-\hat{H}/T}}{Z}\theta(N-\hat{n}), \quad Z = \mathrm{Tr}[e^{-\hat{H}/T}\theta(N-\hat{n})], \quad (5)$$

where a step function $\theta(x)$ is 1 if $x \geqslant 0$ and 0 if $x < 0$. This theory was first proposed in [17] and yields the exact and universal solution to the BEC problem for the ideal gas. We summarize it briefly in this section.

The BEC statistics and thermodynamics can be calculated via the probability distribution $\rho_n = \mathrm{Tr}[\hat{n}\hat{\rho}]$ of the total noncondensate occupation n. It can be represented via its Fourier transform, i.e., the characteristic function $\Theta(u) = \mathrm{Tr}[e^{iu\hat{n}}\hat{\rho}\theta(N-\hat{n})]$,

$$\Theta(u) = \sum_{n=0}^{N} e^{iun}\rho_n, \quad \rho_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-iun}\Theta(u)du. \quad (6)$$

However, that general relation does not establish an explicit solution for ρ_n because a straightforward calculation of a finite sum in Eq. (6) is not practical.

The exact solution for the noncondensate distribution

$$\rho_n = \rho_n^{(\infty)}\theta(N-n)\Big/\sum_{n=0}^{N}\rho_n^{(\infty)}, \quad (7)$$

nevertheless, is found explicitly [17] as merely a $\theta(N-\hat{n})$ cutoff of the auxiliary unconstrained distribution,

$$\rho_n^{(\infty)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-iun}\Theta^{(\infty)}(u)du, \quad (8)$$

$$\Theta^{(\infty)}(u) = \sum_{n=0}^{\infty} e^{iun}\rho_n = \prod_{q\neq 0}\frac{e^{\epsilon_q/T}-1}{e^{\epsilon_q/T}-e^{iu}}. \quad (9)$$

That distribution $\rho_n^{(\infty)}$ is exactly the one, given by a standard grand-canonical-ensemble approximation for the zero value of the chemical potential. A superscript "(∞)" reminds the reader that in the grand-canonical ensemble the noncondensate occupation is allowed to fluctuate within a semi-infinite interval, $n \in [0,\infty)$.

All properties of the distribution in Eq. (6) can be calculated via the cumulants κ_m and the generating cumulants $\tilde{\kappa}_m$, which are defined by the Taylor series of the logarithm of the characteristic function,

$$\ln \Theta(u) = \sum_{m=1}^{\infty} \kappa_m \frac{(iu)^m}{m!} = \sum_{m=1}^{\infty} \tilde{\kappa}_m \frac{(e^{iu}-1)^m}{m!}. \qquad (10)$$

The cumulants can be found from the generating cumulants via the Stirling numbers of the first kind [53],

$$\kappa_r = \sum_{m=1}^{r} \sigma_r^{(m)} \tilde{\kappa}_m, \quad \sigma_r^{(m)} = \sum_{k=0}^{m} \frac{(-1)^{m-k}}{m!} C_m^k k^r, \qquad (11)$$

in particular, $\kappa_1 = \tilde{\kappa}_1, \kappa_2 = \tilde{\kappa}_2 + \tilde{\kappa}_1, \kappa_3 = \tilde{\kappa}_3 + 3\tilde{\kappa}_2 + \tilde{\kappa}_1$. Similar formulas apply to the unconstrained distribution in Eq. (8), for which the cumulants $\kappa_m^{(\infty)}$ and generating cumulants $\tilde{\kappa}_m^{(\infty)}$ are known exactly [54],

$$\tilde{\kappa}_m^{(\infty)} = (m-1)! \sum_{q\neq 0} (e^{\alpha\lambda_q} - 1)^{-m}. \qquad (12)$$

The square of the standard deviation $\sigma^{(\infty)}$ is equal to

$$\kappa_2^{(\infty)} \equiv \sigma^{(\infty)2} = \sum_{q\neq 0} \left[\frac{1}{(e^{\alpha\lambda_q}-1)^2} + \frac{1}{e^{\alpha\lambda_q}-1} \right]. \qquad (13)$$

The unconstrained distribution was first found via its Fourier transform (a characteristic function) and its generating cumulants in [54]. It was also analyzed for the box and harmonic traps in [50,51]. In the early works [6,19,47,55–59] only the first two moments (the mean value and dispersion) were analyzed, though the third and fourth moments were discussed in [26,29]. The actual cutoff noncondensate distribution in Eqs. (7) and (15) and its explicit relation to the statistics and thermodynamics of an ideal gas was first found and analyzed in [17,18]. Even in the later works, including [51], only the auxiliary unconstrained distribution was discussed.

The most convenient way to analyze the BEC phase transition is to study a system with a given one-particle energy spectrum ϵ_q at a fixed temperature T in dependence on the total number of the trapped particles N. That means a fixed unconstrained distribution $\rho_n^{(\infty)}$. The BEC takes place when N exceeds the critical number N_c; that is, $T < T_c$. A case $N \approx N_c$ corresponds to the critical region $T \approx T_c$. The critical temperature T_c is defined by an equality

$$N = N_c(T_c), \quad N_c(T) = \sum_{q\neq 0} \frac{1}{e^{\epsilon_q/T} - 1}. \qquad (14)$$

The thermodynamics of the mesoscopic system for any phase (noncondensed, condensed, or within the critical region) is determined by the partition function $Z = Z^{(\infty)} P^{(\infty)}(N)$. The partition function $Z^{(\infty)}$ for an auxiliary unconstrained ensemble of the excited atoms, usually used in the grand-canonical-ensemble approximation, and a cumulative distribution $P^{(\infty)}(N)$ of the total number of the excited atoms n for the unconstrained statistics, taken at $n = N$, are

$$Z^{(\infty)} = \prod_{q\neq 0} \frac{1}{1 - e^{-\alpha\lambda_q}}, \quad P^{(\infty)}(N) = \sum_{n=0}^{N} \rho_n^{(\infty)}. \qquad (15)$$

Thus, one finds the Gibbs free energy F, average energy E, entropy S, and heat capacity C_V as follows:

$$F = -T \ln Z, \quad S = -\frac{\partial F}{\partial T},$$
$$E = \frac{\partial \ln Z}{\partial (1/T)} = F + TS, \quad C_V = \left(\frac{\partial E}{\partial T} \right)_V. \qquad (16)$$

IV. UNIVERSALITY FOR ARBITRARY TRAP

Let us consider the thermodynamic limit when both the volume of a trap V and the critical number of the trapped atoms in Eq. (14) are increasing. Hence, the dimensionless energy of the first energy level in Eq. (2) tends to zero, $\frac{\epsilon_1}{T} = \alpha \to 0$.

Let us use, instead of n, a new stochastic variable,

$$x = (n - N_c)/\sigma^{(\infty)}, \quad \sigma^{(\infty)} = \sqrt{\kappa_2^{(\infty)}}, \qquad (17)$$

which has the zero mean value and unit dispersion. As shown in [17], its unconstrained probability distribution $\rho_x^{(\infty)}$ quickly converges to a universal function that does not depend on any dimensional physical parameters of the trap and gas (like a volume, temperature, or atomic mass). It depends only on the geometry of the trap, encoded in the dimensionless spectrum $\{\lambda_q\}$ in Eq. (2). That convergence takes place in a wide region of universality, which includes not only the central part of the critical region, $|x| \leqslant 4$, but extends far beyond it.

Here we prove that universality for arbitrary trap via the cumulant analysis and Fourier transformation. Indeed, the cumulants $\kappa_m^{(x)(\infty)}$ and $\kappa_m^{(\infty)}$ for the variables x and n, respectively, are related:

$$\kappa_1^{(x)(\infty)} = 0, \quad \kappa_2^{(x)(\infty)} = 1, \quad \kappa_m^{(x)(\infty)} = \frac{\kappa_m^{(\infty)}}{\kappa_2^{(\infty)m/2}}. \qquad (18)$$

According to Appendix A, the cumulants $\kappa_m^{(\infty)}$ can be represented as the following sum of the residues [28]:

$$\kappa_m^{(\infty)} = \sum_j \operatorname*{Res}_{t=t_j} [\alpha^{-t} \zeta(t+1-m) S(t)]. \qquad (19)$$

Here $\zeta(t+1-m)$ is a Riemann ζ function and the trap function $S(t)$ is defined as

$$S(t) = \Gamma(t) s(t), \quad s(t) \equiv s_t = \sum_{q\neq 0} \frac{1}{(\lambda_q)^t}, \qquad (20)$$

where $\Gamma(t)$ is a Γ function. In the thermodynamic limit, $\alpha \to 0$, each cumulant tends to the residue at the rightmost pole. The latter is either a pole of the ζ function at $t = m$ or a pole of the trap function $S(t)$. If those poles are very close to each other, both of them should be taken into account. Using the rightmost pole of the trap function $t_1 \equiv r$ and its residue $R \equiv \operatorname*{Res}_{t=r} S(t)$, we find the limit

$$\kappa_m^{(\infty)} \to R\, \alpha^{-r} \quad \text{for } m < r, \qquad (21)$$

$$\kappa_m^{(\infty)} \to (m-1)! s_m\, \alpha^{-m} \quad \text{for } m > r, \qquad (22)$$

$$\kappa_m^{(\infty)} \to R\, \alpha^{-r} \ln \alpha^{-1} \quad \text{for } m = r. \qquad (23)$$

[The meaning of the rightmost pole of the trap function can be understood via its relation $r = 3/\nu$ to the index ν of the 3D trap's energy spectrum, Eq. (27).]

Thus, in the thermodynamical limit, the cumulants $\kappa_m^{(x)(\infty)}$ in Eq. (18) are independent on α for any trap (with arbitrary r). Namely, for any $m \geqslant 2$, they tend to some universal constants, which are not zero in the case $r < 2$ and zero (except $\kappa_2^{(x)(\infty)}$) in the case $r \geqslant 2$:

$$\begin{aligned} &\kappa_m^{(x)(\infty)} \to K_m^{(\infty)} = (m-1)! s_m s_2^{-\frac{m}{2}} \quad \text{for } r < 2, \\ &\kappa_m^{(x)(\infty)} \to K_m^{(\infty)} = 0, \quad K_2^{(\infty)} = 1 \quad \text{for } r \geqslant 2. \end{aligned} \tag{24}$$

A limiting distribution of the scaled noncondensate occupation and its characteristic function $\Theta^{(x)} = e^{\phi}$,

$$\rho_x = \int_{-\infty}^{\infty} e^{\phi(u) - iux} \frac{du}{2\pi}, \quad \phi(u) = \sum_{m=1}^{\infty} \frac{K_m^{(\infty)}}{m!} (iu)^m, \tag{25}$$

appear to be some pure mathematical functions, free of any physical parameters and determined only by the dimensionless spectrum λ_q. Those special functions depend on the universality class and are explicitly found in terms of the Gaussian distribution and spectral ζ functions in Sec. VI and Appendix B.

This result means that we calculated the universal effective Hamiltonian, i.e., the Landau function $H^{(\text{univ})}(x) = -\ln \rho_x$ (see Fig. 1). It determines the equilibrium distribution of the fluctuations of the order parameter, or the equilibrium density matrix in Eq. (5). The effective energy of the state with n atoms in the noncondensate, $H^{(\infty)}(n) = -\ln \rho_n^{(\infty)}$, is formally measured relative to a constant, determined by the Gibbs free energy $F^{(\infty)} = -T \ln Z^{(\infty)}$ for the unconstrained system of the excited atoms via its partition function, Eq. (15). The most probable state of the system corresponds to the minimum of the cutoff effective Hamiltonian. The latter is a universal Landau function $H(x) = -\ln[\rho_x \Theta(x-\eta)/P_\eta]$ cut off by a scaled total number of atoms:

$$\eta = \frac{N - N_c}{\sigma^{(\infty)}}; \quad \rho_x^{(\text{univ})} = \frac{\rho_x \theta(\eta - x)}{P_\eta}, \quad P_\eta = \int_{-\infty}^{\eta} \rho_x dx. \tag{26}$$

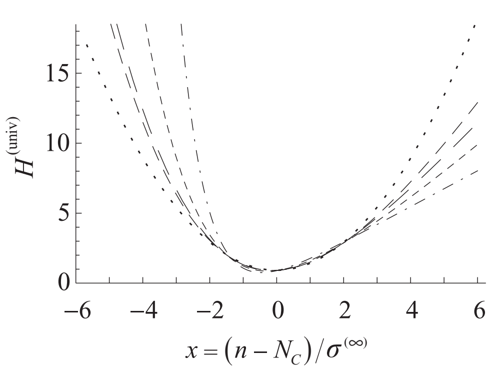

FIG. 1. The universal unconstrained Landau function $H^{(\text{univ})} = -\ln \rho_x$ as a function of the scaled noncondensate occupation x, Eq. (17), for the Gaussian universality class (e.g., the harmonic or linear traps, dotted line) and anomalous universality class: the power-law box (short-dashed line), the boxes with the periodic (medium-dashed line) and zero Dirichlet (long-dashed line) boundary conditions, and the one-dimensional harmonic trap (dot-dashed line).

An increase in the number of atoms N, loaded in a trap, above the critical value N_c or a decrease in the temperature below the critical value T_c each move the cutoff border behind the minimum of the Landau function. In accord with the Ginzburg-Landau free-energy theory of the phase transitions [1], it results in a sharp transition to the BEC phase with a predominant occupation of the ground level due to fixation of the noncondensate occupation at the value N_c, which corresponds to a minimum of the system's energy. The universal cumulative distribution P_η in Eq. (26) and its first two derivatives determine a universal behavior and all subtleties of the BEC critical phenomena.

V. EXAMPLES OF THE TRAPS: ENERGY SPECTRUM VERSUS FORM OF POTENTIAL

This section contains basic information on various traps, which we use in the subsequent sections to specify the general analytical solutions. We demonstrate with these examples both universal and nonuniversal mesoscopic effects in the critical region of BEC. Note that the found universality of the BEC statistics and thermodynamics does not rely on the separability of the traps' potential, but constitutes a robust general property of the critical phenomena in the BEC.

A. Power-law traps: Boundary problem and semiclassical asymptotics

The traps with a power-law energy spectrum [19]

$$\epsilon_{\mathbf{q}} = \epsilon_1 \left(q_x^{\nu} + q_y^{\nu} + q_z^{\nu}\right), \quad \mathbf{q} = \{q_x, q_y, q_z\}, \quad q_i = 0, 1, 2, \ldots, \tag{27}$$

form a model similar to a model with a trap potential

$$U(\mathbf{r}) = u(x) + u(y) + u(z), \quad u(x) = U_0 |x/L|^p. \tag{28}$$

The cases of the harmonic ($p = 2$), linear ($p = 1$), and box ($p = \infty$) traps are exactly solvable. In these and other examples of the three-dimensional (3D) traps we enumerate the eigenstates by the 3D quantum numbers $\mathbf{q}$. Here it is more practical than the use of the 1D notations in Eq. (1). For any specific trap potential in an experiment, one can calculate the trap's spectrum numerically by a standard technique of diagonalizing the tridiagonal matrix, obtained by differencing the one-particle Hamiltonian [60]. Then one can use it instead of Eq. (27) in a completely similar way. For a power-law spectrum, the knowledge of the potential profile is not needed for the analysis of the ideal-gas BEC. Moreover, it cannot be obtained explicitly since the corresponding inverse Sturm-Liouville (scattering) problem does not have a general explicit solution.

The spectral index ν for the high-energy asymptotics and the power p of the potential in Eq. (28) are related: $\nu = 2p/(p +$

2) (see, for example, [61–63]). The latter can be seen from a well-known semiclassical approximation for the high energies in a 1D trap,

$$\oint \sqrt{2m[\varepsilon_q - u(x)]}dx = 2\pi\hbar(q + \beta_0), \tag{29}$$

where q is an integer. The integral is taken over a round-trip path in a classically permitted region and the parameter β_0 depends on the boundary conditions. In particular, for a smooth potential profile and for the box with the zero boundary conditions one has $\beta_0 = 1/2$ and $\beta_0 = 0$, respectively. Indeed, Eq. (29) yields the following energy spectrum asymptotics:

$$\varepsilon_q = \left[\frac{\sqrt{\pi}\hbar U_0^{\frac{1}{p}}\Gamma\left(\frac{3}{2}+\frac{1}{p}\right)}{\sqrt{2m}L\Gamma\left(1+\frac{1}{p}\right)}(q+\beta_0)\right]^{\frac{2p}{p+2}}. \tag{30}$$

Obviously, a given parameter $\hbar U_0^{\frac{1}{p}}/(\sqrt{2m}L)$ defines a family of the isospectral potentials which have, in dimensionless units, the same eigenvalues and the same eigenfunctions. The spatial profiles of the potentials and the corresponding one-particle spectra for the typical power-law traps are shown in Fig. 2.

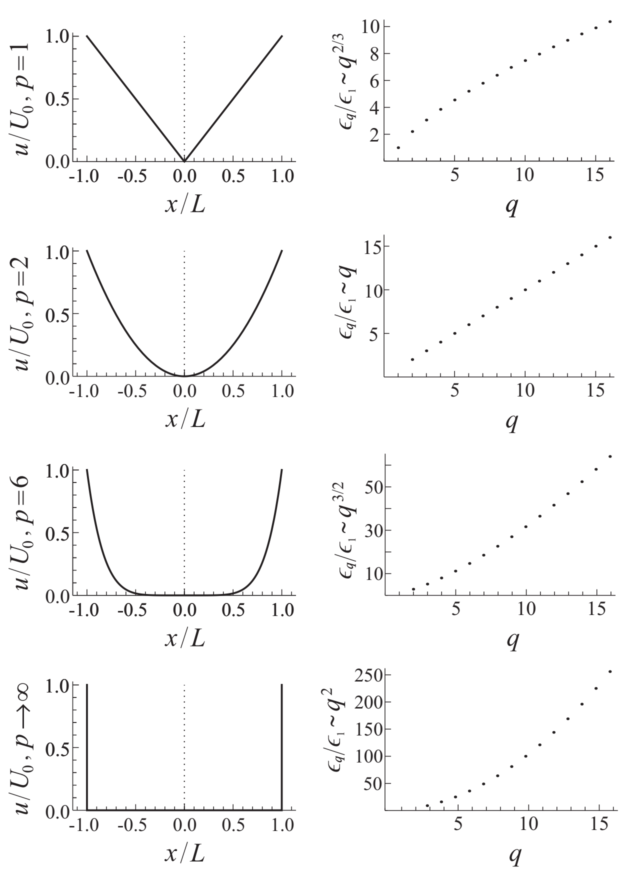

FIG. 2. The trapping potential $u(x) = U_0|x/L|^p$, Eq. (28), and 1D contribution to the energy spectrum $\varepsilon_q = \epsilon_1 q^\nu$ in Eq. (27), $\nu = 2p/(p+2)$, for the power-law linear ($p = 1, \nu = 2/3$), harmonic ($p = 2, \nu = 1$), marginal ($p = 6, \nu = 3/2$), and box ($p = \infty, \nu = 2$) traps.

In Sec. V, we present the spectra and cumulants for the box ($p = \infty$), harmonic ($p = 2$), and linear ($p = 1$) trapping potentials, given by Eq. (28), as well as for their power-law-spectrum counterparts, given by Eq. (27) with the related spectral index $\nu = 2p/(p+2)$. The true and power-law traps differ mainly only by the lower-energy, infrared part of the spectra, determined by the trap's walls and boundary conditions. Their comparison allows us to demonstrate in Secs. VI and VII when and how strong the boundary effects change the BEC statistics and thermodynamics.

In the thermodynamic limit, $\alpha \to 0$, all cumulants $\kappa_m^{(\infty)}$ for any power-law trap in Eq. (27) are given by the representation in Eq. (19). Here the universal numbers $s_t \equiv s(t)$, characterizing the trap function of a trap with a spectral index ν, the rightmost pole r, and the corresponding residue R are, respectively,

$$s_t = \sum_{q_x,q_y,q_z=0}^{\infty}{}' \frac{1}{\left(q_x^\nu + q_y^\nu + q_z^\nu\right)^t}, \quad r = \frac{3}{\nu},$$
$$R = \Gamma^3\left[1 + \frac{1}{\nu}\right]. \tag{31}$$

The prime indicates that the singular term $q_x + q_y + q_z = 0$ is excluded from the sum. The values r and R follow from Appendix C. So, Eqs. (21)–(23) yield

$$\begin{aligned}
\kappa_m^{(\infty)} &\simeq \Gamma^3\left(1+\frac{1}{\nu}\right)\zeta\left(\frac{3}{\nu}+1-m\right)\alpha^{-\frac{3}{\nu}} \quad \text{for } m < \frac{3}{\nu},\\
\kappa_m^{(\infty)} &\simeq \Gamma^3\left(1+\frac{m}{3}\right)\alpha^{-m}\ln(\alpha^{-1}) \quad \text{for } m = 3/\nu,\\
\kappa_m^{(\infty)} &\simeq (m-1)!s_m\alpha^{-m} \quad \text{for } m > 3/\nu,
\end{aligned} \tag{32}$$

where ζ is a Riemann ζ function. The universal numbers $s_m(\nu)$ are shown in Fig. 3. For all higher-order cumulants they are determined mainly by the contribution from the first energy level (which has a degeneracy $g_1 = 3$): $s_m(\nu) \to g_1 = 3$, when $2^{m\nu} \gg 1$.

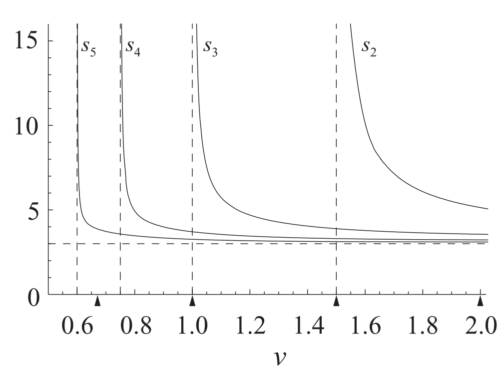

FIG. 3. The exact values of the universal numbers $s_m(\nu)$, Eq. (31), related to the main cumulants $\kappa_m^{(\infty)}$ of the orders $m = 2,3,4,5$ via Eq. (32), for the power-law traps as the functions of the spectral index ν in Eq. (27). The arrows mark the box ($\nu = 2$), marginal ($\nu = 3/2$), isotropic harmonic ($\nu = 1$), and linear ($\nu = 2/3$) traps.

B. Box traps with the periodic and Dirichlet boundary conditions and power-law box trap

The box traps with the periodic and zero Dirichlet boundary conditions and the power-law box trap have the following one-particle energy spectra, respectively:

$$\epsilon_{\mathbf{q}}^{(p)} = 4\epsilon^{(b)}\lambda_{\mathbf{q}}^{(p)}, \quad \lambda_{\mathbf{q}}^{(p)} = \sum_{i=x,y,z} q_i^2, \quad q_i = 0,\pm1,\pm2,\ldots, \tag{33}$$

$$\epsilon_{\mathbf{q}}^{(z)} = 3\epsilon^{(b)}\lambda_{\mathbf{q}}^{(z)}, \quad \lambda_{\mathbf{q}}^{(z)} = \sum_{i=x,y,z} \frac{q_i^2}{3} - 1, \quad q_i = 1,2,\ldots, \tag{34}$$

$$\epsilon_{\mathbf{q}}^{(\nu=2)} = \epsilon^{(b)}\lambda_{\mathbf{q}}^{(\nu=2)}, \quad \lambda_{\mathbf{q}}^{(\nu=2)} = \sum_{i=x,y,z} q_i^2, \quad q_i = 0,1,\ldots. \tag{35}$$

Here the energy scale common for all three box traps, $\epsilon^{(b)} = (\pi\hbar)^2/(2mL^2)$, is determined by the size of the box $2L$. The first excited energy ϵ_1 and the parameter $\alpha = \epsilon_1/T$ in Eq. (2) are different for different boxes.

In order to find the cumulants via Eq. (19) and asymptotics of the BEC statistics (Secs. IV and VI) for any trap we use a known technique [28] (Appendix C) that allows one to fully analyze the trap function in Eq. (20) via the structure of its poles. The box with the periodic boundary conditions has only two poles ($t_1 = 3/2, t_2 = 0$) with the residues $c_1 = \pi^{3/2}, c_2 = -1$. The power-law box has four poles ($t_1 = 3/2, t_2 = 1, t_3 = 0.5, t_4 = 0$) with the residues $c_1 = \pi^{3/2}/8, c_2 = 3\pi/8, c_3 = 3\pi^{1/2}/8, c_4 = -7/8$. The box with the zero Dirichlet boundary conditions has the same four poles with different residues, $c_1 = \frac{(3\pi)^{\frac{3}{2}}}{8}, c_2 = -\frac{9\pi}{8}, c_3 = \frac{3^{\frac{3}{2}}\pi^{\frac{1}{2}}(1+\pi)}{8}, c_4 = -\frac{9(1+\pi)}{8}$, as well as an infinite series of poles $t_j = (4-j)/2, j = 5,6,\ldots$.

In the bulk limit all three boxes have a similar leading term in each cumulant in Eqs. (10)–(13) [28]:

$$\kappa_1^{(\infty)} \simeq N_c^{(b)} \equiv \zeta\left(\frac{3}{2}\right)\left(\frac{\pi}{4\alpha_b}\right)^{\frac{3}{2}}, \quad \alpha_b = \frac{\epsilon^{(b)}}{T}, \tag{36}$$

$$\kappa_m^{(\infty)} \simeq (m-1)!s_m\alpha^{-m}, \quad m \geqslant 2. \tag{37}$$

However, the parameters α (note that α_b is not α) and numbers s_m in Eq. (37) are different for different boxes, in particular, $s_2^{(p)} = 16.53, s_2^{(z)} = 8.01, s_2^{(\nu=2)} = 5.14$. Thus, a change in the boundary conditions affects the cumulants and BEC statistics in Eq. (25). However, it does not change the leading term $N_c^{(b)}$ in the critical number of atoms in Eq. (36); hence the BEC critical temperature, altering only the next-to-leading terms in the mean occupation:

$$N_c^{(p)} = N_c^{(b)} + \frac{s^{(p)}(1)}{4\alpha_b}, \; N_c^{(z)} = N_c^{(b)} + \frac{3\pi\,\ln\alpha_b}{8\alpha_b} + \frac{\delta^{(z)}}{\alpha_b}, \tag{38}$$

$$N_c^{(\nu=2)} = N_c^{(b)} + \frac{3\pi\,\ln\alpha_b^{-1}}{8\alpha_b} + \frac{\delta^{(\nu=2)}}{\alpha_b}. \tag{39}$$

Here $s^{(p)}(1) \simeq -8.91, \delta^{(z)} \simeq 0.76, \delta^{(\nu=2)} \simeq 1.09$, and all terms on order of α^0 and smaller are omitted.

C. Isotropic harmonic trap

For a true isotropic harmonic trap [i.e., a parabolic potential in Eq. (28), $p = 2$], the result in Eq. (30) with $\beta_0 = 1/2$ is an exact one. It coincides with the spectrum of a power-law harmonic ($\nu = 1$) trap:

$$\epsilon_{\mathbf{q}}^{(h)} = \epsilon^{(h)}\lambda_{\mathbf{q}}^{(h)}, \quad \lambda_{\mathbf{q}}^{(h)} = \sum_{i=x,y,z} q_i, q_i = 0,1,2,\ldots. \tag{40}$$

It is an equidistant ladder with the energy step $\epsilon^{(h)} = \frac{\hbar}{L}(\frac{2U_0}{m})^{1/2}$, parameter $\alpha = \alpha_h = \epsilon^{(h)}/T$ in Eq. (2), and energy degeneracy $g_{\mathbf{q}}^{(h)} = (\lambda_{\mathbf{q}}^{(h)} + 1)(\lambda_{\mathbf{q}}^{(h)} + 2)/2$.

For this trap the cumulants are known (see [26,28]).

D. Anisotropic harmonic trap

An anisotropic harmonic trap has different energy steps $\epsilon_x \leqslant \epsilon_y \leqslant \epsilon_z$ along the x, y, z axes. Its spectrum

$$\epsilon_{\mathbf{q}}^{(ah)} = \epsilon_x\lambda_{\mathbf{q}}^{(ah)}, \quad \lambda_{\mathbf{q}}^{(ah)} = Aq_x + Bq_y + Cq_z, \tag{41}$$

is similar to the one in Eq. (40). Here we keep the $A = \epsilon_x/\epsilon_x \equiv 1$ along with the $B = \epsilon_y/\epsilon_x$ and $C = \epsilon_z/\epsilon_x$ to make the subsequent Eq. (44) symmetric.

The universal numbers in Eqs. (31) and (20) are

$$s_m^{(ah)} = \sum_{q_x,q_y,q_z=0}^{\infty}{}' \frac{1}{(Aq_x + Bq_y + Cq_z)^m}. \tag{42}$$

The thermodynamic-limit asymptotics of the cumulants for the 3D anisotropic harmonic trap [26,28],

$$\begin{aligned}
\kappa_1^{(\infty)} &\simeq \frac{\zeta(3)c_3}{\alpha^3} + \frac{\pi^2 c_2}{6\alpha^2} + \frac{c_1(\gamma - \ln\alpha) + \delta_1}{\alpha} - \frac{c_0}{2} + \cdots,\\
\kappa_2^{(\infty)} &\simeq \frac{\zeta(2)c_3}{\alpha^3} + \frac{c_2(\gamma - \ln\alpha) + \delta_2}{\alpha^2} - \frac{c_1}{2\alpha} - \frac{c_0}{12} + \cdots,\\
\kappa_3^{(\infty)} &\simeq \frac{c_3(\gamma - \ln\alpha) + \delta_3}{\alpha^3} - \frac{c_2}{2\alpha^2} - \frac{c_1}{12\alpha} + \cdots,\\
\kappa_m^{(\infty)} &\simeq \frac{\Gamma(m)s_m^{(ah)}}{\alpha^m} + \frac{c_3\zeta(4-m)}{\alpha^3} + \frac{c_2\zeta(3-m)}{\alpha^2}\\
&\quad + \frac{c_1\zeta(2-m)}{\alpha} + \cdots, \quad m \geqslant 4,
\end{aligned} \tag{43}$$

includes the parameters c_i and δ_i of the trap function $s_t^{(ah)}\Gamma(t) \approx \frac{c_i}{t-i} + \delta_i + O(t-i)$ near a pole $t = i$. According to Appendices A–C, the main residues are

$$\begin{aligned}
c_3 &= \frac{1}{ABC}, \quad c_2 = \frac{A+B+C}{2ABC},\\
c_1 &= \frac{A^2 + B^2 + C^2 + 3AB + 3AC + 3BC}{12ABC},\\
c_0 &= \frac{A^2B + AB^2 + A^2C + B^2C + AC^2 + BC^2}{24ABC} - \frac{7}{8}.
\end{aligned} \tag{44}$$

The $\delta_i \sim \alpha^0$ can be calculated via a Mellin transform.

An extremely anisotropic 1D trap has $s_m^{(ah)} = \zeta(m)$,

$$\kappa_1^{(\infty)} \simeq \frac{\gamma - \ln\alpha}{\alpha} + \frac{1}{4} + \frac{\alpha}{144} + \cdots,$$
$$\kappa_m^{(\infty)} \simeq \frac{\Gamma(m)\zeta(m)}{\alpha^m} + \frac{\zeta(2-m)}{\alpha} + \sum_{n=0}^{\infty}\frac{\zeta(1-n-m)\zeta(-n)}{n!}\alpha^n,$$
$$m \geqslant 2. \quad (45)$$

E. Traps with a linear potential

A trap with a linear potential [$p = 1$ in Eq. (28)], called also a triangular trap [63], has a well-known exact solution of its Schrödinger equation (1) in terms of the Airy functions [63] with the energy spectrum

$$\epsilon_{\mathbf{q}}^{(t)} = \epsilon^{(t)}\lambda_{\mathbf{q}}^{(t)}, \quad \lambda_{\mathbf{q}}^{(t)} = \frac{3z_0 - \sum_{i=x,y,z} z_{q_i}}{z_0 - z_1}, \quad q_i = 0,1,\ldots. \quad (46)$$

It is set by a gap $\epsilon^{(t)} = (z_0 - z_1)[(\hbar U_0/L)^2/(2m)]^{\frac{1}{3}}$ via a sequence of the zeros $z_{q_i} < 0$ of the Airy function and its first derivative: $z_0 = -1.0188, z_1 = -2.3381, z_2 = -3.2482, \ldots$. Using asymptotics of the Airy function [53], it can be approximated as

$$\lambda_{\mathbf{q}}^{(t)} \approx \left(\frac{3\pi}{4}\right)^{\frac{2}{3}} \sum_{i=x,y,z} \frac{\Lambda(q_i)}{z_0 - z_1}, \quad q_i = 0,1,2,\ldots, \quad (47)$$

where $\Lambda(0) = 0$, $\Lambda(q_i > 0) = (q_i + \frac{1}{2})^{\frac{2}{3}} + z_0(\frac{4}{3\pi})^{\frac{2}{3}}$.

Only the renormalization and shifts tell that spectrum from the spectrum of the power-law linear trap with the index $\nu = 2p/(p+2) = 2/3$ in Eq. (27),

$$\epsilon_{\mathbf{q}}^{(2/3)} = \epsilon^{(2/3)}\lambda_{\mathbf{q}}^{(2/3)}, \quad \epsilon^{(2/3)} = \left(\frac{3\pi\hbar U_0}{4L\sqrt{2m}}\right)^{2/3},$$
$$\lambda_{\mathbf{q}}^{(2/3)} = \sum_{i=x,y,z} q_i^{2/3}, \quad q_i = 0,1,2,\ldots. \quad (48)$$

In the thermodynamic limit, the true and power-law linear traps have the similar asymptotics for each cumulant in Eqs. (10)–(13) (Appendix C) in terms of the parameter $\alpha = \epsilon^{(2/3)}/T$ (same for the both traps):

$$\kappa_m^{(\infty)} \simeq \left(\frac{3\sqrt{\pi}}{4}\right)^3 \frac{\zeta\left(\frac{11}{2} - m\right)}{\alpha^{9/2}} + \frac{s_4\delta_{m,4}}{\alpha^4}$$
$$- \frac{3^{10/3}\zeta\left(\frac{11}{2} - m\right)\pi^{5/6} z_0}{4^{7/3}\alpha^{7/2}} + \cdots \quad \text{for } m \leqslant 4,$$
$$\kappa_m^{(\infty)} \simeq \frac{\Gamma(m)s_m}{\alpha^m} + \cdots \quad \text{for } m \geqslant 5. \quad (49)$$

They differ only by the numbers s_m and the term with z_0, which should be set to zero for the power-law linear trap. The $\delta_{m,4}$ is the Kronecker's δ.

VI. THE TWO UNIVERSALITY CLASSES

Let us find all universality classes of the BEC statistics using the general solution in Eqs. (24)–(26).

A. The Gaussian universality class

For all traps, which have the rightmost pole of the trap function larger than two, $r > 2$ [see Eqs. (19)–(23)], all cumulants of order $m \geqslant 3$ in Eq. (24) vanish in the thermodynamic limit as $K_m^{(\infty)} \sim \alpha^{m(\frac{r}{2}-1)} \to 0$, or $\sim\alpha^{m(\frac{3}{2\nu}-1)} \to 0$ due to Eq. (32). These are the traps, the spectrum of which can be majorated by the power-law spectrum with the index $\nu = 3/2$, and, in particular, all 3D traps with a small spectrum index $\nu < 3/2$ in Eq. (27). Thus, a remarkable result is that all such traps have exactly the same Gaussian universal unconstrained statistics in the critical region,

$$\rho_x = \rho_x^{(\text{Gauss})(\infty)} = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}. \quad (50)$$

In this case the dispersion of the fluctuations in Eq. (32) is normal and scales as $\sigma^{(\infty)} \sim \sqrt{N_c}$, that is, similar to any standard thermodynamic fluctuations. This universality class includes the isotropic harmonic and linear traps and all anisotropic harmonic traps (Secs. V C–V E), for which the main contribution to the cumulant $\kappa_2^{(\infty)}$ in Eq. (43) comes from the residue c_3 at the rightmost pole $r = 3$ of the trap function.

B. The anomalous universality class

In the opposite case, when the rightmost pole of the trap function is less than two, $r < 2$, all cumulants of order $m \geqslant 3$ tend to the nonzero constants, Eq. (24). Hence, all such traps have essentially non-Gaussian universal statistics described by some special function in Eq. (25). The latter depends on and keeps the memory of the boundaries and form of trap despite the thermodynamic limit. In this case, according to Eqs. (21) and (22), the BEC fluctuations $\sigma^{(\infty)} \sim N_c^{1/r}$ are always anomalously large, compared to the standard thermodynamic fluctuations $\sim N_c^{1/2}$.

The anomalous universality class includes, in particular, all power-law traps with a large spectral index $\nu > 3/2$ in Eq. (27). These 3D traps have $r = \frac{3}{\nu} < 2$ that implies the anomalous fluctuations $\sigma^{(\infty)} \sim N_c^{\nu/3}$.

All strongly anisotropic harmonic traps (Sec. V D), for which the main contribution to the cumulant $\kappa_2^{(\infty)}$ in Eq. (43) comes from δ_2/α^2, while contributions from the residues $c_{2,3}$ at the two rightmost poles $t = 2,3$ are strongly suppressed by the anisotropy $B, C \gg 1$, also belong to the anomalous universality class.

We find an exact explicit formula for the logarithm of the universal characteristic function in Eq. (25) [see Appendix B, Eq. (B7)] for arbitrary traps within the anomalous universality class,

$$\phi(u) = S_R\left(0, \frac{u}{\sqrt{s_2}}\right). \quad (51)$$

Here a regularized trap function $S_R(t,u)$ in Eq. (B3) is a known in mathematics spectral ζ function (see Appendix C and reviews [64–66]), associated with the spectrum of the trap. This result allows one to find explicitly, by simple canonical calculations, all properties of the universal distribution for any given trap.

A general asymptotics of the universal unconstrained probability distribution in the undercritical region of the noncondensate occupation, $-x \gg 1$,

$$\rho_x \simeq \frac{e^{-iu_s x+\phi(u_s)}}{\sqrt{-2\pi\phi^{(2)}(u_s)}}, \quad \phi^{(2)}(u_s) = \left.\frac{d^2\phi}{du^2}\right|_{u=u_s(x)}, \tag{52}$$

follows from the Fourier integral in Eq. (25) via the method of a stationary phase. A stationary point $u_s(x)$ is defined by the equation $d\phi/du = ix$ and tends to a large, pure imaginary value at $x \to -\infty$. Its asymptotics at $|u| \to \infty$ (Appendices B and C) yields

$$\frac{d^2\phi}{du^2} = -S\left(2,\frac{u}{\sqrt{s_2}}\right) \simeq -\sum_{j=1}^{\infty}\frac{c_j\Gamma(2-t_j)}{\left(-ius_2^{-\frac{1}{2}}\right)^{2-t_j}}. \tag{53}$$

Here t_j and c_j stand for the positions and residues of the poles of the trap functions $S(t)$ or $S(t,u)$.

For a power-law trap in Eq. (27), the leading-order asymptotics for the distribution in Eq. (52) is

$$\rho_x \sim e^{-\gamma|x|^{\frac{3}{3-\nu}}}; \quad \gamma = \frac{(1-\nu/3)s_2^{\frac{3}{2(3-\nu)}}}{\Gamma^{\frac{3\nu}{3-\nu}}\left(1+\frac{1}{\nu}\right)\left|\Gamma^{\frac{\nu}{3-\nu}}\left(1-\frac{3}{\nu}\right)\right|}. \tag{54}$$

The result in Eq. (54) proves an evolution of the noncondensed-phase asymptotics across the anomalous universality class from the Gaussian one, $\sim\exp(-\gamma x^2)$, at $\nu \to 3/2$ to an anomalous asymptotics with much steeper decaying exponent, $\sim\exp(-\gamma|x|^3)$, characteristic for the box traps at $\nu \to 2$. That conclusion is very general since it is predetermined by the position of the rightmost pole $r = 3/\nu$ of the trap function. Hence, it is valid for a wide variety of traps with various perturbations of the power-law spectrum.

A general asymptotics of the universal distribution in the overcritical region of the noncondensate occupation, $x \gg 1$, is similar to that for a box case [17],

$$\rho_x \simeq \wp_{g_1}(\{x_m\})e^{-x\sqrt{s_2}-g_1+s_0'}, \quad x\sqrt{s_2}(\lambda_2-1) \to \infty. \tag{55}$$

It has a perfect exponential accuracy and is a product of $e^{-x\sqrt{s_2}} \equiv e^{-\alpha(n-N_c)}$ and a polynomial of degree p,

$$\wp_{g_1}(\{x_m\}) = \sum_{m=0}^{p}\sum^{(m,p)}\frac{x_1^{a_1}\cdots x_p^{a_p}}{a_1!1^{a_1}\cdots a_p!p^{a_p}}, \quad p = g_1-1. \tag{56}$$

The polynomial depends on the noncondensate occupation n only through a variable $x_1 = x\sqrt{s_2} - x_0'$. The parameters x_0', s_0', and $x_m, m \geqslant 2$, are the constants:

$$x_1 = x' - x_0' \simeq \tilde{x}_1, \quad x_0' = -g_1 + \sum_{q\neq 0,1}\frac{1}{\lambda_q(\lambda_q-1)},$$
$$x_m = (-1)^m s_m' \simeq \tilde{x}_m, \quad m \geqslant 2, \tag{57}$$

$$s_0' = \sum_{q\neq 0,1}\left(\ln\frac{\lambda_q}{\lambda_q-1} - \frac{1}{\lambda_q}\right) \simeq \tilde{s}_0',$$
$$s_m' = \sum_{q\neq 0,1}\frac{1}{(\lambda_q-1)^m} \simeq \tilde{s}_m', \quad m \geqslant 2. \tag{58}$$

The sum $\sum^{(m,p)}$ runs over all non-negative integers $a_1,\ldots,a_p$, which satisfy the following two conditions: $a_1 + a_2 + \cdots + a_p = m$, $a_1 + 2a_2 + \cdots + pa_p = p$.

A detailed analytical calculation of the anomalous universal statistics is illustrated in Appendix D.

Finally, we present an exact formula for the universal statistics in the case of the 1D harmonic trap:

$$\rho_x^{(1)} = \frac{\pi}{\sqrt{6}}e^{-\frac{\pi x}{\sqrt{6}}-\gamma}P_x^{(1)}, \quad P_\eta^{(1)} = \exp(-e^{-\frac{\pi\eta}{\sqrt{6}}-\gamma}). \tag{59}$$

It follows from the thermodynamic limit of its cumulants in Eq. (45) if one calculates a Fourier transform of its characteristic function. (The γ is the Euler's constant.) That double-exponent distribution is the most asymmetric among all (shown in Fig. 1) distributions in the anomalous universality class. As is known, due to the anomalous fluctuations on order of a mean value [47,54,55], the 1D harmonic trap has only a quasicondensation instead of a true BEC phase transition. Nevertheless, the universality analysis presented in this paper is valid and perfectly describes the critical phenomena in this case as well.

C. The marginal nonuniversality subclass

The Gaussian and anomalous universality classes are separated by the marginal subclass of the traps with the trap-function rightmost pole $r = 2$. For the power-law traps that means the spectral index $\nu = 3/2$. In this subclass, according to Eqs. (22) and (23), the higher-order cumulants in Eq. (18) decrease in the thermodynamic limit $\alpha \to 0$ only logarithmically,

$$\kappa_m^{(x)(\infty)} \simeq \frac{\Gamma(m)s_m}{[\Gamma^3(5/3)\ln(\alpha^{-1})]^{m/2}}, \quad m \geqslant 3. \tag{60}$$

Formally, at $\alpha = 0$ (i.e., $L = \infty$) the limit is the Gaussian statistics. However, for any realistic, even macroscopically large trap with r or ν equal or close to 2 or 3/2, respectively, the scaled statistics in Eq. (25) remains neither Gaussian nor universal. It depends on a scale of a trap via the parameter $\alpha = \epsilon_1/T$.

The 2D and moderately anisotropic harmonic traps with the cumulant $\kappa_2^{(\infty)}$, determined mainly by the residue c_2 at the trap-function pole $t = 2$, represent an important class of such marginal statistics.

VII. UNIVERSAL CRITICAL FUNCTIONS

A. The λ structures' variety and self-similarity

A structure of various thermodynamic quantities around the critical point shows a tremendous variety depending on the form, boundary conditions, and size of the trap as well as the number of trapped atoms. To make the problem clear, let us consider the famous λ structure of the specific heat for the different traps and numbers of the trapped atoms (Figs. 4 and 5); for the traps' details see Sec. V. For a relatively small number of atoms, the graphs can be calculated numerically by means of a formal recursion for a partition function [34,35,43,46,47,55]. We see that in the wide region around the critical temperature T_c the λ structure essentially changes from trap to trap as well as with a change of the number of atoms. The standard grand-canonical-ensemble mean-field approach, which omits fluctuations, fails to describe the smooth λ structure in the critical region. It yields only its high- and

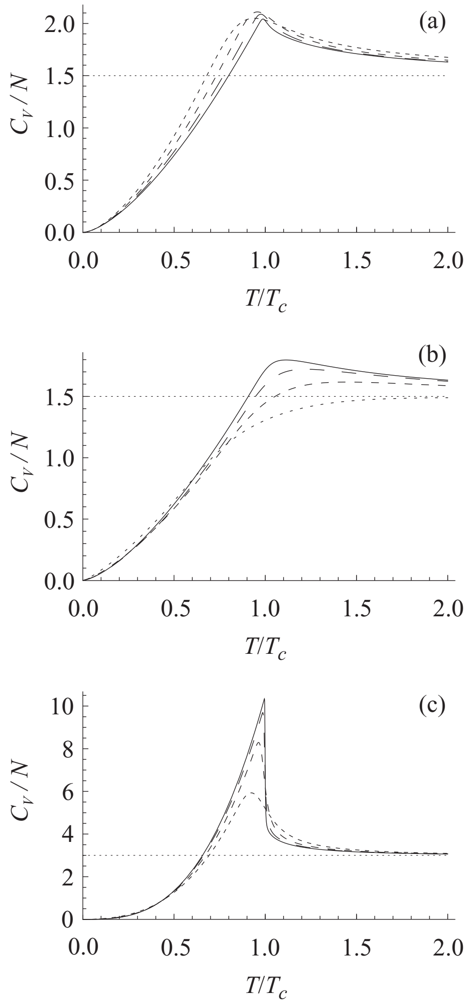

FIG. 4. Evolution of the specific heat λ structure as a function of temperature given by the exact numerical plots for the increasing numbers of atoms $N = 10^2$ (short-dashed line), 10^3 (medium-dashed line), 10^4 (long-dashed line), and $N = 10^5$ (solid line): (a) the box with the zero Dirichlet boundary conditions, (b) the box with power-law energy spectrum, (c) the isotropic harmonic trap.

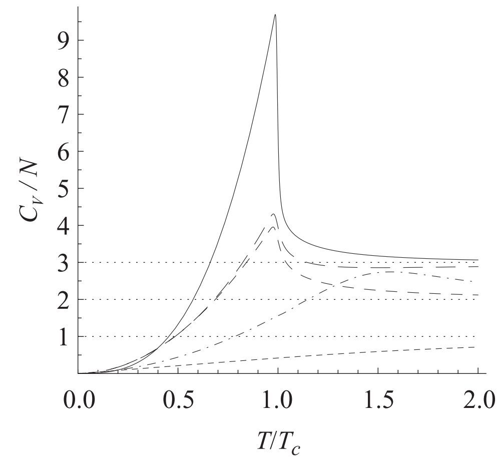

FIG. 5. Evolution of the specific heat λ structure as a function of temperature for the anisotropic harmonic traps in Eq. (41) with increasing degree of anisotropy: $\epsilon_x = \epsilon_y = \epsilon_z \ll T$ (3D isotropic trap, solid line), $\epsilon_x = \epsilon_y \ll \epsilon_z = 300\epsilon_x = 4T$ (long-dashed line), $\epsilon_x = \epsilon_y \ll T, \epsilon_z = \infty$ (2D trap, medium-dashed line), $\epsilon_x \ll \epsilon_y = 600\epsilon_x = T/2, \epsilon_z = \infty$ (dot-dashed line), $\epsilon_x \ll T, \epsilon_y = \epsilon_z = \infty$ (1D trap, short-dashed line). All curves are the exact numerical plots for $N = 10^4$ atoms.

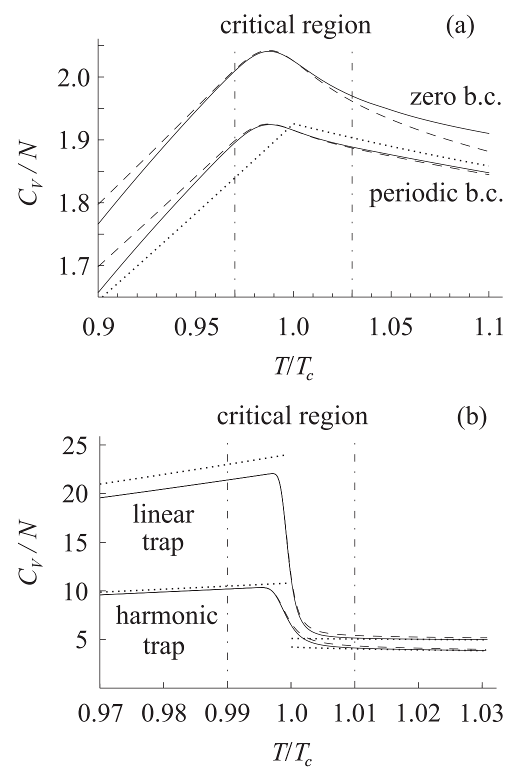

FIG. 6. The universal λ structure of the specific heat (solid lines), given by the universal function in Eq. (74) or (83), versus the exact numerical graphs (dashed lines) for the different traps with $N = 10^5$ atoms: (a) the boxes with the zero Dirichlet and periodic boundary conditions as well as (b) the traps with the isotropic harmonic and linear potentials. The analytical curves in (b) are plotted on the basis of Eq. (85), which includes also a slow temperature dependence outside the critical region. The previously known, grand-canonical-ensemble ansatz in Eq. (76) (dotted lines) yields a discontinuity [2,19,20,22,23] and is incorrect in the critical region.

low-temperature asymptotics (see Fig. 6x and the graphs and discussions in [34]).

Our method [17], developed in the present paper for traps with any form of confining potential and boundary conditions, yields an amazingly simple analytical solution to this problem. It is given by the explicit universal function [Eq. (71)] of the self-similar variable in Eq. (26). Being plotted as a function of the temperature T (Fig. 6), that universal function yields all the variety of λ structures for all traps with different numbers of the loaded atoms. It works starting from even small mesoscopic number of atoms $\sim 10^2$ until the bulk limit $N \to \infty$. It reveals a large, order-of-unity contribution of fluctuations to the specific heat in the critical region. The width of the critical region over T/T_c is finite, but self-similarly shrinks in the thermodynamic limit as $\sigma^{(\infty)}/N \to 0$.

B. The power-law traps

First, we write the general formulas for all universal critical functions for an arbitrary trap, then specify them for

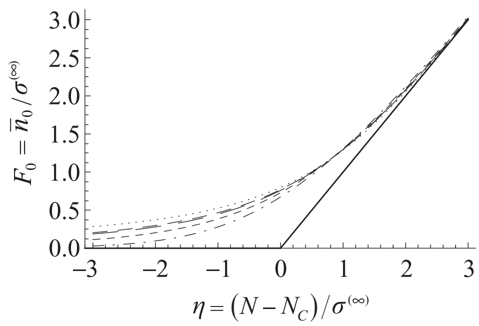

FIG. 7. The BEC order parameter as a universal critical function $F_0(\eta)$, Eq. (61), of a scaled total number of atoms η for the isotropic harmonic trap [dotted line, Eq. (79)], the boxes with the periodic (medium-dashed line) and zero (long-dashed line) boundary conditions, the power-law box (short-dashed line), and the 1D harmonic trap (dot-dashed line). The solid line is the Landau mean-field approximation.

power-law traps with any spectral index ν in Eq. (27) and, in the subsequent sections, calculate and compare them for the harmonic, linear, and box traps. We define the critical function for any statistical or thermodynamic quantity as an appropriately scaled and centered quantity in order that it would be well-defined and finite in the thermodynamic limit in the whole critical region. That approach allows us to resolve a structure of the statistical or thermodynamic quantities in the critical region of the BEC. Otherwise, it would appear only as a structureless singularity or discontinuity in their value or derivatives.

The universal critical function for the order parameter, i.e., for the mean condensate occupation, is a function of the scaled number of trapped atoms:

$$F_0(\eta) = \frac{\bar{n}_0}{\sigma^{(\infty)}} = \eta - \bar{x} \equiv \eta - \int_{-\infty}^{\eta} x\rho_x^{(\mathrm{univ})}dx. \quad (61)$$

As is shown in Fig. 7, the BEC order parameter follows a smooth universal curve with the nonzero values on the order of the dispersion of fluctuations $\sigma^{(\infty)}$ in Eq. (32) on both sides from the critical point. This is the case not only in the mesoscopic system, but also in the thermodynamic limit as well. That picture resolves the universal fine structure of the order parameter in the critical region, contrary to the Landau mean-field approximation picture of a curve with a discontinuity of its derivative at the critical point. The difference in the universal behavior of the order parameter in the critical region for the different universality classes and different traps is small and is barely seen in Fig. 7. However, it is very essential for other quantities, e.g., for the specific heat.

The universal critical functions for the initial moments $A_m = \alpha_m/(\sigma^{(\infty)})^m$, central moments $M_m = \mu_m/(\sigma^{(\infty)})^m$, and cumulants $K_m = \kappa_m/(\sigma^{(\infty)})^m$ of the universal distribution of the noncondensate occupation $\rho_x^{(\mathrm{univ})}$ in Eq. (26) are defined similarly, e.g.,

$$M_m(\eta) \equiv \frac{\mu_m}{(\sigma^{(\infty)})^m} = \int_{-\infty}^{\eta} (x - \bar{x})^m \rho_x^{(\mathrm{univ})}dx. \quad (62)$$

We start the analysis of the universal thermodynamic functions with the Gibbs free energy, since all the main thermodynamic quantities are determined by its derivatives. According to Eqs. (15) and (16), it is

$$\frac{F}{T} = \frac{F^{(\infty)}}{T} - \ln P_\eta, \quad \frac{F^{(\infty)}}{T} = -\ln Z^{(\infty)}, \quad (63)$$

and contains a constant, which is independent on the number of atoms and is well known in the statistical physics [1]. In the thermodynamic limit it is related to the average energy $\bar{E}^{(\infty)}$ and the entropy $S^{(\infty)}$ of the unconstrained ensemble of the excited atoms:

$$\frac{F^{(\infty)}}{T} = -\frac{\nu \bar{E}^{(\infty)}}{3T} = -\frac{\nu S^{(\infty)}}{\nu + 3} = -\frac{\zeta(1 + 3/\nu)}{\zeta(3/\nu)}N_c. \quad (64)$$

That trivial constant does not contain any essential information on the BEC critical phenomena, although being a factor of $N_c \gg 1$ greater than the term $-\ln P_\eta$, which is responsible for the BEC phase transition via the constraint-cutoff mechanism in Eq. (26). To exclude it, we introduce the universal critical function for the Gibbs free energy per unit temperature centered to its value at the critical point:

$$F_F(\eta) = \frac{F}{T} - \frac{F}{T}\bigg|_{\eta=0} = \ln P_{\eta=0} - \ln P_\eta. \quad (65)$$

The average energy of N atoms in any power-law trap is equal, according to Eqs. (16) and (63), to

$$\bar{E} = \bar{E}^{(\infty)} + T^2\frac{\partial}{\partial T}\ln P_\eta. \quad (66)$$

It also contains a trivial constant, given in Eq. (64), and responsible for the critical behavior term, determined by $\ln P_\eta$. In the thermodynamic limit the temperature derivative is equal to the derivative with respect to the universal variable η times a large factor:

$$T\frac{\partial}{\partial T} \simeq -\frac{3}{\nu}\frac{N_c}{\sigma^{(\infty)}}\frac{\partial}{\partial \eta}. \quad (67)$$

That yields a simple approximation for average energy

$$\bar{E} = \bar{E}^{(\infty)} - \frac{3TN_c}{\nu\sigma^{(\infty)}}\frac{\rho_\eta}{P_\eta}. \quad (68)$$

Hence, we can introduce the universal critical function for the average energy per unit temperature,

$$F_E = \frac{\nu\sigma^{(\infty)}}{3N_c}\left(\frac{\bar{E}}{T} - \frac{\bar{E}^{(\infty)}}{T}\right) = -\frac{\partial}{\partial\eta}\ln P_\eta \equiv -\frac{\rho_\eta}{P_\eta}. \quad (69)$$

The heat capacity of N atoms in any power-law trap is equal, according to Eqs. (16) and (66), to

$$C_V = C_V^{(\infty)} + \frac{\partial}{\partial T}T^2\frac{\partial}{\partial T}\ln P_\eta. \quad (70)$$

Its universal form in the approximation of Eq. (67),

$$\frac{C_V}{N} = \frac{C_V^{(\infty)}}{N} + \frac{9N_c^2}{(\nu\sigma^{(\infty)})^2N}\frac{\partial^2 \ln P_\eta}{\partial\eta^2} - \frac{3}{\nu N}\frac{\partial}{\partial T}\left(\frac{TN_c}{\sigma^{(\infty)}}\right)\frac{\rho_\eta}{P_\eta}, \quad (71)$$

contains a constant, given by an analog of Eq. (A6),

$$C_V^{(\infty)} = \sum_j \operatorname*{Res}_{t=t_j} t(t+1)\alpha^{-t}\zeta(t+1)S(t). \quad (72)$$

That result yields an explicit asymptotic expansion of $C_V^{(\infty)}$ as a series of α^{-t_j} terms, the leading of which is

$$C_V^{(\infty)} \simeq \frac{3(3+\nu)}{\nu^2}\frac{\zeta(1+3/\nu)}{\zeta(3/\nu)}N_c. \tag{73}$$

In the leading order Eq. (71) yields the following universal critical function for the specific heat:

$$\begin{aligned} F_C(\eta) &= \frac{(\nu\sigma^{(\infty)})^2}{9N_c}\left[\frac{C_V}{N} - \frac{C_V(N=N_c)}{N_c}\right] \\ &\approx \frac{\partial^2 \ln P_\eta}{\partial\eta^2} - \frac{\partial^2 \ln P_\eta}{\partial\eta^2}\bigg|_{\eta=0} \\ &\quad - \frac{\left(1+\frac{\nu}{3}\right)\zeta\left(1+\frac{3}{\nu}\right)\sigma^{(\infty)3}}{\zeta\left(\frac{3}{\nu}\right)N_c^2}\eta. \end{aligned} \tag{74}$$

It describes a deviation of the specific heat from its critical value in the central part of the critical region, that is, the universal fine structure of the λ point.

The universal behavior of the thermodynamic quantities, given by Eqs. (63), (68), (71) or similar ones, can be smoothly matched outside the central critical region with a mean-field or grand-canonical-ensemble approximation, obtained via a density of states $g(\epsilon)$,

$$\begin{aligned} E^{(gc)}(T) &\simeq \int_0^\epsilon \frac{\epsilon g(\epsilon)d\epsilon}{z^{-1}e^{\epsilon/T}-1} \\ &= \frac{3\epsilon_1}{\nu}\Gamma^3\left(1+\frac{1}{\nu}\right)g_{1+3/\nu}(z)\left(\frac{T}{\epsilon_1}\right)^{1+3/\nu}, \end{aligned} \tag{75}$$

$$\begin{aligned} c^{(gc)}(T) \equiv \frac{C_V^{(gc)}}{N} &\simeq \frac{3}{\nu N}\Gamma^3\left(1+\frac{1}{\nu}\right)\left(\frac{T}{\epsilon_1}\right)^{\frac{3}{\nu}} \\ &\times\left[\left(1+\frac{3}{\nu}\right)g_{1+3/\nu}(z) - \frac{3}{\nu}\frac{g_{3/\nu}^2(z)}{g_{3/\nu-1}(z)}\theta(T-T_c)\right]. \end{aligned} \tag{76}$$

Here $\theta(T-T_c)$ is a step function and fugacity $z = e^{\mu/T}$ is set via a Bose-Einstein function [53]:

$$g_{\frac{3}{\nu}}(z) = \zeta\left(\frac{3}{\nu}\right)\left(\frac{T_c}{T}\right)^{\frac{3}{\nu}}, \quad g_p(z) = \frac{1}{\Gamma(p)}\int_0^\infty \frac{x^{p-1}dx}{z^{-1}e^x - 1}. \tag{77}$$

Thus, the universal critical functions of the main thermodynamic quantities, including the Gibbs free energy, average energy, and specific heat in Eqs.(65), (69), (74), are determined by the logarithm of the universal cumulative distribution of the scaled noncondensate occupation, $\ln P_\eta$, and its first two derivatives. The universal distribution ρ_x in Eq. (25) and its cumulative distribution P_η in Eq. (26) are found for the both universality classes analytically (Secs. IV–VI). It allows us to find analytically and plot the universal behavior of all main statistical and thermodynamic quantities. It can be matched with the mean-field asymptotics in both the high-temperature and the low-temperature phases by the general formulas in Eqs. (63), (68), and (71). We present that analysis first for the Gaussian universality class and then for the different traps within the anomalous universality class.

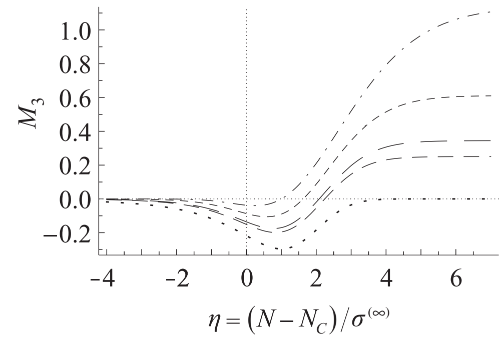

FIG. 8. The universal central moment $M_3(\eta)$ of the BEC fluctuations (the asymmetry coefficient) in Eq. (62) as a function of a scaled total number of the trapped atoms η in the critical region for the isotropic harmonic trap [dotted line, Eq. (81)], the power-law box (short-dashed line), the boxes with the periodic (medium-dashed line) and zero Dirichlet (long-dashed line) boundary conditions, and the 1D harmonic trap (dot-dashed line).

C. The Gaussian universality class: The isotropic harmonic and similar traps

The universal cumulative distribution in this case is

$$P_\eta^{(\text{Gauss})} = \frac{1}{2}\left(1+\text{erf}\frac{\eta}{\sqrt{2}}\right), \quad \text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2}dy. \tag{78}$$

The universal critical function for the order parameter is a simple analytical function, shown in Fig. 7,

$$F_0^{(\text{Gauss})} \equiv \frac{\bar{n}_{\mathbf{q_0}}}{\sigma^{(\infty)}} = \eta + G, \quad G = \sqrt{\frac{2}{\pi}}\frac{\exp\left(-\frac{\eta^2}{2}\right)}{1+\text{erf}\frac{\eta}{\sqrt{2}}}. \tag{79}$$

The dispersion of the cutoff distribution $\rho_x^{(\text{univ})}$ is

$$D^{(\text{Gauss})}(\eta) \equiv \frac{\sigma^2}{\sigma^{(\infty)2}} = 1 - \eta G - G^2. \tag{80}$$

The universal asymmetry coefficient [see Eq. (62)]

$$M_3^{(\text{Gauss})}(\eta) \equiv \frac{\mu_3}{\sigma^{(\infty)3}} = (1-\eta^2)G - 3\eta G^2 - 2G^3 \tag{81}$$

is another simple analytical function, shown in Fig. 8.

The universal critical functions for the Gibbs free energy, the average energy, and the specific heat in Eqs. (65), (69), and (74) also can be found analytically:

$$F_F^{(\text{Gauss})}(\eta) = -\ln\left(1+\text{erf}\frac{\eta}{\sqrt{2}}\right), \quad F_E^{(\text{Gauss})}(\eta) = -G, \tag{82}$$

$$F_C^{(\text{Gauss})}(\eta) = \frac{\partial^2 \ln P_\eta}{\partial\eta^2} - \frac{\partial^2 \ln P_\eta}{\partial\eta^2}\bigg|_{\eta=0} = \frac{2}{\pi} - \eta G - G^2. \tag{83}$$

The analytical result in Eq. (83) is quite interesting because it reveals, for the first time, the smooth universal fine structure of the famous discontinuity in the specific heat at the λ point. It is shown in Fig. 9 as a function of the scaled number of atoms in the trap and in Figs. 4–6 as a function of the temperature. For

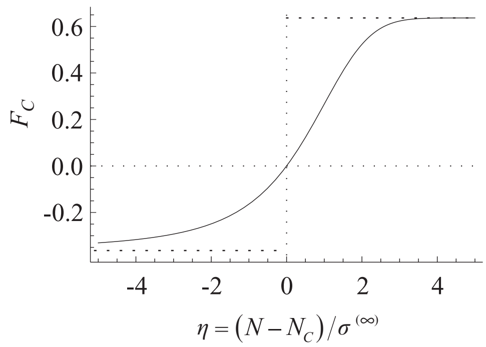

FIG. 9. The usual grand-canonical-ensemble ansatz (dotted lines) versus the universal fine structure of the "jump" in the specific heat at the λ point for the Gaussian universality class, plotted by means of Eq. (74), reduced to Eq. (83), as a function of the scaled total number of atoms η (solid line).

the harmonic ($\nu = 1$) and linear ($\nu = 2/3$) traps in Fig. 6, the specific heat is given by Eq. (71). Since the universal critical function F_C "jumps" from $\frac{2}{\pi} - 1$ to $\frac{2}{\pi}$, the actual specific heat of the ideal gas in the trap with an index $\nu < 3/2$ experiences a jump

$$\Delta\left(\frac{C_V}{N}\right) = \left(\frac{3}{\nu}\right)^2 \frac{N_c}{\sigma^{(\infty)2}} \simeq \frac{9\zeta(3/\nu)}{\nu^2\zeta(3/\nu - 1)}. \tag{84}$$

For the case of the harmonic trap, that discontinuity was demonstrated in [19] (see also [20,22,23] and Eq. (10.20) in [2]) within the grand-canonical-ensemble approach (which is incorrect in the critical region [6,17,34]) and numerically in [24,25,36,37,46]. Its universal smooth structure was not found analytically.

Now we can write down a simple formula for the specific heat of any trap with the spectral index $\nu < \frac{3}{2}$, which would be valid for any temperature and number of atoms, both inside and outside the critical region. Let us use a canonical ansatz for a superposition of the two functions with the essentially different scales:

$$c \equiv \frac{C_V}{N} = c^{(gc)}(T)\left[\frac{\theta(T - T_c)}{c^+} + \frac{\theta(T_c - T)}{c^-}\right] c^{(cr)}(T),$$
$$c^- = c^{(gc)}(T_c - 0), \quad c^+ = c^{(gc)}(T_c + 0). \tag{85}$$

An extremely fast jump of the specific heat in the critical region from a condensed value c^- to a noncondensed value c^+ is described by a universal function,

$$c^{(cr)}(T) = c^- - (c^- - c^+)\left[\frac{2}{\pi} - F_C^{(\text{Gauss})}(\eta)\right]. \tag{86}$$

A slow, mean-field variation of the specific heat outside the critical region is described by the grand-canonical-ensemble approximation [1,2,5,6,67] $c^{(gc)}(T)$. In the continuous approximation for the power-law traps [Eq. (27)], it is given by Eq. (76).

As examples, let us consider the critical behavior for the specific heat of the isotropic harmonic and linear traps (Sec. V C), shown in Fig. 6(b). Here we also go beyond the grand-canonical-ensemble approximation by taking into account the fact that the number of atoms is finite. Therefore, we keep a few main terms in the asymptotic expansion of the discrete sum for the specific heat in Eq. (72) at $T < T_c$. Then, using Eq. (76), we find for the harmonic trap

$$c^{(gc)}(T) \simeq \frac{1}{N}\left[\frac{12\zeta(4)}{\alpha^3} + \frac{9\zeta(3)}{\alpha^2} + \frac{2\zeta(2)}{\alpha}\right] \quad \text{at } T < T_c, \tag{87}$$

$$c^{(gc)}(T) \simeq \frac{1}{N\alpha^3}\left[12g_4(z) - \frac{9g_3^2(z)}{g_2(z)}\right] \quad \text{at } T > T_c. \tag{88}$$

For the true linear trap, Eq. (46), the result is similar,

$$c^{(gc)} \simeq \frac{9\Gamma^3\left(\frac{5}{2}\right)}{NA_0^{\frac{9}{2}}}\left[\frac{11\zeta\left(\frac{11}{2}\right)}{4\alpha^{9/2}} - \frac{21B_0\zeta\left(\frac{9}{2}\right)}{4\alpha^{7/2}} + \frac{35B_0^2\zeta\left(\frac{7}{2}\right)}{8\alpha^{5/2}}\right] \tag{89}$$

at $T < T_c, A_0 = \frac{(3\pi)^{\frac{2}{3}}B_0}{2\sqrt{2}z_0}, B_0 = \frac{4z_0}{(9\pi)^{\frac{2}{3}} + 4z_0}, z_0 \simeq -1.02;$

$$c^{(gc)} \simeq \frac{9\Gamma^3\left(\frac{5}{2}\right)}{2N\alpha^{\frac{9}{2}}}\left[\frac{11}{2}g_{\frac{11}{2}}(z) - \frac{9g_{\frac{9}{2}}^2(z)}{2g_{\frac{7}{2}}(z)}\right], \quad T > T_c. \tag{90}$$

As is seen in Fig. 6(b), the universal analytical result in Eqs. (83)–(86) describes remarkably well the fine λ structure of the specific heat both in the critical region and beyond it. Note that in order to show in Fig. 6(b) a magnitude of the mesoscopic effect, we intensionally introduced a related (few-percent) discrepancy between the analytical and numerical curves at $T > T_c$ by using only the simplest continuous-integral approximation for the grand-canonical specific heat in the noncondensed phase in Eqs. (88) and (90).

Remarkably, the universal Gaussian statistics (and thermodynamics) is very robust. Any change in the trap's spectrum, that keeps the rightmost pole r larger than 2 and ensures a domination of its contribution to the dispersion of fluctuations, can change only the scaling in Eqs. (17) or (26) for the self-similar variable η, but does not change the Gaussian universality. Such irrelevant changes may be related to an anisotropy of the trap, a gap between the ground and excited levels, or an addition of a few lower energy levels. In particular, all anisotropic 3D traps (Sec. V D) have exactly the same universal statistics and thermodynamics in the critical region, until $r > 2$. Thus, the analysis of the isotropic harmonic trap applies to the λ structure of any such anisotropic trap.

D. The anomalous universality class: A comparison of the different box traps (power-law, periodic, Dirichlet)

Within the anomalous universality class, the traps have different universal distribution in Eqs. (25) and (26), depending on the form of the confining potential and boundary conditions. The latter dependence is truly remarkable for it persists even in the thermodynamic limit. Now we can explicitly reveal it by a comparison of the universal critical functions for the main statistical and thermodynamic quantities, for example, for the box traps with the different boundary conditions in Eqs. (33)–(35). Similar to Sec. VII C, it is straightforward to calculate the analytical formulas for all universal critical functions, derived in Sec. VII B, via the universal distributions

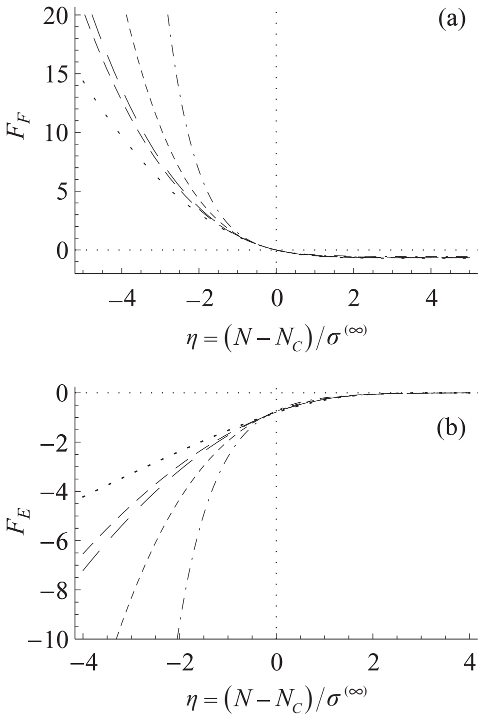

FIG. 10. The universal critical functions (a) $F_F(\eta)$ in Eq. (65) for the Gibbs free energy and (b) $F_E(\eta)$ in Eq. (69) for the average energy for the isotropic harmonic trap [dotted line, Eq. (82)], the boxes with the periodic (medium-dashed line) or zero (long-dashed line) boundary conditions, the power-law box (short-dashed line), and the 1D harmonic trap (dot-dashed line).

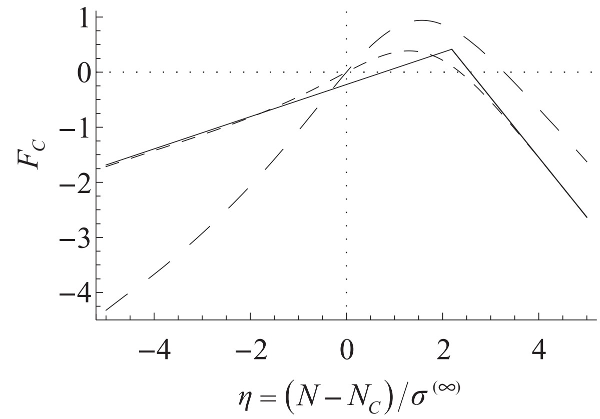

FIG. 11. The universal critical function $F_C(\eta)$ in Eq. (74) for the specific heat, given by Eq. (71), for the boxes with the periodic (medium-dashed line) or zero (long-dashed line) boundary conditions. The solid line is a linear asymptotics, the slopes of which match the ones given by the standard grand-canonical-ensemble ansatz.

ρ_x, explicitly found in Sec. VI B and shown in Fig. 1. We skip here these formulas and directly present the universal functions for the BEC order parameter [Eq. (61)], asymmetry [Eq. (62)], Gibbs free energy [Eq. (65)], average energy [Eq. (69)], and specific heat [Eq. (74)] in Figs. 7–11.

The boundary conditions change the structure of the spectrum λ_q and, hence, the universal numbers s_m in Eq. (20) as well as the scaled cumulants $K_m^{(\infty)}$ in Eq. (24). In a result, there are clear differences in the distribution ρ_x in Fig. 1, its integral P_η, the order parameter in Fig. 7, the Gibbs free energy, and the average energy in Fig. 10 for the different boxes. Yet, due to a global, averaged nature of those quantities, the differences do not appear to be very large.

Surely, for the more refined statistical and thermodynamic quantities the effect of the boundaries is drastically more pronounced and can be directly measured by experiment. The examples are the third central moment, shown in Fig. 8, and the specific heat, defined by the second derivative of the cumulative distribution P_η and shown in Figs. 4–6 and 11.

A similar strong effect of the form of a trapping potential on the critical BEC statistics and thermodynamics can be observed with the anisotropic traps readily available in many laboratories (Sec. V D). They change their universal statistics from the Gaussian to the anomalous one with an increase in an anisotropy and acquire the double-exponent statistics in Eq. (59) in the limit of the extremely anisotropic, 1D harmonic trap. This effect is shown in Figs. 5, 6–8, and 10.

Since the original works by Bose and Einstein in 1924, the specific heat of the ideal gas in a trap was analyzed by many authors (see, for example, [2,68] and references therein). Usually, it was done within the grand-canonical-ensemble approach, which is incorrect in the critical region. The result was limited to a main term in the asymptotics that shrank the structure of the specific heat in the critical region to a discontinuity of its derivative [see [67] and Eq. (76)].

Here we find the universal λ structure of the heat capacity in the whole critical region for any trap (for example, see Fig. 11). The explicit comparison with the exact numerical curves presented in Figs. 4–6 clearly demonstrates that the obtained universal solution for the specific heat in Eqs. (74) and (71) nicely describes all variety of the specific heat λ structures for various box traps with any parameters, starting from even a small critical number of atoms $N_c \sim 10^2$.

VIII. THE MESOSCOPIC EFFECTS BEYOND THE UNIVERSALITY: A CANONICAL SOLUTION FOR THE CRITICAL REGION VIA A CONFLUENT HYPERGEOMETRIC DISTRIBUTION

On top of the universal dependence on the variable η via the universal function P_η, all statistical and thermodynamic quantities include a next-to-the-main-order slow dependence on α in Eq. (2). It means a beyond-the-universality dependence on the temperature, size, and other parameters of a mesoscopic system. This effect is due to a deviation of the cumulants for the actual mesoscopic distribution at small, but finite $\alpha \neq 0$ from their limiting values in Eqs. (21)–(24).

The BEC statistics in the most interesting central part of the critical region, where $|\eta|$ is not very large (say, $|\eta| \leqslant 4$), is set by a few low-order cumulants. We find it as a confluent hypergeometric distribution,

$$\rho_x^{(3)} = \frac{e_1^{g_1} e_2^{g_2} X^{g_1+g_2-1} e^{-e_2 X}}{\Gamma(g_1+g_2)} M(g_1, g_1+g_2, (e_2-e_1)X). \tag{91}$$

It is based [17] on a Kummer's $M(a,b;z)$ function [53] with $X = x + \frac{g_1}{e_1} + \frac{g_2}{e_2}$ and exactly matches the first five cumulants $\kappa_m^{(\infty)} = (m-1)!b_m$ in Eq. (19) if

$$e_{1,2} = \frac{a_1 \pm \sqrt{a_1^2 - 4a_0 a_2}}{2(a_0/\sigma^{(\infty)})}, \quad g_{1,2} = \frac{e_{1,2}^3(b_3 e_{2,1} - b_2\sigma^{(\infty)})}{(e_{2,1} - e_{1,2})\sigma^{(\infty)3}};$$

$a_0 = b_3 b_5 - b_4^2$, $a_1 = b_2 b_5 - b_3 b_4$, $a_2 = b_2 b_4 - b_3^2$.

This canonical solution is perfectly valid for any trap. One has to use $\rho_x^{(3)}$ instead of the universal ρ_x in the formulas of Sec. VII [Eqs. (63), (66), (70)]. We do not present the related plots for various quantities since in the critical region they are indistinguishable from the exact curves in the figures, like Fig. 6.

IX. CONCLUSIONS

We conclude that the probability distribution of the condensate occupation as well as all statistical and thermodynamic quantities for the canonical ideal gas in a system of a finite, mesoscopic number of atoms in arbitrary trap can be scaled to the appropriate regular, nonsingular critical functions. They resolve the structure of the BEC phase transition in the critical region and converge fast to the corresponding universal functions in the thermodynamic limit. The universal statistics and functions depend on the spectrum of the trap. They keep a memory on the boundaries and form of the trap even in the thermodynamic limit. The obtained explicit analytical solution for the BEC λ structure is a long-wanted replacement of the usually used, but incorrect within the critical region, mean-field or grand-canonical-ensemble ansatz.

We find that there are two universality classes of the BEC statistics and thermodynamics, separated by the position $r = 2$ of the rightmost pole of the trap function, as is explained in the discussion of Eqs. (19)–(23). We describe them analytically for the 3D traps with any spectral index ν, related to the rightmost pole as $\nu = \frac{3}{r}$ and to the power p of the spatial asymptotics of a trapping potential $U(r) \sim r^p$ as $\nu = \frac{2p}{p+2}$.

The first, *Gaussian universality class*, contains all traps with a relatively small spectral index $\nu < \frac{3}{2}$, i.e., $r > 2$. They have the same universal cutoff-Gaussian statistics in the central part of the critical region and behave similar to the isotropic harmonic trap.

The second, *anomalous universality class* contains all traps with a larger spectral index, $\nu > \frac{3}{2}$, i.e., $r < 2$. Those traps have the different universal functions, depending on the form and the boundary conditions of the trap. They demonstrate a strongly non-Gaussian statistics with an anomalously large dispersion of fluctuations, dominated by a singular contribution from the excitations in the infrared limit of the spectrum. An example is a box trap, which has $\nu = 2$. We compare the different box traps (the power-law box and the boxes with the periodic and zero Dirichlet boundary conditions) and find their universal critical functions to be remarkably different.

We consider mainly the 3D traps, but the analysis can be easily extended to any trap's dimension d. In particular, a case of the strongly anisotropic harmonic traps (Sec. V D) is quite interesting experimentally. In a general d-dimensional case, the marginal value of the spectral index, separating the two universality classes of the normal and anomalous statistics, is equal to $\nu = \frac{d}{2}$. It corresponds to the confining potential profile with the power $p = \frac{2d}{4-d}$. Thus, the 2D harmonic trap [$\nu = 1$, Eq. (43)], which is parabolic in space ($p = 2$), and the 1D trap with the fractional power $p = \frac{2}{3}$ ($\nu = \frac{1}{2}$) both have the potential with the marginal power $p = \frac{2d}{4-d}$ and belong to the marginal nonuniversality subclass. An extremely anisotropic 1D harmonic trap belongs to the anomalous universality class and is perfectly described by the exact double-exponent solution in Eq. (59).

The obtained universal statistics constitute a starting point (i.e., the zeroth-order approximation) of a long-wanted consistent microscopic theory of the critical BEC phenomena in an interacting gas. This statistics should be used in a theorem on the nonpolynomial averages and appropriate diagram technique [69], that provide a nonperturbative-in-fluctuations way to account for the particle interaction. Such a theory includes the effect of a long-wavelength-modes instability, involved in the spontaneous symmetry breaking. The latter appears in the interacting gas on top of the constraint-cutoff mechanism of the nonanalyticity discussed in the present paper. Both points are missed in the commonly used grand-canonical-ensemble theory, based on the Dyson-type Belyaev-Popov equations for the Green's functions and the Gross-Pitaevskii equation for the order parameter. This theory will be published in a separate paper.

A final and very interesting conclusion is that the effect of the boundaries and form of the trap on the BEC critical phenomena manifests itself in the global thermodynamic quantities even in the bulk limit and can be directly measured in the experiments. Particularly, one can observe that effect in a BEC trap with steep enough and controllable walls, similar to a box trap in the Raizen's experiments [11,70,71]. Another possibility is to use an anisotropic 2D harmonic or similar trap, which is fundamentally different from the isotropic harmonic trap and belongs to the anomalous universality class. The specific heat is one of the most suitable for that purpose quantity since, as shown above (see Figs. 4–6 and 11), its famous λ structure is strongly subjected to that effect.

For the anisotropic 3D harmonic, linear, or similar traps within the Gaussian universality class there is also the effect of the form of the potential on the BEC phase transition. It can be observed, in particular, by measuring a change in the value of the specific-heat jump at the λ point, given by Eq. (84).

Another experiment on that remarkable phenomenon can be implemented in the traps with an additional small-size localized potential well. The latter allows one to control the energy gap between the ground and excited levels by shifting the ground level or adding a few extra lower-energy levels. The very measurement of the specific heat λ structure is a remarkably challenging problem for BEC laboratories worldwide. The discussed effects should be observable in both ideal and interacting gases.

Note added. The same universal scaling and two universality classes of the BEC statistics were described from a mathematical point of view in the important paper [72], which was recently published. (The BEC thermodynamics and, in particular, the specific heat were not studied in [72].) A series of rigorous theorems, proven in [72], confirms our analytical theory of BEC universality, published a few years ago in [17,18], and is in agreement with the above explicit cumulant analysis.

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APPENDIX A: SERIES REPRESENTATION FOR THE CUMULANTS AND THE CHARACTERISTIC FUNCTION

Full information on the unconstrained probability distribution $\rho_n^{(\infty)}$and its characteristic function $\Theta^{(\infty)}$ in Eqs. (8) and (9) can be retrieved from the cumulants

$$\kappa_m^{(\infty)} = \frac{\partial^m \varphi^{(\infty)}}{\partial (iu)^m}\bigg|_{u=0}, \quad \varphi^{(\infty)}(u) \equiv \ln \Theta^{(\infty)}(u). \qquad \text{(A1)}$$

In order to find the universal BEC statistics [Eq. (25)] in the thermodynamic limit, when the dimensionless energy of the first excited level $\alpha \equiv \epsilon_1/T \to 0$ is a small parameter, one needs a formula for the cumulants in the form of a power series in α. Such representation has been derived in a general form and analyzed for the particular cases of the harmonic and box traps in [26–29]. Here we recall it in an adopted way.

The key point for its derivation is the Mellin-Barnes integral representation for the exponent,

$$e^{-a} = \frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} a^{-t}\Gamma(t)dt, \qquad \text{(A2)}$$

where the constant $\tau > 0$ determines the path of integration, which has to lie on the right of all poles of the integrand, and $\Gamma(t)$ is a Γ function.

Let us use the Taylor expansion of the logarithm of the characteristic function in Eq. (9),

$$\begin{aligned}\varphi^{(\infty)}(u) &= \sum_{q\neq 0}[\ln(1-e^{-\alpha\lambda_q}) - \ln(1-e^{-\alpha\lambda_q+iu})]\\ &= \sum_{q\neq 0}\sum_{m=1}^{\infty}\frac{e^{-\alpha\lambda_q m+ium}-e^{-\alpha\lambda_q m}}{m},\end{aligned}$$

and the Mellin-Barnes formula to get

$$\varphi^{(\infty)} = \sum_{q\neq 0}\sum_{m=1}^{\infty}\int_{\tau-i\infty}^{\tau+i\infty}\frac{(\alpha\lambda_q - iu)^{-t}-(\alpha\lambda_q)^{-t}}{2\pi i m^{t+1}}\Gamma(t)dt.$$

Let us set the constant τ to provide the absolute convergence of the sums and the integral. Thus, we can interchange them and find the integral representation

$$\varphi^{(\infty)}(u) = \int_{\tau-i\infty}^{\tau+i\infty}\frac{\zeta(t+1)\Gamma(t)}{2\pi i\alpha^t}\sum_{q\neq 0}\left[\frac{1}{\left(\lambda_q - \frac{iu}{\alpha}\right)^t} - \frac{1}{\lambda_q^t}\right]dt, \qquad \text{(A3)}$$

where $\zeta(n) = \sum_{l=1}^{\infty}\frac{1}{l^n}$ is a Riemann ζ function. By completing the path of integration to the closed contour with an addition of an arc in the left half plane, we can calculate the whole contour integral immediately as a sum of the residues at the integrand poles. For the small values of α the integral along the big arc tends to zero. (Note that this result is achieved for each value of α smaller than some constant close to 1; therefore, the condition $\alpha \to 0$ itself is not required.) Thus, we get the desired representation,

$$\varphi^{(\infty)}(u) = \sum_j \operatorname*{Res}_{t=t_j}\alpha^{-t}\zeta(t+1)\left[S\left(t,\frac{u}{\alpha}\right) - S(t)\right], \qquad \text{(A4)}$$

where $S(t,u)$ and $S(t)$ are the so-called extended and pure trap functions, respectively:

$$S(t,u) = \Gamma(t)\sum_{q\neq 0}\frac{1}{(\lambda_q - iu)^t}, \quad S(t) \equiv S(t,0). \qquad \text{(A5)}$$

The expression for the cumulants [28],

$$\kappa_m^{(\infty)} = \sum_j \operatorname*{Res}_{t=t_j}\ \alpha^{-t}\zeta(t+1-m)S(t), \qquad \text{(A6)}$$

immediately follows from Eqs. (A1) and (A4) and an obvious property of the extended trap function

$$\partial S(t,u)/\partial(iu) = S(t+1,u). \qquad \text{(A7)}$$

The crucial point is that we have separated dependence of the cumulants on all trap and gas parameters (like the size of the trap as well as the temperature and mass of the atoms) in just one factor α, while all information about the energy spectrum is allocated in the trap function $S(t)$. This fact turns the result in Eq. (A6) into a very useful and powerful instrument.

The residues in Eq. (A6) for cumulants are provided either by the Riemann ζ function, which goes as $\zeta(x) = \frac{1}{x-1} + \gamma$ at $x \to 1$, or by the trap function $S(t)$ (see Appendix C). Each pole yields a power term in α. In the thermodynamic limit $\alpha \to 0$, the leading term in the dependence of the residues on α is due to the rightmost pole located at $t_1 \equiv r$.

APPENDIX B: ANALYTICAL SOLUTION FOR UNIVERSAL CHARACTERISTIC FUNCTION: THE ANOMALOUS UNIVERSALITY CLASS

Let us derive an explicit analytical solution for a logarithm of a characteristic function $\phi \equiv \ln \Theta^{(\infty)(\text{univ})}$,

$$\phi(u) = \sum_{m=2}^{\infty}\frac{S(m)}{m!}\left(\frac{iu}{\sqrt{s_2}}\right)^m, \quad S(m) = \sum_{q\neq 0}\frac{\Gamma(m)}{\lambda_q^m}, \qquad \text{(B1)}$$

of the universal probability distribution in Eq. (25) for the traps within the anomalous universality class, for which the rightmost pole in Eqs. (A4)–(A6) is located at the point $t_1 \equiv r < 2$ (see Sec. VI B). Since the stochastic variable $x = (n - \kappa_1^{(\infty)})/\sigma^{(\infty)}$ [Eq. (17)] of this universal distribution is centered to the exact mean value of the noncondensate occupation $N_c \equiv \kappa_1^{(\infty)}$ and scaled by the dispersion $\sigma^{(\infty)}$, the first term $m=1$ in the sum in Eq. (B1) is absent and Fourier variable u is scaled by a factor $\sigma^{(\infty)} \underset{\alpha\to 0}{\longrightarrow} \sqrt{s_2}/\alpha$, which corresponds to the subtraction of the term $(i\alpha u/\sqrt{s_2})\kappa_1^{(\infty)}$ from the function $\varphi^{(\infty)}(\alpha u/\sqrt{s_2})$ since $\varphi^{(\infty)}(u)$ was originally introduced in Eq. (A1) as the logarithm of the characteristic function for the noncondensate occupation n. Hence, Eqs. (A4)–(A6) immediately yield

$$\begin{aligned}\phi(u) &= \lim_{\alpha\to 0}\left[\varphi^{(\infty)}\left(\frac{\alpha u}{\sqrt{s_2}}\right) - \frac{i\alpha u\kappa_1^{(\infty)}}{\sqrt{s_2}}\right] \\ &= \lim_{\alpha\to 0}\sum_j \operatorname*{Res}_{t=t_j} \alpha^{-t}\zeta(t+1)S_R\left(t,\frac{u}{\sqrt{s_2}}\right), \end{aligned}\tag{B2}$$

where we introduced a regularized trap function

$$S_R(t,u) = S(t,u) - S(t) - iuS(t+1), \tag{B3}$$

which does not have any singularities at $t \geqslant 0$ for any trap within the anomalous universality class.

The latter remarkable property can be deduced from an analysis of an analytical continuation of the trap function, briefly presented in Appendix C, and can be seen already from a formula

$$S_R(t,u) = \Gamma(t)\sum_{q\neq 0}\left\{\frac{1}{\lambda_q^t}\left[\left(1-\frac{iu}{\lambda_q}\right)^{-t} - 1\right] - \frac{iut}{\lambda_q^{t+1}}\right\}, \tag{B4}$$

which follows from Eq. (B3) and the definition of the trap functions in Eq. (A5). Using the Newton's generalized negative binomial theorem,

$$\left(1-\frac{iu}{\lambda_q}\right)^{-t} = \sum_{m=0}^{\infty}\frac{\Gamma(t+m)}{m!\,\Gamma(t)}\left(\frac{iu}{\lambda_q}\right)^m, \tag{B5}$$

we find that by subtracting $S(t) + iuS(t+1)$ in Eq. (B3) we exactly cancel all ultraviolet divergent terms (which are the first two terms in the Taylor series),

$$S_R = \sum_{m=2}^{\infty}\sum_{q\neq 0}\frac{\Gamma(t+m)}{m!\lambda_q^{t+m}}(iu)^m = \sum_{m=2}^{\infty}\frac{S(t+m)}{m!}(iu)^m, \tag{B6}$$

leaving in Eq. (B6) only the finite, well-defined terms of orders $m \geqslant 2$. The latter terms are finite because they are determined by the shifted functions $S(t+m)$, which do not have any singularities to the right from the rightmost pole $t_1 = r$ at $t+m > r$, that is at $t \geqslant 0$, for $r < 2$ within the anomalous universality class.

Therefore, in Eq. (B2), there is only one non-negative pole coming from the Riemann ζ function at $t=0$, which yields an exact final formula,

$$\phi(u) = S_R\left(0,\frac{u}{\sqrt{s_2}}\right). \tag{B7}$$

The same result follows from the comparison of Eqs. (B1) and (B6) at $t=0$.

The final result in Eq. (B7) is very general and truly remarkable since it is an explicit analytical formula for the logarithm of the characteristic function of the universal probability distribution of the unconstrained noncondensate occupation in terms of the known in mathematics special functions, namely, the spectral ζ functions (see reviews [64–66]) associated with arbitrary spectrum of the trap, that is a growing sequence of numbers $\lambda_q \to \infty$.

APPENDIX C: TRAP FUNCTIONS ANALYSIS

The practical use of the general formula for the universal characteristic function in Eq. (B7) as well as the integral representations for the characteristic function in Eq. (A4) and the cumulants in Eq. (A6) requires the detailed information on the extended trap function $S(t,u)$, which is defined by the dimensionless one-particle spectrum λ_q [Eq. (2)] and is given by Eq. (A5) inside the convergence region of the variable t. Let us construct its analytical continuation and find poles and residues of this function.

In fact, the trap function $S(t,u)$ associated with the spectrum λ_q in Eqs. (1) and (2) is known in mathematics as the spectral ζ function and there are some techniques for its analysis (see reviews [64–66]). Here we briefly describe one of those methods.

Let us introduce a partition function associated with the spectrum λ_q by the following rule:

$$\Xi(\beta) = \sum_{q\neq 0} e^{-\beta\lambda_q}. \tag{C1}$$

We assume that it has an analytical expansion

$$\Xi(\beta) \simeq \sum_{j=1}^{\infty} c_j\beta^{-t_j} \quad \text{at } \beta \to 0, \tag{C2}$$

where $\{t_j\}$ is a sequence of real numbers decreasing to $-\infty$; $j=1,2,\ldots$. As follows from Eqs. (A2) and (A5), the function $\Xi(\beta)$ is related to the pure trap function $S(t)$ via the Mellin transformation,

$$\Xi(\beta) = \frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty}\beta^{-t}S(t)dt, \quad S(t) = \int_0^{\infty}\beta^{t-1}\Xi(\beta)d\beta, \tag{C3}$$

which is valid at the condition $t > t_1$, which guarantees convergence of the integral in Eq. (C3) at $\beta \to 0$. To construct an analytical continuation of the trap function $S(t)$, we rewrite it using integration by parts:

$$S(t) = \int_0^{\infty}\frac{d}{d\beta}[\beta^{t_1}\Xi(\beta)]\frac{\beta^{t-t_1}}{t_1-t}d\beta. \tag{C4}$$

The latter coincides with Eq. (C3) at $t > t_1$, but converges in a wider region $t > t_2$. It can be expanded to any $t > t_{J+1}$ if we repeat that procedure J times,

$$S(t) = \int_0^\infty \beta^{t-t_J}(\hat{D}_J \cdots \hat{D}_2\hat{D}_1)\Xi(\beta)d\beta, \tag{C5}$$

where we introduce a differential operator $\hat{D}_j$,

$$\hat{D}_j f(\beta) = \frac{d}{d\beta}\left[\frac{\beta^{t_j-t_{j-1}+1}f(\beta)}{t_j - t}\right],$$

and set $t_0 = 1$ in order to simplify notations.

That procedure reveals also all singular points of the trap function $S(t)$ which are just poles, located at $t = t_j$ and having the residues

$$\operatorname*{Res}_{t=t_j} S(t) = c_j. \tag{C6}$$

The extended trap function $S(t,u)$ can be analyzed with exactly the same approach. Since it is defined by the shifted sequence $\{\lambda_q - iu\}$, the corresponding partition function is just $\Xi(\beta,u) = \Xi(\beta)e^{iu\beta}$. So, $S(t,u)$ has a simple analytical asymptotics at $|u| \gg 1, \mathrm{Im}(u) > 0$, which can be found if one considers purely imaginary values of the argument $u = iv, v \gg 1$ and uses the Mellin transform similar to Eq. (C3):

$$S(t,iv) = \int_0^\infty \beta^{t-1}\Xi(\beta)e^{-v\beta}d\beta. \tag{C7}$$

Since the integrand is a fast decreasing function of β, the main contribution to the integral comes from a region of small β for which the expansion in Eq. (C2) is valid. On this basis, we immediately get

$$S(t,u) \simeq \sum_{j=1}^{\infty} c_j\Gamma(t-t_j)(-iu)^{t_j-t} \quad \text{at } |u| \to \infty. \tag{C8}$$

That final formula is valid for arbitrary t and can be analytically continued at least to any u with a positive imaginary part and large absolute value.

The expansion of the partition function $\Xi(\beta)$ in Eq. (C2) can be easily evaluated since for $\beta \to 0$ the sum $\Xi(\beta)$ is related to an integral over the λ space.

In particular, if the trap's Schrödinger equation allows one to separate variables, the problem simplifies because $\Xi(\beta)$ can be factorized into 1D partition functions. In that case the result can be obtained for almost any sequence f_n by well-known summation formulas, for example, by the Euler-MacLaurin formula,

$$\sum_{n=a}^{b} f_n = \int_a^b f_x dx + f_b + \sum_{k=1}^{\infty}\frac{B_k}{k!}\frac{\partial^{k-1}f_x}{\partial x^{k-1}}\bigg|_a^b, \tag{C9}$$

where B_k are the Bernoulli numbers. Another very useful method (immediately yielding an answer for arbitrary power-law trap) is a residue technique that includes the Mellin transform and is described in [73].

APPENDIX D: UNIVERSAL STATISTICS FOR THE THREE DIFFERENT BOX TRAPS

Here, in addition to the exact solution in Eq. (51) and its asymptotics, we present the results for the universal functions in Eq. (25) for three different box traps in Eqs. (33)–(35). In the central part of the critical region we find their analytical approximations in terms of the parabolic cylinder function in Eq. (36) of [17] or in terms of the canonical solution in Eq. (91).

The parameters in Eq. (91) are $e_1^{(p)} \approx 4.30$, $g_1^{(p)} \approx 8.5$, $e_2^{(p)} \approx 29.57$, $g_2^{(p)} \approx 473$ for the periodic boundary conditions, $e_1^{(z)} \approx 2.95$, $g_1^{(z)} \approx 3.85$, $e_2^{(z)} \approx 23.67$, $g_2^{(z)} \approx 312$ for the zero boundary conditions, and $e_1^{(\nu=2)} \approx 2.33$, $g_1^{(\nu=2)} \approx 3.53$, $e_2^{(\nu=2)} \approx 13.54$, $g_2^{(\nu=2)} \approx 63.94$ for a power-law box. These results are derived by matching the first four or five cumulants [17], respectively. Their validity interval overlaps the validity intervals of the undercritical ($x < -3$) and overcritical ($x > 3$) asymptotics:

(a) for the box with the periodic boundary conditions,

$$\rho_x \approx \frac{s_2^{\frac{7}{4}}(x_0 - x)^{\frac{5}{2}}}{8\pi^{13/2}} \exp\left[\phi_0' + \frac{s_2^{\frac{3}{2}}(x - x_0)^3}{12\pi^4}\right], \quad x < -1, \tag{D1}$$

where $x_0 \approx 2.2$ and $\phi_0' \approx 2.2$ are the shift and normalization parameters, respectively,

$$\rho_x \approx \sqrt{s_2}\wp_6\, e^{-\sqrt{s_2}x - g_1^{(p)} + s_0'}, \quad x > 3, \tag{D2}$$

$$\wp_6 = \frac{x_1^5}{5!} + \frac{x_2 x_1^3}{12} + \frac{x_3 x_1^2}{6} + \left[\frac{x_2^2}{2} + x_4\right]\frac{x_1}{4} + \frac{x_3 x_2}{6} + \frac{x_5}{5}, \tag{D3}$$

where $x_1 = \sqrt{s_2}x - x_0', x_0' \approx 8.7$, $s_0' \approx 6.45$, $x_2 \approx 22.44$, $x_3 \approx -14.04$, $x_4 \approx 12.72$, $x_5 \approx -12.3$, $g_1^{(p)} = 6$;

(b) for the box with zero boundary conditions,

$$\begin{aligned}
\rho_x &\approx \frac{2\sqrt{s_2}}{\pi^{3/2}3^{3/4}}\tilde{w}^{5/2} \times \exp[\tilde{g}(x)], \quad x < -3.5,\\
\tilde{w}(x) &= -\frac{\sqrt{3}}{\pi}W_{-1}\left(-\frac{\pi}{\sqrt{3}}e^{\frac{4\sqrt{s_2}}{9\pi}(x-x_0)}\right),\\
\tilde{g}(x) &= -\frac{\sqrt{3}\pi^2}{4}\tilde{w}^3 + \frac{9\pi}{8}\tilde{w}^2 - \frac{3\sqrt{3}(\pi+\pi^2)}{4}\tilde{w}\\
&\quad + \frac{1+9\pi}{4}\ln\tilde{w} + \phi_0',\\
x_0 &\approx 0.55, \quad \phi_0' \approx 5.59;
\end{aligned} \tag{D4}$$

$$\rho_x \approx \sqrt{s_2}\wp_2 e^{-\sqrt{s_2}x - g_1^{(z)} + s_0'}, \quad x > 2, \tag{D5}$$

$$\wp_2 = \left(x_1^2 + x_2\right)/2, \quad s_0' \approx 2.96, \quad g_1^{(z)} = 3, \tag{D6}$$

where $x_1 = \sqrt{s_2}x - x_0'$, $x_0' \approx 3.54$, $x_2 \approx 9.08$; and

(c) for the power-law box

$$\begin{aligned}
\rho_x &\approx \frac{2\sqrt{s_2}}{\pi^{3/2}}w^{5/2} \times \exp[g(x)], \quad x < -3.5,\\
w(x) &= \frac{3}{\pi}W_0\left(\frac{\pi}{3}e^{-\frac{4\sqrt{s_2}}{3\pi}(x-x_0)}\right),\\
g(x) &= \phi_0' - \frac{\pi^2}{12}w^3 - \frac{3\pi}{8}w^2 - \frac{3\pi}{4}w - \frac{1}{4}\ln w,
\end{aligned} \tag{D7}$$

where $\phi_0' = 2.63$, $x_0 = -0.48$;

$$\rho_x \approx \sqrt{s_2}\wp_2 e^{-\sqrt{s_2}x - g_1^{(\nu=2)} + s_0'}, \quad x > 1, \tag{D8}$$

where polynomial $\wp_2$ is given by Eq. (D6) with $x_2 \approx 3.96$ and $x_1 = \sqrt{s_2}x - x_0'$, $x_0' \approx -6.13$, $s_0' \approx 1.35$, $g_1^{(\nu=2)} = 3$. In Eqs. (D4) and (D7), W_{-1} and W_0 are the lower and upper branches of a Lambert W function [53], which is inverse to the function xe^x.

Thus, the universal non-Gaussian statistics ρ_x is given by these formulas analytically for all three different box traps in the whole critical region.

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[1] L. D. Landau and E. M. Lifshitz, *Statistical Physics, Part 1* (Pergamon, Oxford, 1981).
[2] L. Pitaevskii and S. Stringary, *Bose-Einstein Condensation* (Clarendone, Oxford, 2003).
[3] V. V. Kocharovsky, Vl. V. Kocharovsky, M. Holthaus, C. H. R. Ooi, A. A. Svidzinsky, W. Ketterle, and M. O. Scully, Adv. At. Mol. Opt. Phys. **53**, 291 (2006).
[4] A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinskii, *Methods of Quantum Field Theory in Statistical Physics* (Prentice-Hall, Englewood Cliffs, N. J., 1963).
[5] R. K. Pathria, *Statistical Mechanics*, 2nd ed. (Elsevier, Oxford, 1996).
[6] R. M. Ziff, G. E. Uhlenbeck, and M. Kac, Phys. Rep. **32**, 169 (1977).
[7] E. M. Lifshitz and L. P. Pitaevskii, *Statistical Physics, Part 2* (Pergamon, Oxford, 1981).
[8] P. B. Weichman, M. Rasolt, M. E. Fisher, and M. J. Stephen, Phys. Rev. B **33**, 4632 (1986).
[9] V. L. Ginzburg, Sov. Phys. Solid State **2**, 1824 (1960).
[10] A. P. Levanyuk, Soviet Physics - JETP **9**, 571 (1959).
[11] T. P. Meyrath, F. Schreck, J. L. Hanssen, C.-S. Chuu, and M. G. Raizen, Phys. Rev. A **71**, 041604 (2005).
[12] R. L. D. Campbell, R. P. Smith, N. Tammuz, S. Beattie, S. Moulder, and Z. Hadzibabic, Phys. Rev. A **82**, 063611 (2010).
[13] T. Berrada, S. van Frank, R. Bucker, T. Schumm, J.-F. Schaff, and J. Schmiedmayer, Nature Communications **4**, 2077 (2013).
[14] K. Nho and D. Blume, Phys. Rev. Lett. **95**, 193601 (2005).
[15] H. Kleinert, *Gauge Fields in Condensed Matter* (World Scientific, Singapore, 1989).
[16] V. L. Ginzburg, Rev. Mod. Phys. **76**, 981 (2004).
[17] V. V. Kocharovsky and Vl. V. Kocharovsky, Phys. Rev. A **81**, 033615 (2010).
[18] V. V. Kocharovsky and Vl. V. Kocharovsky, J. Phys. A: Math. Theor. **43**, 225001 (2010).
[19] S. R. de Groot, G. J. Hooyman, and C. A. ten Seldam, Proc. Roy. Soc. London A **203**, 266 (1950).
[20] S. Grossmann and M. Holthaus, Phys. Lett. A **208**, 188 (1995).
[21] H. D. Politzer, Phys. Rev. A **54**, 5048 (1996).
[22] W. Ketterle and N. J. van Druten, Phys. Rev. A **54**, 656 (1996).
[23] R. Napolitano, J. De Luca, V. S. Bagnato, and G. C. Marques, Phys. Rev. A **55**, 3954 (1997).
[24] H. Haugerud, T. Haugset, and F. Ravndal, Phys. Lett. A **225**, 18 (1997).
[25] W. Deng and P. M. Hui, Solid State Commun. **104**, 729 (1997).
[26] M. Holthaus, E. Kalinowski, and K. Kirsten, Ann. Phys. (NY) **270**, 198 (1998).
[27] K. Kirsten and D. J. Toms, Phys. Rev. E **59**, 158 (1999).
[28] M. Holthaus, K. T. Kapale, V. V. Kocharovsky, and M. O. Scully, Physica A **300**, 433 (2001).
[29] C. Weiss, M. Block, M. Holthaus, and G. Schmieder, J. Phys. A: Math. Gen. **36**, 1827 (2003).
[30] L. Salasnich, J. Math. Phys. **41**, 8016 (2000).
[31] Yee-Mou Kao, D. H. Lin, Pin Han, and Pi-Gang Luan, Eur. Phys. J. B **34**, 55 (2003).
[32] W. Daia and Mi Xie, Phys. Lett. A **311**, 340 (2003).
[33] J. V. Pulé and V. A. Zagrebnov, J. Math. Phys. **45**, 3565 (2004).
[34] K. Glaum, H. Kleinert, and A. Pelster, Phys. Rev. A **76**, 063604 (2007).
[35] Jian-hui Wang and Yong-li Ma, J. Phys. B: At. Mol. Opt. Phys. **42**, 245301 (2009).
[36] Jian-hui Wang, Yong-li Ma, and J. He, J. Low Temp. Phys. **162**, 23 (2011).
[37] Jian-hui Wang, Hui-yi Tang, and Yong-li Ma, Ann. Phys. **326**, 634 (2011).
[38] Jian-hui Wang, B. Zhuang, and J. He, J. Low Temp. Phys. **170**, 99 (2013).
[39] N. Goldenfeld, *Lectures on Phase Transitions and the Renormalization Group* (Addison-Wesley, Reading, MA, 1992).
[40] A. Z. Patashinskii and V. L. Pokrovskii, *Fluctuation Theory of Phase Transitions* (Nauka, Moscow, 1982).
[41] F. M. Gasparini, M. O. Komball, K. P. Mooney, and M. Diaz-Avila, Rev. Mod. Phys. **80**, 1009 (2008).
[42] M. E. Fisher, Rev. Mod. Phys. **46**, 597 (1974).
[43] P. T. Landsberg, *Thermodynamics- With Quantum Statistical Illustrations* (Interscience, New York, 1961).
[44] P. T. Landsberg, Proc. Cam. Phil. Soc. **50**, 65 (1953).
[45] W. J. Mullin and J. P. Fernandez, Am. J. Phys. **71**, 661 (2003).
[46] N. L. Balazs and T. Bergeman, Phys. Rev. A **58**, 2359 (1998).
[47] S. Grossmann and M. Holthaus, Phys. Rev. Lett. **79**, 3557 (1997); Opt. Express **1**, 262 (1997).
[48] A. Sinner, F. Schutz, and P. Kopietz, Phys. Rev. A **74**, 023608 (2006).
[49] D. J. Bedingham, Phys. Rev. D **68**, 105007 (2003).
[50] V. A. Alekseev, Quantum Electronics **31**, 16 (2001).
[51] V. A. Alekseev, Soviet Physics - JETP **112**, 932 (2011).
[52] G. Su and J. Chen, Eur. J. Phys. **31**, 143 (2010).
[53] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1972).
[54] V. V. Kocharovsky, Vl. V. Kocharovsky, and M. O. Scully, Phys. Rev. A **61**, 053606 (2000); Phys. Rev. Lett. **84**, 2306 (2000).
[55] M. Wilkens and C. Weiss, J. Mod. Optics **44**, 1801 (1997); Opt. Express **1**, 272 (1997).
[56] R. B. Dingle, Proc. Camb. Phil. Soc. **45**, 275 (1949); Adv. Phys. **1**, 111 (1952); *Asymptotic Expansions* (Academic Press, London, NY, 1973), p. 267.
[57] A. R. Fraser, Philosophical Magazine **42**, 165 (1951).
[58] F. Reif, *Fundamentals in Thermal Physics* (McGraw-Hill, N.Y., 1965).

[59] E. H. Hauge, Physica Norvegica **4**, 19 (1969).
[60] Numerical Recipes in FORTRAN, The Art of Scientific Computing (Cambridge Univ. Press, New York, 1992).
[61] J. S. de Wet and F. Mandl, Proc. R. Soc. Lond. A **200**, 572 (1950).
[62] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Pergamon, Oxford, 1991).
[63] K. Tamvakis, *Problems and Solutions in Quantum Mechanics* (Cambridge Univ. Press, New York, 2005).
[64] A. Voros, Commun. Math. Phys. **110**, 439 (1987).
[65] A. Voros, Adv. Studies in Pure Math. **21**, 358 (1992).
[66] K. Kirsten, J. Math. Phys. **32**, 3008 (1991).
[67] F. Y.-H. Wang, Am. J. Phys. **72**, 1193 (2004).
[68] P. Borrmann, O. Mulken, and J. Harting, Phys. Rev. Lett. **84**, 3511 (2000).
[69] V. V. Kocharovsky and Vl. V. Kocharovsky, Laser Phys. **17**, 700 (2007); J. Mod. Optics **54**, 2491 (2007).
[70] M. Pons, A. del Campo, J. G. Muga, and M. G. Raizen, Phys. Rev. A **79**, 033629 (2009).
[71] A. M. Dudarev, M. G. Raizen, and Qian Niu, Phys. Rev. Lett. **98**, 063001 (2007).
[72] S. Chatterjee and P. Diaconis, J. Phys. A: Math. Theor. **47**, 085201 (2014).
[73] A. O. Gelfond, *Residues and their applications* (Mir, Moscow, 1971).