Quantum traversal time through a double barrier

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The generalized Hartmann effect (GHE) predicts a strict inequality between the traversal times across a contiguous and a separated double-barrier system. This is compared to the implications of the time-of-arrival (TOA) operator approach to barrier traversal time [E. A. Galapon, Phys. Rev. Lett. **108**, 170402 (2012)]. It is shown that, for initial wave packets with compact supports in the far incident side of the barrier system, the expectation value of the traversal time is independent of the separation between the barriers. On the other hand, for wave packets with supports extending inside the first barrier, the contribution of the barrier separation to the traversal time exponentially increases with the barrier height. Our result shows that if the support of the incident wave packet is far from the barrier region, the GHE inequality is violated. However, if the support of the wave packet extends inside the barrier region, the GHE inequality is consistent with the TOA operator approach, but only when the particle's incident energy is very small.

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I. INTRODUCTION

The problem of predicting the duration of a quantum particle to tunnel through and emerge from a potential barrier is known as the quantum tunneling time problem [1–4]. The problem remains one of the most contentious and unresolved fundamental problems of quantum mechanics [5–12]. While there are diverse approaches and contradicting opinions on the tunneling time, there is a consensus that tunneling time is the phase time itself, which is the energy derivative of the phase of the transmitted wave packet through the barrier [13,14]. Consideration of the phase time predicts the presence of superluminal or greater-than-the-speed-of-light tunneling velocities. This superluminality arises from the linear asymptotic behavior of the phase time with the barrier length. This implies that tunneling time becomes independent of the barrier length as the barrier becomes spatially opaque, so that a tunneling particle is superluminal for a sufficiently thick barrier. This phenomenon is the well known Hartman effect.

The analysis of Hartman has been applied to multiple barriers separated by free space [15–17]. It has been determined that the traversal time across the barrier and the intervening free space is not only independent of the barrier widths but also independent of the barrier separation. That is, while current is finite and the wave function is oscillatory in the interbarrier separation, the group velocities there are infinite. The phenomenon is referred to as the generalized Hartman effect (GHE) [15]. However, recent analyses of the stationary phase method (SPM) used in the derivation of the GHE [18,19] have shown that the GHE holds only for wave packets with sufficiently small energy widths [20] or equivalently for sufficiently small interbarrier separations [21]. These imply that for wave packets with sufficiently broad energy distributions or for sufficiently well-separated barriers the GHE disappears. Moreover, a more rigorous use of SPM yields a subtle dependence of the phase time on the the barrier width [22]. However, there are claimed strong theoretical [23,24] and experimental [25] confirmations of the generalized Hartman effect; but there are equally strong theoretical arguments that deny the correctness of the interpretation of the experiments purportedly supporting the GHE [26,27].

In this paper we investigate quantum traversal time across two potential barriers using the theory of quantum first time of arrival operators developed in [28–31] and compare its results with the predictions of the GHE. In [32] the theory was applied to quantum traversal time across a single potential barrier. There it was found that only the above-barrier components of the momentum distribution of the incident wave packet contribute to any measurable barrier traversal time, and that below-the-barrier components are transmitted without delay. In simple terms, the theory predicts that tunneling occurs instantaneously. This is consistent with the recent experiments on quantum tunneling which yielded vanishing tunneling time within experimental accuracy [33,34]. The TOA operator approach to quantum tunneling already does not jibe with the Hartmann effect in which the tunneling time is nonvanishing and finite, albeit the tunneling is superluminal according to the effect for sufficiently opaque barriers. In this paper we apply the TOA operator approach to quantum traversal across two barriers. For incident wave packets with support away from the barriers, we find that instantaneous tunneling occurs only across the barriers, with the quantum particle essentially behaving as a free particle in the interbarrier region. On the other hand, for wave packets with support extending inside the barrier, the traversal time can be exponentially large. We will find these results not consistent with the expectations from the GHE.

II. BARRIER TRAVERSAL TIME

Before we proceed we recall from [32] the operational definition of the barrier traversal time that is theoretically described through TOA operators: A detector D_T to announce the arrival of a particle is located at the origin. A similar detector D_R is located at the far left of D_T. A potential barrier $V(q)$ with length L is placed between D_T and D_R. A localized wave packet $\psi(q)$ is prepared between D_R and $V(q)$ and

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launched at $t = 0$ towards the barrier. The TOA is recorded when D_T clicks; otherwise, no data are collected when D_R clicks. This is repeated a large number of times, with $\psi(q)$ as the initial state for every repeat and the average TOA $\bar{\tau}_B$ at D_T is computed. The same experiment is performed without the barrier, and the average free time of arrival $\bar{\tau}_F$ at D_T is computed from the new TOA data. The quantities $\bar{\tau}_F$ and $\bar{\tau}_B$ are then compared. (What we have just described coincides with the operational definition of the tunneling delay time by Steinberg, Kwiat, and Chiao in their well-known photonic tunneling experiment [35] except that they measured $-\Delta\tau$.)

The comparison between $\bar{\tau}_F$ and $\bar{\tau}_B$ is made by means of the difference of the expected TOAs given by

$$\Delta\tau = \bar{\tau}_F - \bar{\tau}_B. \tag{1}$$

When $\Delta\tau > 0$ the particle that passed through the barrier arrived earlier than the free particle or is advanced otherwise, it arrived later or is delayed. $\Delta\tau$ is not the traversal time across the barrier, but the traversal time can be deduced from it. If $\delta\tau_f$ is the expected traversal time across the length of the barrier for a free particle, then the barrier traversal time is given by $\Delta\tau_{\text{trav}} = \delta t_f - \Delta\tau$. Note that $\Delta\tau_{\text{trav}}$ is not necessarily the tunneling time $\Delta\tau_{\text{tun}}$ because the wave packet $\psi(q)$ may have an incident kinetic energy distribution that has components both above and below the maximum barrier height. Only the incident kinetic energy component that is below the barrier height contributes to the tunneling time $\Delta\tau_{\text{tun}}$. The traversal time approaches the tunneling time as the above-barrier incident energy components become arbitrarily negligible. We assume that this condition is satisfied so that the traversal time is the tunneling time.

III. GENERALIZED HARTMAN EFFECT

Now let us see how the generalized Hartman effect may manifest itself in the above operational definition of the barrier traversal time as applied to the system of two barriers depicted in Fig. 1. The key assumption of the GHE is that the phase time, as calculated by the application of the stationary phase method on a localized wave packet, is the traversal time or tunneling time across the barrier system [15,16]. Of course, not everyone agrees with this interpretation of the phase time [1,11], but let us assume for the moment that the phase time is the traversal time and let us see where it leads us. The GHE predicts that the traversal time across a system of two barriers is asymptotically independent of the total width of the barriers plus the separation between them as the barrier's width becomes arbitrarily large. This implies that there is a difference between the configurations when the two barriers are contiguous and when they are separated.

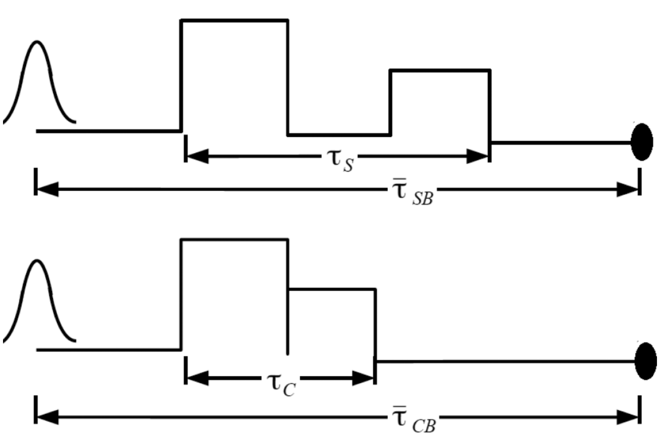

FIG. 1. The total traversal time for a quantum particle to reach the arrival point for the contiguous $\bar{\tau}_{\text{CB}}$ and the separated $\bar{\tau}_{\text{SB}}$ barrier system. The GHE claims that the tunneling time from the leftmost to the rightmost edges of the barrier region for the contiguous and separated cases are the same, i.e., $\tau_S = \tau_C$.

To demonstrate this, consider the illustration in Fig. 1 where we measure the time for a quantum particle to arrive at the detector's location as it encounters the barrier system. Let $\bar{\tau}_{\text{CB}}$ and $\bar{\tau}_{\text{SB}}$ be the respective expected arrival times for the contiguous and separated barriers, which are ensemble averages in accordance with the previous paragraph. Moreover, let τ_C and τ_S be the respective traversal times (the phase times) across the barrier system when the barriers are contiguous and separated. In the asymptotic regime where the widths of the barriers become arbitrarily large, the GHE implies the equality $\tau_C = \tau_S$. Then it follows that $\bar{\tau}_{\text{SB}} < \bar{\tau}_{\text{CB}}$ since the free region outside the barrier system for the separated case is now smaller than that of the contiguous case. Let $\bar{\tau}_F$ be the expected TOA at the detector in the absence of the two barriers, and $\Delta\tau_C = \bar{\tau}_F - \bar{\tau}_{\text{CB}} > 0$ and $\Delta\tau_S = \bar{\tau}_F - \bar{\tau}_{\text{SB}} > 0$, with the positivity of the TOA difference due to the superluminality of the GHE. Then GHE predicts the strict inequality

$$0 < \Delta\tau_C < \Delta\tau_S. \tag{2}$$

The above experimental setup involving a system of two barriers falls within the purview of the TOA operator approach to the barrier traversal time. The approach gives definite predictions on the quantities $\Delta\tau_C$ and $\Delta\tau_S$, and questions arise whether its predictions square with inequality (2) or not. We now devote the rest of the paper in showing that the TOA operator approach predicts a set of relations between $\Delta\tau_C$ and $\Delta\tau_S$ that generally exclude inequality (2).

IV. TIME-OF-ARRIVAL OPERATOR

The theory of TOA operators as developed in [28–32,36–40] requires constructing the TOA operator T for a given arrival point in the configuration space, for a given interaction potential. It postulates to model the situation where at time $t = 0$ a wave packet $\psi(q)$ is launched in the presence of a potential $V(q)$ toward a detector located at x. Then the average time elapsed between the launching of the wave packet and a successful registration of the particle at the detector is given by the expectation value $\langle\psi|\mathsf{T}|\psi\rangle$, where T is the TOA operator corresponding to $V(q)$. Now let T_B be the TOA operator in the presence of the barrier system and T_F the TOA operator in the absence of the barrier. We then make the identifications $\bar{\tau}_B = \langle\psi|\mathsf{T}_B|\psi\rangle$ and $\bar{\tau}_F = \langle\psi|\mathsf{T}_F|\psi\rangle$. The time of arrival difference is given by

$$\Delta\tau = \langle\psi|\mathsf{T}_F|\psi\rangle - \langle\psi|\mathsf{T}_B|\psi\rangle. \tag{3}$$

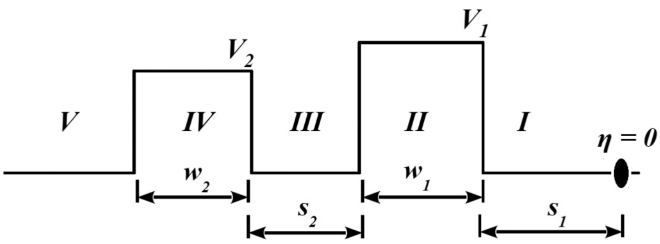

FIG. 2. The double barrier with potential height of V_1 and V_2 and width of w_1 and w_2, respectively. The barriers are separated by a distance of s_2, with s_1 being the distance of the rightmost barrier edge to the arrival point at $\eta = 0$. The different regions of integration are labeled using Roman numerals.

Equation (3) is postulated to be the theoretical value of Eq. (1) [32]. The TOA differences $\Delta\tau_C$ and $\Delta\tau_S$ can then be computed from Eq. (3) with the appropriate TOA operators corresponding to the separated and contiguous configurations of the two barriers.

Given an analytic or piecewise constant potential $V(q)$, a TOA operator can be constructed by quantization. Without loss of generality we will fix the arrival point at the origin. The quantized TOA operator for arrival at the origin is the integral operator

$$(\mathsf{T}_0\varphi)(q) = \int_{-\infty}^{\infty} \frac{\mu}{i\hbar} T_0(q,q')\mathrm{sgn}(q-q')\varphi(q')dq', \quad (4)$$

where sgn(x) is the sign function and $T_0(q,q') = \tilde{T}_0(\eta,\zeta)$, with

$$\tilde{T}_0(\eta,\zeta) = \frac{1}{2}\int_0^{\eta} d\eta'\, {}_0F_1\left(;1;\frac{\mu}{2\hbar^2}\zeta^2\{V(\eta) - V(\eta')\}\right) \quad (5)$$

in which ${}_0F_1(;1;x)$ is a specific hypergeometric function, and $\zeta = (q - q')$ and $\eta = (q + q')/2$. From Eq. (5) we obtain the free particle TOA operator T_F by substituting $V(q) = 0$ and using the value ${}_0F_1(;1;0) = 1$. We obtain $\tilde{T}_F(\eta,\zeta) = \eta/2$ or

$$T_F(q,q') = \tfrac{1}{4}(q + q'). \quad (6)$$

The substitution of Eq. (6) back into Eq. (4) gives the free TOA operator.

We now construct the TOA operator T_B across the double potential barrier depicted in Fig. 2. By inspection, the potential $V(q)$ in configuration space is mapped into the same potential in the η coordinate, i.e., a system of barriers with the same potential heights and supports. To obtain $\tilde{T}_0(\eta,\zeta)$, we divide the η coordinate into five nonoverlapping regions demarcated by the edges of the barriers. The kernel $\tilde{T}_0(\eta,\zeta)$ will then have five pieces corresponding to the five regions where η may fall. Use is made of the following identity ${}_0F_1(;,1,x) = I_0(2\sqrt{x})$ for $x > 0$ and ${}_0F_1(;,1,x) = J_0(2\sqrt{|x|})$ for $x < 0$, where $I_0(x)$ and $J_0(x)$ are Bessel functions.

The five pieces corresponding to the five regions are given by

$$\tilde{T}_I(\eta,\zeta) = \frac{\eta}{2},$$

$$\tilde{T}_{II}(\eta,\zeta) = \frac{\eta}{2} - \frac{s_1}{2}[I_0(\kappa_1|\zeta|) - 1],$$

$$\tilde{T}_{III}(\eta,\zeta) = \frac{\eta}{2} - \frac{w_1}{2}[J_0(\kappa_1|\zeta|) - 1],$$

$$\tilde{T}_{IV}(\eta,\zeta) = \frac{\eta}{2} - \frac{(s_1+s_2)}{2}[I_0(\kappa_2|\zeta|) - 1)] - \frac{w_1}{2}\left[{}_0F_1\left(;1;\kappa_{21}^2\zeta^2/4\right) - 1\right],$$

$$\tilde{T}_V(\eta,\zeta) = \frac{\eta}{2} - \frac{w_1}{2}[J_0(\kappa_1|\zeta|) - 1] - \frac{w_2}{2}[J_0(\kappa_2|\zeta|) - 1],$$

where $\kappa_n = \sqrt{2\mu V_n}/\hbar$ for $n = 1,2$, and $\kappa_{21}^2 = 2\mu(V_2 - V_1)/\hbar^2$. Notice that $\tilde{T}_{IV}$ has two possible behaviors depending on the arrangement of the two barriers or on the sign of $(V_2 - V_1)$. The pieces can be written in the form $\tilde{T}_r(\eta,\zeta) = \eta/2 - F_r(\zeta)$; we have in particular, $F_I(\zeta) = 0$. Taking all the regions simultaneously, we write $\tilde{T}(\eta,\zeta) = \eta/2 - F(\eta,\zeta)$ such that $F(\eta,\zeta) = F_r(\zeta)$ whenever η is in region r.

We now show that the classical limit of the TOA operator T_B is the classical TOA at the origin when the classical TOA is defined. The limit is obtained by taking the inverse Weyl-Wigner transform of the kernel $\langle q|\mathsf{T}_0|q'\rangle = (\mu/i\hbar)T_0(q,q')\mathrm{sgn}(q - q')$. It is given by $t_0 = \int_{-\infty}^{\infty}\langle q_0 + \frac{v}{2}|\mathsf{T}_0|q_0 - \frac{v}{2}\rangle e^{-ip_0v/\hbar}dv$, where q_0 and p_0 are now the respective classical position and momentum at $t = 0$. Explicitly the classical limit assumes the form

$$t_0(q_0,p_0) = \frac{\mu}{i\hbar}\int_{-\infty}^{\infty}\tilde{T}_0(q_0,\zeta)\mathrm{sgn}(\zeta)\,e^{-ip_0\zeta/\hbar}d\zeta, \quad (7)$$

where the integral should be understood in the distributional sense.

The classical limit will depend on where the initial position q_0 lies in the five possible regions. The classical limit for an initial position q_0 in region r is obtained from Eq. (7). Explicitly it is given by

$$t_r(q_0,p_0) = \frac{\mu}{i\hbar}\int_{-\infty}^{\infty}\tilde{T}_r(q_0,\zeta)\mathrm{sgn}(\zeta)e^{-ip_0\zeta/\hbar}d\zeta. \quad (8)$$

Integrals involving the Bessel functions are obtained by expanding them in their series representations and then performing a term by term integration using the integral identity

$$\int_{-\infty}^{\infty} v^{m-1}\mathrm{sgn}(v)e^{-ixv}dv = \frac{2(m-1)!}{i^m x^m}, \quad m = 1,2,\ldots \quad (9)$$

(the inverse Fourier transform of [41, p. 360, no. 18]) to obtain the classical limit. The resulting series is summed by analytic extension.

Performing the required operations, the classical limit corresponding to the five possible locations of q_0 is

$$t_I(q_0,p_0) = -\frac{\mu}{p_0}q_0,$$

$$t_{II}(q_0,p_0) = -\frac{\mu}{p_0}(q_0 + s_1) + \frac{\mu}{\sqrt{p_0^2 + 2\mu V_1}}s_1,$$

$$t_{III}(q_0,p_0) = -\frac{\mu}{p_0}(q_0 + w_1) + \frac{\mu}{\sqrt{p_0^2 - 2\mu V_1}}s_1,$$

$$t_{IV}(q_0,p_0) = -\frac{\mu}{p_0}(q_0+s_1+s_2+w_1) + \frac{\mu}{\sqrt{p_0^2+2\mu V_2}}(s_1+s_2) + \frac{\mu}{\sqrt{p_0^2 \pm 2\mu\Delta V}}w_1,$$

$$t_V(q_0,p_0) = -\frac{\mu}{p_0}(q_0+w_1+w_2) + \frac{\mu}{\sqrt{p_0^2-2\mu V_1}}w_1 + \frac{\mu}{\sqrt{p_0^2-2\mu V_2}}w_2,$$

where $\Delta V = |V_2 - V_1|$ and the sign in $\sqrt{p_0^2 \pm 2\mu\Delta V}$ takes the sign of $(V_2 - V_1)$.

Clearly t_I is just the free classical TOA across a free region in s_1. On the other hand, the first term of t_{II} is the traversal time on top of the first barrier and the second term is the traversal time across the free segment s_1, so that t_{II} is the TOA at the origin. For t_{III} the first term is the traversal time in the free region before and after the barrier whose length is $(q_0 + w_1)$, on the other hand, the second term is the traversal time on top of the entire length of the first barrier. For t_{IV} the first term is the traversal time on top of the second barrier, the second term is the traversal time for the free regions s_1 and s_2, and the third term is the traversal time across the first barrier. For t_V the first term is the traversal time across the free region, the second term the traversal time across the first barrier, and the third term is the traversal time across the second barrier. Hence the traversal time operator has the correct classical limit.

V. TWO-BARRIER TRAVERSAL TIME CALCULATIONS

Now let the incident wave packet be $\psi(q) = \varphi(q)e^{ik_0q}$ with a momentum expectation value of $\hbar k_0$ and $\varphi(q)$ satisfying $\int_{-\infty}^{+\infty} dq\varphi(q)\varphi'(q) = 0$. The condition on $\varphi(q)$ is necessary to have the desired momentum expectation value $\hbar k_0$. The expectation value of the TOA operator for this given incident wave packet is given by

$$\langle\psi|T|\psi\rangle = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dq'dq\bar{\psi}(q)\psi(q')\frac{\mu}{i\hbar} \times T_0(q,q')\text{sgn}(q-q'). \tag{10}$$

Changing the variables from (q,q') to (η,ζ), one can show that $\langle\psi|T|\psi\rangle = \text{Im}(\tau^*)$ where τ^* is the complex-expected TOA given by

$$\tau^* = -\frac{2\mu}{\hbar}\int_0^{\infty} d\zeta \int_{-\infty}^{+\infty} d\eta\bar{\varphi}\left(\eta-\frac{\zeta}{2}\right)\varphi\left(\eta+\frac{\zeta}{2}\right) \times \tilde{T}_0(\eta,\zeta)e^{ik_0\zeta}. \tag{11}$$

Given that the free and the two-barrier kernels are given by $\tilde{T}_F(\eta,\zeta) = \eta/2$ and $\tilde{T}_B(\eta,\zeta) = \eta/2 - F(\eta,\zeta)$, we obtain the complex TOA difference

$$\Delta\tau^* = -\frac{2\mu}{\hbar}\int_0^{\infty} d\zeta \int_{-\infty}^{+\infty} d\eta\bar{\varphi}\left(\eta-\frac{\zeta}{2}\right) \times \varphi\left(\eta+\frac{\zeta}{2}\right)F(\eta,\zeta)e^{ik_0\zeta}. \tag{12}$$

The measurable TOA difference is given by $\Delta\tau = \text{Im}(\Delta\tau^*)$.

A. Vanishing support inside the barrier region

We now consider the case when the support of $\varphi(q)$ lies entirely to the left of the barrier system, which is the typical situation for a tunneling particle. Under this condition, only region V will contribute in the expectation value. We let $\Phi(\zeta) = \int_{-\infty}^{+\infty} d\eta\bar{\varphi}(\eta-\zeta/2)\varphi(\eta+\zeta/2)$ and $\Phi_r(\zeta) = \int_r d\eta\bar{\varphi}(\eta-\zeta/2)\varphi(\eta+\zeta/2)$, where the integration is performed only in region r. Performing the η integration in Eq. (12) gives $\Phi_V(\zeta)F_V(\zeta)$. We can extend the integration to $+\infty$ since the integrand will just vanish beyond region V. Thus,

$$\Delta\tau_V^* = \sum_{n=1}^{2}\frac{w_n}{v_0}k_0\int_0^{+\infty} d\zeta\, e^{ik_0\zeta}\Phi(\zeta)[1-J_0(\kappa_n\zeta)] \tag{13}$$

where $v_0 = \hbar k_0/\mu$ is the initial velocity of the incident wave packet, and the subscript V is to emphasize that the support of the incident wave packet is entirely in region V. Notice that the obtained complex traversal time difference in Eq. (13) has no dependence on the barrier spacing s_2 nor on the distance of the barrier system to the arrival point s_1. We can attribute this to the cancellation of the s_2 term in the interbarrier traversal time with that of the free TOA, this implies that the quantum particle is behaving as a free particle in this region. Only the barrier widths w_1 and w_2 and their respective potential heights affect the traversal time. This implies that the contiguous and separated barrier system will give the same TOA difference, that is, $\Delta\tau_C = \Delta\tau_S$, so that GHE is not observed in this particular configuration.

Let us consider the physical content of Eq. (13). Similar to the single barrier case with $L = w_1 + w_2$, we can write

$$\Delta\tau_V^* = (L/v_0)Q^* - \sum_{n=1}^{2}(w_n/v_0)R_n^*, \tag{14}$$

where $Q^* = k_0\int_0^{+\infty} e^{ik_0\zeta}\Phi(\zeta)d\zeta$ and $R_n^* = k_0\int_0^{+\infty} e^{ik_0\zeta}\Phi(\zeta)J_0(\kappa_n\zeta)d\zeta$. For arbitrarily large k_0 or for high incident energy $k_0^2\hbar^2/2m$, the first term of Eq. (14) reduces to the free traversal time across a region with length L. Also, if we let $\hbar \to 0$ we will get $(w_n/v_0)\text{Im}R_n^* \sim w_n/v_n$ where v_n is the velocity of the particle on top of the barrier with width w_n. It follows that in the classical limit $R_n = \text{Im}R_n^* \sim v_0/v_n$ identifies R_n as the effective index of refraction of the nth barrier with respect to the incident wave packet and the quantity $(w_n/v_0)R_n$ lends to the interpretation as the quantum traversal time across the barrier. This is just an extension of our results in [32] for the single barrier case. Furthermore, if $\tilde{\psi}(k)$ is the Fourier transform of the incident wave packet $\psi(q)$, then the index of refraction can be cast into the form

$$R_n = k_0\int_{\kappa_n}^{+\infty}\frac{|\tilde{\psi}(k)|^2}{\sqrt{k^2-k_0^2}}dk - k_0\int_{\kappa_n}^{+\infty}\frac{|\tilde{\psi}(-k)|^2}{\sqrt{k^2-k_0^2}}dk. \tag{15}$$

Equation (15) shows that only above-barrier energy components contribute in any measurable traversal time in keeping with the original result in [32]. If the distribution $|\hat{\psi}(k)|^2$ has a compact support in $0 < k < \min\{\kappa_1, \kappa_2\}$, then $\Delta\tau_V > 0$, that is, the tunneling particle arrives earlier compared to the particle traveling freely in space. This result is consistent with the observation made by the authors of [35].

B. Nonvanishing support inside the barrier region

The equality of the contiguous and separated-barrier arrival time difference is due to the absence of the separation parameter s_2 in the relevant kernel $\tilde{T}_V(\eta,\zeta)$. To obtain a traversal time with possible dependence on the separation distance s_2, the incident wave function should have a nonvanishing component inside the barrier system. This can be done if we assume that the support of the incident wave packet extends inside the barrier (in region IV). Upon inspection of the barrier system's kernel it is clear that the expected TOA acquires dependence on the barrier separation. The piecewisely defined $F(\eta,\zeta)$ compelled us to split the η integration in Eq. (12) into five integrals. This means that for the rth region we have $\Phi_r(\zeta)F_r(\zeta)$. Our analysis can be facilitated if we let our incident wave be a Gaussian packet, i.e., $\varphi(q) = (2\pi\sigma^2)^{-1/4}e^{-(q-q_0)^2/(4\sigma^2)}$ where σ^2 is its initial position variance. This will give us $\Phi(\zeta) = e^{-\zeta^2/(8\sigma^2)}$ and $\Phi_r(\zeta) = p_r\Phi(\zeta)$ where

$$p_r = (2\pi\sigma^2)^{-\frac{1}{2}} \int_r e^{-(\eta-q_0)^2/(2\sigma^2)} d\eta, \tag{16}$$

which we recognize as the probability of finding the particle in region r at time $t = 0$.

To guarantee that only below-the-barrier energy components contribute in the traversal time, we will let $V_1 = V_2 = V$ and take V to go arbitrarily large. In this regime, we have ${}_0F_1(;1;\kappa_{21}^2\zeta^2/4) = 1$ and the dependence of $F_{IV}(\zeta)$ on w_1 vanishes. It follows that the kernel $F_{IV}(\zeta) \propto [I_0(\kappa\zeta) - 1] \sim e^{\kappa\zeta}/\sqrt{2\pi\kappa\zeta}$ where $\kappa^2 = 2\mu V/\hbar^2$. Under this condition, the contribution from regions I, II, and III becomes negligible. Upon neglecting the contributions of p_I, p_{II}, and p_{III}, which are also negligible in comparison to p_{IV} and p_V, the TOA difference reduces to

$$\Delta\tau = p_V \cdot \Delta\tau_V + p_{IV} \cdot \Delta\tau_{IV}. \tag{17}$$

For a large value of κ, we can write

$$\Delta\tau_{IV} \sim (s_1 + s_2)\frac{\mu}{\kappa\hbar}e^{2\sigma^2(\kappa^2-k_0^2)} \times \left[\sin(4\sigma^2 k_0\kappa) - \frac{k_0}{2\kappa}\cos(4\sigma^2 k_0\kappa)\right] \tag{18}$$

(the asymptotic approximation of $\Delta\tau_{IV}$ is detailed in the Appendix.)

Notice that the expression in Eq. (18) has a propensity to change in sign, and for sufficiently large κ the sign of $\Delta\tau_{IV}$ will dominate the full $\Delta\tau$ in Eq. (17). Whenever $\Delta\tau_{IV}$ vanishes, only the $\Delta\tau_V$ term will contribute to the overall time of arrival difference and our previous result will hold, i.e., $\Delta\tau_C = \Delta\tau_S$. At this point, we just have to consider two cases, i.e., when $\Delta\tau$ is negative and when it is positive. The case when $\Delta\tau < 0$ means that $\langle\psi|\mathsf{T}_B|\psi\rangle > \langle\psi|\mathsf{T}_F|\psi\rangle$ or the barrier causes some delay in the arrival time of the incident particle. No superluminality is observed for this case, and by setting $s_2 = 0$ in Eq. (18) we will get the inequality $\Delta\tau_S < \Delta\tau_C < 0$, which contradicts the prediction of GHE in Eq. (2). On the other hand, the case $\Delta\tau > 0$ implies two possibilities, that is, the particle arrived earlier upon encountering a barrier system or the arrival time for the barrier case is negative. The latter implies that the particle is already at the arrival point prior to time $t = 0$ and is now moving away from it. We can think of it as a particle being reflected away from the barrier system and moving to the left as if it has already passed the arrival point.

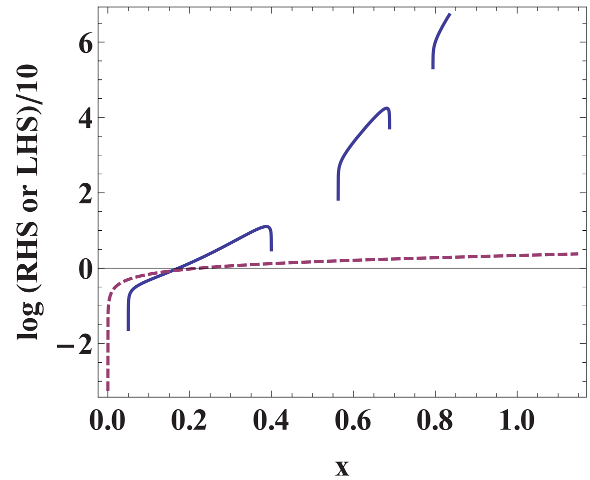

FIG. 3. (Color online) Comparison of the Eq. (20) left-hand side (solid line) and Eq. (20) right-hand side (dashed line). Only those cases where $\Delta\tau > 0$ are shown in the scaled logarithmic plot. The condition in Eq. (20) is satisfied whenever the dashed line is above the solid line.

Whatever the negative arrival time means for the barrier case, it will be meaningless to compare it to the free arrival time, which, for our setup, is always positive. Let us look for the condition such that $\langle\psi|\mathsf{T}_B|\psi\rangle > 0$ or $0 < \Delta\tau < \langle\psi|\mathsf{T}_F|\psi\rangle$ is satisfied. This requires the calculation of the free arrival time using Eq. (10). This was already obtained in [42]

$$\langle\psi|\mathsf{T}_F|\psi\rangle = \frac{|q_0|\mu}{\hbar}\text{erfi}(\sqrt{2}\sigma k_0)\sigma e^{-2\sigma^2 k_0^2}. \tag{19}$$

The comparison between Eqs. (18) and (19) can be simplified by the following parametrization: let $\alpha = \kappa/k_0 > 1$, $\beta = |q_0|/(s_1 + s_2) > 1$, and $x = \sqrt{2}\sigma k_0$. The condition that we seek translates into

$$\left[\frac{\sin(2\alpha x^2)}{\alpha} - \frac{\cos(2\alpha x^2)}{2\alpha^2}\right] e^{\alpha^2 x^2} < \frac{\beta x}{\sqrt{2}}\text{erfi}(x). \tag{20}$$

We plot simultaneously the left- and right-hand sides of Eq. (20) in Fig. 3 using the following parameters: $\alpha = 10$ and $\beta = 25$.

The condition above is satisfied whenever the solid line graph is below the dashed line graph. This only happens in the lowest interval where $\sin(2\alpha x^2) > 0$, i.e., $x \in (0, \sqrt{\pi/(2\alpha)})$, and note that this region becomes smaller as α is increased. For this small interval where $\langle\psi|\mathsf{T}_B|\psi\rangle > 0$ holds, we obtain an inequality that is consistent with GHE. However, the assumption that κ is large requires a large value for α, making the region where the inequality of the time differences consistent with Eq. (2) very small. Outside this region the left-hand side (LHS) becomes greater than the right-hand side (RHS) of Eq. (20), which will result into a negative barrier-case arrival time. Thus, for the case when the support of the incident wave packet extends inside the barrier region, the predicted tunneling phenomenon of the GHE is satisfied only when $k_0 < (\pi/4)(1/\kappa\sigma^2)$ for some large finite κ. Otherwise, the arrival time $\langle\psi|\mathsf{T}_B|\psi\rangle$ is negative and the particle will most likely be reflected by the barrier.

VI. CONCLUSION

We have applied the TOA approach to the double-barrier quantum traversal time problem. We find that, for incident (initial) wave packets with support outside the barrier system, the expected arrival time is independent of the barrier separation and that the particle in the interbarrier region is essentially free. However, for wave packets with support extending inside the leading barrier, the expected arrival time becomes dependent on the barrier separation. The former case violates the GHE strict inequality in Eq. (2) where the TOA difference of the contiguous and of the separated barrier configurations are the same. The GHE inequality is observed in the latter case only when the initial momentum of the incident particle satisfies $k_0 < (\pi/4)(1/\kappa\sigma^2)$.

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APPENDIX

Let $\Delta\tau_{IV} = (s_1 + s_2)(\mu/\hbar)I$. The factor I is the integral

$$I = \int_0^\infty d\zeta\, \sin(k_0\zeta)\frac{e^{-\zeta^2/(8\sigma^2)+\kappa\zeta}}{\sqrt{2\pi\kappa\zeta}}. \tag{A1}$$

Upon evaluating the integral I for large κ we will get the asymptotic form of $\Delta\tau_{IV}$ in Eq. (18). The asymptotic form of I is outlined as follows. First, completing the squares of the exponents will give a factor of $e^{2\sigma^2\kappa^2}$. Then, changing variables from ζ to $u = \zeta - 4\sigma^2\kappa$ will allow us to rewrite the integral in the form

$$I = \int_0^{4\sigma^2\kappa} [f(-u) + f(u)]e^{-u^2/(8\sigma^2)}du + \int_{4\sigma^2\kappa}^{+\infty} f(u)e^{-u^2/(8\sigma^2)}du, \tag{A2}$$

where

$$f(u) = \frac{1}{2\sigma\kappa\sqrt{2\pi}}\frac{\sin(4\sigma^2k_0\kappa + k_0u)}{\sqrt{1+u/4\sigma^2\kappa}}.$$

As κ increases indefinitely, the last integral in Eq. (A2) becomes negligible. This will give us

$$I \propto \int_0^x du\left[\frac{\sin(k_0x - k_0u)}{(1-u/x)^{1/2}} + \frac{\sin(k_0x + k_0u)}{(1+u/x)^{1/2}}\right]e^{-yu^2}, \tag{A3}$$

where we set $x = 4\sigma^2\kappa$ and $y = 1/(8\sigma^2)$. Since $0 < u < x$ in the first integral, then we can use

$$\left(1 \pm \frac{u}{x}\right)^{-1/2} \sim 1 \mp \frac{1}{2}\frac{u}{x} + \frac{3}{8}\left(\frac{u}{x}\right)^2 \mp \frac{5}{16}\left(\frac{u}{x}\right)^3 + \cdots. \tag{A4}$$

Plug this expansion back to Eq. (A3) and group according to powers of (u/x). Using the identity $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$, the combination of the sine functions for the even powers of (u/x) becomes $2\sin(k_0x)\cos(k_0u)$ and for odd powers of (u/x) we have $-2\cos(k_0x)\sin(k_0u)$. The integral I can now be evaluated as $I \sim I_1 + I_2$ where

$$\begin{aligned} I_1 &= 2\sin(k_0x)\int_0^x du[1+O(x^{-2})]e^{-yu^2}\cos(k_0u) \\ &\sim 2\sin(k_0x)\int_0^\infty du e^{-yu^2}\cos(k_0u) + O(x^{-2}) \\ &\sim \sqrt{\frac{\pi}{y}}\sin(k_0x)e^{-k_0^2/(4y)} + O(x^{-2}) \end{aligned}$$

and

$$\begin{aligned} I_2 &= -\frac{\cos(k_0x)}{x}\int_0^x du[1+O(x^{-2})]e^{-yu^2}u\sin(k_0u) \\ &\sim -\frac{\cos(k_0x)}{x}\int_0^\infty du e^{-yu^2}u\sin(k_0u) + O(x^{-3}) \\ &\sim -\sqrt{\frac{\pi}{y}}\frac{k_0}{4xy}\cos(k_0x)e^{-k_0^2/(4y)} + O(x^{-3}). \end{aligned}$$

The second lines in I_1 and I_2 are obtained by taking $x \to \infty$. The improper integrals are evaluated using the known identities. Putting all the dropped constants back and using the original set of variables we will finally get

$$I \sim \frac{1}{\kappa}\left[\sin(4\sigma^2k_0\kappa) - \frac{k_0}{2\kappa}\cos(4\sigma^2k_0\kappa)\right] \times e^{2\sigma^2(\kappa^2-k_0^2)}[1+O(\kappa^{-2})] \tag{A5}$$

and the TOA difference $\Delta\tau_{IV}$ reads

$$\Delta\tau_{IV} \sim (s_1+s_2)\frac{\mu}{\hbar\kappa}\left[\sin(4\sigma^2k_0\kappa) - \frac{k_0}{2\kappa}\cos(4\sigma^2k_0\kappa)\right] \times e^{2\sigma^2(\kappa^2-k_0^2)}[1+O(\kappa^{-2})].$$

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