Coherent-state path integrals in the continuum

G. Kordas,[1] S. I. Mistakidis,[2,1] and A. I. Karanikas[1]

[1] *University of Athens, Physics Department, Panepistimiopolis, Ilissia, 15771 Athens, Greece*

[2] *Zentrum für Optische Quantentechnologien, Universität Hamburg, Luruper Chaussee 149, 22761 Hamburg, Germany*

(Received 4 July 2014; published 3 September 2014)

We discuss the time continuous path integration in the coherent-state basis in a way that is free from inconsistencies. Employing this notion we reproduce known and exact results working directly in the continuum. Such a formalism can set the basis to develop perturbative and nonperturbative approximations already known in the quantum-field-theory community. These techniques can be proven useful in a great variety of problems where bosonic Hamiltonians are used.

DOI: 10.1103/PhysRevA.90.032104 PACS number(s): 03.65.Db, 03.65.Sq, 67.85.Hj

I. INTRODUCTION

The widely known path-integral formalism that was pioneered by Feynman [1,2] almost 70 years ago has been proven an extremely helpful tool for understanding and handling quantum mechanics, quantum field theory, statistical mechanics, even polymer physics and financial markets [3]. The introduction of the overcomplete base of coherent states [4–9] has expanded the concept of path integration into a complexified phase space enlarging its range of possible applications in many areas of physics and chemistry, mainly as a tool for semiclassical approximations. The path integration in terms of coherent states has been discussed in detail in many excellent papers [4–13]. In most of them both the definition and the calculations are based on lattice regularization and the continuum limit is taken only after the relevant calculations have been performed. On the other hand, quantitative differences with exact results have been reported [14] when attempts have been made to handle coherent-state path integrals and perform calculations directly in the continuum. A recent attempt [15] to solve the problem offers only corrections to a questionable leading term and does not give a definitive solution. However, the continuum form of coherent-state-based path integration has been extensively used in quantum field theory for perturbative approximations, for resuming perturbative series or for applying nonperturbative techniques. In this sense, the time continuous integration in a complexified phase space is problematic.

When dealing with path-integral expressions in the continuum we have to take into account that such expressions must be considered as formal unless a definite regularization prescription has been given [9]. In this work we undertake the task of establishing a time continuous formulation of path integration in the coherent-state basis and a corresponding time-sliced definition. In the context of the proposed formulation, the path integration can be performed directly in the continuum without facing inconsistencies and reproduces the exact results at least for the cases in which the relevant Hamiltonian is expressed as a polynomial of creation and annihilation operators. Such bosonic Hamiltonians are used in a great variety of important physical problems, e.g., ultracold atoms in optical lattices [16], cavity optomechanical systems [17,18], nonequilibrium transport [19,20], and other phenomena [21–23]. Thus, our formalism may be proven a powerful tool both for analytical and numerical applications since it allows the use of the quantum-field-theory toolbox. These techniques may be proven helpful for extending the study of many-body dynamics beyond the usual approximate methods.

The paper is organized as follows. In Sec. II we reproduce known results, such as the partition function for the simple case of a harmonic oscillator, using path integration in the complexified phase space. Then, in Sec. III we calculate the partition function for the case of the one-site Bose-Hubbard (BH) model with time continuous coherent-state path integrals, while in Sec. IV we use this method in order to find the exact expression for the propagator. Finally, in Sec. V we discuss the semiclassical calculation for a Hamiltonian that depends only on the number operator. We summarize our findings and give an outlook in Sec. VI.

II. A SIMPLE EXAMPLE

To set the stage, we begin with the trivial case of a harmonic oscillator:

$$\hat{H}_0 = \frac{\hat{p}^2}{2} + \frac{\hat{q}^2}{2}. \tag{1}$$

The partition function of this system, $Z_0 = \mathrm{Tr}e^{-\beta\hat{H}_0} = \sum_{n=0}^{\infty} e^{-\beta(n+1/2)}$, can be expressed as a Feynman phase-space integral:

$$Z_0 = \int \mathcal{D}p \int_{q(0)=q(\beta)} \mathcal{D}q \, \exp\left\{-\int_0^\beta d\tau[-ip\dot{q} + H_0(p,q)]\right\}$$
$$= \frac{e^{-\beta/2}}{1-e^{-\beta/2}} = \sum_{n=0}^{\infty} e^{-\beta(n+1/2)}. \tag{2}$$

The integral in the left-hand side (lhs) of the above expression acquires a full meaning through its time-sliced definition. However, in the simple case of the harmonic oscillator, the result of Eq. (2) can be derived directly in the continuum [3]. In the phase-space path integral that appears in Eq. (2) we can make the canonical change of variables:

$$q = \frac{1}{\sqrt{2}}(z^* + z), \quad p = \frac{i}{\sqrt{2}}(z^* - z). \tag{3}$$

In terms of these complex variables, Eq. (2) is transcribed into the following form:

$$Z_0 = \int_{\text{periodic}} \mathcal{D}^2 z \exp\left\{-\int_0^\beta d\tau \left[\frac{1}{2}(z^*\dot{z} - \dot{z}^* z) + |z|^2\right]\right\}$$
$$= \sum_{n=0}^{\infty} e^{-\beta(n+1/2)}. \tag{4}$$

A comment is needed at this point. In the phase-space integral (2) the integration over $q(\tau)$ is restricted by the periodic condition $q(0) = q(\beta)$ while the $p(\tau)$ integration is unrestricted. For the time-sliced expression that defines the integral, this means that we are dealing with $(q_0,\dots,q_N; q_0 = q_N)$ "position" and $(p_1,\dots,p_N)$ "momentum" integrations. To arrive at the periodic conditions accompanying the integral (4), one [3] introduces a fictitious p_0 variable which is set identically equal to p_N.

However, the partition function (4) can also be calculated by using the coherent-state basis:

$$Z_0 = \int \frac{dzdz^*}{2\pi i} \langle z|\, e^{-\beta \hat{H}_0} |z\rangle$$
$$= e^{-\beta/2} \int \frac{dzdz^*}{2\pi i} \langle z| e^{-\beta \hat{a}^\dagger \hat{a}} \,|z\rangle \,. \tag{5}$$

Splitting the exponential into N factors and using the following resolution of the identity operator in terms of coherent states [10–12],

$$\hat{I} = \int \frac{d^2 z}{\pi} |z\rangle\langle z| := \int \frac{dzdz^*}{2\pi i} |z\rangle\langle z|$$
$$= \int \frac{d\mathrm{Re}z\, d\mathrm{Im}z}{\pi} |z\rangle\langle z|, \tag{6}$$

we arrive at the expression

$$\langle z| e^{-\beta \hat{a}^\dagger \hat{a}} |z\rangle = \lim_{N\to\infty} \prod_{j=1}^{N-1} \int \frac{dz_j dz_j^*}{2\pi i} e^{-f_0(z^*,z)}, \tag{7}$$

where the exponent has the form

$$f_0(z^*,z) = \sum_{j=0}^{N-1} \left[\frac{1}{2}(z_{j+1} - z_j) z_{j+1}^* - \frac{1}{2}(z_{j+1}^* - z_j^*) z_j + \varepsilon z_{j+1}^* z_j\right], \tag{8}$$

and $\varepsilon = \beta/N$. Note the boundary conditions in Eq. (7) that follow from the trace operation $z_N^* = z^*, z_0 = z$. The integrations can be explicitly performed [10,11], and comparing the result with Eq. (4) we conclude that

$$\int_{\text{periodic}} \mathcal{D}^2 z \exp\left\{-\int_0^\beta d\tau \left[\frac{1}{2}(z^*\dot{z} - \dot{z}^* z) + |z|^2\right]\right\}$$
$$= e^{-\beta/2} \lim_{N\to\infty} \prod_{j=0}^{N} \int \frac{dz_j dz_j^*}{2\pi i} e^{-f_0(z^*,z)}. \tag{9}$$

It is a simple exercise [3] to confirm that the factor appearing in the right-hand side (rhs) of the last equation can be absorbed into the discretized expression by symmetrizing the time slicing of the Hamiltonian from $z_{j+1}^* z_j$ to $z_j^* z_j$:

$$\int_{\text{periodic}} \mathcal{D}^2 z \exp\left\{-\int_0^\beta d\tau \left[\frac{1}{2}(z^*\dot{z} - \dot{z}^* z) + |z|^2\right]\right\}$$
$$= \lim_{N\to\infty} \prod_{j=0}^{N} \int \frac{dz_j dz_j^*}{2\pi i} \exp\left[-f_0^{(s)}(z^*,z)\right], \tag{10}$$

where

$$f_0^{(s)}(z^*,z) = \sum_{j=0}^{N-1} \left[\frac{1}{2}(z_{j+1} - z_j) z_{j+1}^* - \frac{1}{2}(z_{j+1}^* - z_j^*) z_j + \varepsilon z_j^* z_j\right]. \tag{11}$$

Despite the fact that the two sides in Eq. (10) have been calculated independently, we consider this relation as a *definition* in the sense that it gives a concrete meaning to the formal integration over paths that go through a complexified phase space.

As a definition, Eq. (10) can also be read from a different point of view. Suppose that we are given the normal ordered Hamiltonian $\hat{H}_1 = \hat{a}^\dagger \hat{a}$ and we want to find the relevant time continuous coherent-state path integral. The previous analysis dictates that we must begin by finding the position-momentum expression for the Hamiltonian in hand $\hat{H}_1 = \hat{p}^2/2 + \hat{q}^2/2 - 1/2$. Then, we have to construct the Feynman phase-space path integral in which this Hamiltonian assumes its classical version $H_1^F = p^2/2 + q^2/2 - 1/2$. Making in this integral the variable change (3) we get $H_1^F = |z|^2 - 1/2$, thus obtaining the continuous path integral we are looking for. The discretized definition of this integral can be read from Eq. (10):

$$\mathrm{Tr} e^{-\beta \hat{H}_1} = \int_{\text{periodic}} \mathcal{D}^2 z\, e^{-\int_0^\beta d\tau [\frac{1}{2}(z^*\dot{z} - \dot{z}^* z) + H_1^F(z^*,z)]}$$
$$= e^{\beta/2} \lim_{N\to\infty} \prod_{j=0}^{N} \int \frac{dz_j dz_j^*}{2\pi i} \exp\left[-f_0^{(s)}(z^*,z)\right]. \tag{12}$$

In this trivial example it is useful to point out that, although we began from a normal ordered Hamiltonian, the Hamiltonian entering into the continuous path integral is the Weyl symbol $H_W(z^*,z)$, which, in the present case, coincides with $H_1^F(z^*,z)$.

III. THE ONE-SITE BOSE-HUBBARD MODEL

As a less trivial example let us consider the one-site BH model:

$$\hat{H}_{\mathrm{BH}} = -\mu \hat{n} + \frac{U}{2} \hat{n}(\hat{n} - 1), \tag{13}$$

where $\hat{n} = \hat{a}^\dagger \hat{a}$ denotes the particle number operator, μ is the chemical potential, and U is the corresponding interparticle interaction strength. The partition function of this system is readily seen to be

$$Z_{\mathrm{BH}} = \mathrm{Tr} e^{-\beta \hat{H}_{\mathrm{BH}}} = \sum_{n=0}^{\infty} e^{-\beta[-\mu n + \frac{U}{2} n(n-1)]}. \tag{14}$$

The same result can be obtained by going directly through path integration. As the above discussion has shown, the route begins by using the "position" and "momentum" operators to rewrite Eq. (13) in the form

$$\hat{H}_{\text{BH}} = -\frac{1}{2}(\mu + U)(\hat{p}^2 + \hat{q}^2) + \frac{U}{8}(\hat{p}^2 + \hat{q}^2)^2 + \frac{\mu}{2} + \frac{3U}{8}. \tag{15}$$

The partition function of the system can now be expressed as a Feynman phase-space path integral:

$$Z_{\text{BH}} = \int \mathcal{D}p \int_{q(0)=q(\beta)} \mathcal{D}q \, \exp\left\{-ip\dot{q} + H^F_{\text{BH}}(p,q)\right\}. \tag{16}$$

It is obvious that, in the last expression, H^F_{BH} stands for the classical version of the quantum Hamiltonian (15). Introducing the complex variables (3), we obtain

$$\begin{aligned} Z_{\text{BH}} &= e^{-\beta(\frac{\mu}{2}+\frac{3U}{8})} \\ &\quad \times \int_{\text{periodic}} \mathcal{D}^2 z \, e^{-\int_0^\beta d\tau [\frac{1}{2}(z^*\dot{z} - \dot{z}^* z) - (\mu+U)|z|^2 + \frac{U}{2}|z|^4]} \\ &= e^{-\beta(\frac{\mu}{2}+\frac{3U}{8})} \lim_{N\to\infty} \prod_{j=0}^{N} \int \frac{dz_j dz_j^*}{2\pi i} e^{-f^{(s)}_{\text{BH}}(z^*,z)}, \end{aligned} \tag{17}$$

where

$$\begin{aligned} f^{(s)}_{\text{BH}}(z^*,z) = \sum_{j=0}^{N-1} \bigg[&\frac{1}{2}(z_{j+1} - z_j) z^*_{j+1} \\ &- \frac{1}{2}(z^*_{j+1} - z^*_j) z_j - \varepsilon(\mu + U)|z_j|^2 + \varepsilon \frac{U}{2}|z_j|^4 \bigg]. \end{aligned} \tag{18}$$

We shall prove that the above integral can be exactly calculated yielding the result of Eq. (14). Before this, however, a comment is in order. The Hamiltonian entering in the last expression,

$$H^F_{\text{BH}}(z^*,z) = -(\mu + U)|z|^2 + \frac{U}{2}|z|^4 + \frac{\mu}{2} + \frac{3U}{8}, \tag{19}$$

constitutes (apart from a constant) the Weyl-symbol Hamiltonian $H_{\text{BH},W}$ for the system under consideration. To understand this point we must take a closer look at the proposed technique that follows the route

$$\hat{H}(\hat{a}^\dagger,\hat{a}) \to \hat{H}(\hat{q},\hat{p}) \to H^F(q,p), \tag{20}$$

which is a recipe for associating an arbitrary quantum Hamiltonian with a classical function. The key observation is that when the quantum Hamiltonian is a polynomial in $\hat{a}$ and $\hat{a}^\dagger$ the respective time slicing of the Feynman path integrals [3] leads to expressions that differ from the Wigner transformation

$$H_W(p,q) = \int_{-\infty}^{\infty} ds e^{ips} \left\langle q - \frac{s}{2} \middle| \hat{H} \middle| q + \frac{s}{2} \right\rangle, \tag{21}$$

which defines the Weyl symbol, by at most a constant.

The calculation of the integral (17) proceeds with the use of a Hubbard-Stratonovich [24–28] transformation. This can be realized by the introduction of the collective field $\zeta = |z|^2$ and the use of the functional identities

$$\begin{aligned} 1 &= \int \mathcal{D}\zeta \, \delta[\zeta - |z|^2], \\ \delta[\zeta - |z|^2] &= \int \mathcal{D}\sigma \, e^{-i\int_0^\beta d\tau \sigma(\zeta - |z|^2)}. \end{aligned} \tag{22}$$

In this way the integral under consideration takes the form

$$\begin{aligned} Z_{\text{BH}} = e^{-\beta(\frac{\mu}{2}+\frac{3U}{8})} &\int \mathcal{D}\zeta \int \mathcal{D}\sigma \\ &\times e^{-i\int_0^\beta d\tau \sigma\zeta - \frac{U}{2}\int_0^\beta d\tau \zeta^2 + (\mu+U)\int_0^\beta d\tau \zeta} \\ &\times \int_{\text{periodic}} \mathcal{D}^2 z \, e^{-\int_0^\beta d\tau [\frac{1}{2}(z^*\dot{z} - \dot{z}^* z) - i\sigma|z|^2]}. \end{aligned} \tag{23}$$

Here, the last functional integration can be performed directly in the continuum [3]. The result reads as follows:

$$\int_{\text{periodic}} \mathcal{D}^2 z \, e^{-\int_0^\beta d\tau [\frac{1}{2}(z^*\dot{z} - \dot{z}^* z) - i\sigma|z|^2]} = \frac{e^{\frac{i}{2}\int_0^\beta d\tau\sigma}}{1 - e^{i\int_0^\beta d\tau\sigma}}. \tag{24}$$

Inserting this into Eq. (23), and assuming that a small positive imaginary part accompanies the field σ, we can immediately find that

$$\begin{aligned} Z_{\text{BH}} &= e^{-\beta(\frac{\mu}{2}+\frac{3U}{8})} \int \mathcal{D}\zeta \int \mathcal{D}\sigma \\ &\quad \times e^{-i\int_0^\beta d\tau\sigma\zeta - \frac{U}{2}\int_0^\beta d\tau\zeta^2 + (U+\mu)\int_0^\beta d\tau\zeta} \sum_{n=0}^{\infty} e^{i(n+\frac{1}{2})\int_0^\beta d\tau\sigma} \\ &= e^{-\beta(\frac{\mu}{2}+\frac{3U}{8})} \int \mathcal{D}\zeta \, e^{-\frac{U}{2}\int_0^\beta d\tau\zeta^2 + (U+\mu)\int_0^\beta d\tau\zeta} \\ &\quad \times \sum_{n=0}^{\infty} \int \mathcal{D}\sigma \, e^{-i\int_0^\beta d\tau\sigma(\zeta - n - 1/2)}. \end{aligned} \tag{25}$$

The integration over the field σ results in a functional delta function that forces the field ζ to be a constant: $\zeta = n + 1/2$. Thus we get

$$\begin{aligned} Z_{\text{BH}} &= e^{-\beta\frac{3U}{8}} \sum_{n=0}^{\infty} e^{-\frac{U}{2}\beta(n+\frac{1}{2})^2 + U\beta(n+\frac{1}{2}) + \mu\beta n} \\ &= \sum_{n=0}^{\infty} e^{-\beta[-\mu n + \frac{U}{2}n(n-1)]}. \end{aligned} \tag{26}$$

Before proceeding, a comment is needed. Let us suppose that one tries to calculate the integral (17) by using polar coordinates $z = \sqrt{r}e^{i\theta}$ [14]. In this case the continuum action is supposed to have the form

$$\begin{aligned} &\int_0^\beta d\tau \left[ir\dot{\theta} - (\mu + U) r + \frac{U}{2} r^2 \right] \\ &\quad = ir(\beta)\theta(\beta) - ir(0)\theta(0) \\ &\qquad + \int_0^\beta d\tau \left[-i\dot{r}\theta - (\mu + U) r + \frac{U}{2} r^2 \right]. \end{aligned} \tag{27}$$

The measure of the functional integration is taken to be

$$\int \mathcal{D}^2 z = \int \mathcal{D}r\mathcal{D}\theta = \lim_{N\to\infty}\prod_{j=0}^{N}\int_0^\infty dr_j \int_0^{2\pi}\frac{d\theta_j}{2\pi}. \quad (28)$$

The integral over θ ensures that r is a constant and the first term in the lhs of Eq. (28) forces this constant to be an integer. In this manner, one arrives at the wrong conclusion that

$$Z_{\rm BH} = e^{-\beta(\frac{\mu}{2}+\frac{3U}{8})}\sum_{n=0}^{\infty} e^{-(\mu+U)n\beta-\frac{U}{2}n^2\beta}. \quad (29)$$

The problem has nothing to do with the BH path integral (17); it persists even for the trivial case of the simple harmonic oscillator in Eq. (4) and the well-known result of Eq. (14) is not reproduced. The culprit for these wrong results is the fact that the parameter $\theta(t)$, being the phase of $z(t)$, is a multivalued function: At every instant t it is possible to add an arbitrary integer multiple of 2π without changing $e^{i\theta(t)}$. Thus the use of the Leibnitz rule that led to the expression (28) was completely illegal [29]. The problem persists even in the discrete version of the relevant integral: A calculation based on the use of polar coordinates fails to reproduce the correct continuum limit. The proper way to take into account the periodicity of z is by writing [3]

$$z(\tau) = \frac{1}{\sqrt{\beta}}\sum_{m=-\infty}^{\infty} z_m e^{-i\frac{2\pi m}{\beta}\tau}. \quad (30)$$

In this way, the correct results emerge in both the continuum and the discretized versions of the path integral.

IV. CORRELATION FUNCTIONS

As long as we are interested in the partition function of a system, the measure of integration in terms of the (p,q) variables can be immediately translated into the measure in terms of the (z,z^*) variables. The situation changes when we are interested in calculating path integrals with specific boundary conditions in the complexified phase space. This kind of calculation is tightly related with correlation functions that are the basic tools needed in any actual calculation pertaining to systems with interactions.

We can express propagators in the coherent-state language beginning with the definition

$$\langle z_b|\,\hat{U}(T,0)\,|z_a\rangle = \int_{\substack{z^*(T)=z_b^*\\ z(0)=z_a}} \mathcal{D}^2 z\; e^{-\Gamma_{ba}} e^{i\int_0^T dt[\frac{i}{2}(z^*\dot{z}-\dot{z}^*z)-H^F(z^*,z)]}. \quad (31)$$

In this expression we have denoted the time evolution operator as

$$\hat{U}(t_b,t_a) = \hat{T}\exp\left\{-i\int_0^T dt\,\hat{H}(t)\right\}, \quad (32)$$

and we have used the abbreviation

$$\Gamma_{ba} = \tfrac{1}{2}(|z_b|^2+|z_a|^2) - \tfrac{1}{2}[z_b^* z(T) + z^*(0)z_a]. \quad (33)$$

The interpretation of Eq. (31) is the following: In the lhs one begins by dividing the time interval $(T,0)$ into small pieces $\varepsilon = T/N$; inserting in each step the coherent-state resolution of the identity operator and following the standard [10,11] procedure leads to the symmetric time-sliced version of the coherent-state path integral. The limit $N\to\infty$ of this discretized expression defines the path integral that appears in the rhs in Eq. (31).

The consequences of the definition (31) can be trivially checked in the case of a harmonic oscillator with a frequency ω. Starting from the rhs we solve the classical equations of motion with the boundary conditions $z_{\rm cl}^*(T) = z_b^*, z_{\rm cl}(0) = z_a$, finding that

$$z_{\rm cl} = z_a e^{i\omega t}, \quad z_{\rm cl}^* = z_b^* e^{-i\omega(T-t)}. \quad (34)$$

Then we perform the replacements $z \to z + z_{cl}$ and $z^* \to z^* + z_{\rm cl}^*$ in order to find

$$\int_{z_b^*,z_a} \mathcal{D}^2 z\, e^{-\Gamma_{ab}} e^{i\int_0^T dt[\frac{i}{2}(z^*\dot{z}-\dot{z}^*z)+\omega|z|^2]} = \exp\left\{z_b^* z_a e^{i\omega T} - \frac{1}{2}(|z_b|^2+|z_a|^2)\right\} \times \int_{z_b^*=z_a=0}\mathcal{D}^2 z\, e^{i\int_0^T dt[\frac{i}{2}(z^*\dot{z}-\dot{z}^*z)+\omega|z|^2]}. \quad (35)$$

According to Eq. (31) the functional integral in the rhs of Eq. (35) is the vacuum expectation value of the time evolution operator of the harmonic oscillator:

$$\langle 0|\,\hat{U}(T,0)\,|0\rangle = e^{-i\omega T/2}. \quad (36)$$

Inserting Eq. (36) into Eq. (35) we can derive the harmonic oscillator propagator in the coherent-state representation. This result could also have been produced [10,11] directly from the lhs of the definition (31).

Another simple case in which the definition (31) can be used for calculations directly in the continuum is the case of the BH model of Eq. (13). In this framework, the propagator

$$K_{ba} = \langle z_b|\, e^{-iT\hat{H}_{\rm BH}}\,|z_a\rangle, \quad (37)$$

is immediately seen to have the form

$$K_{ba} = \sum_{n,m}\langle z_b|n\rangle\langle n|e^{-iT\hat{H}_{\rm BH}}|m\rangle\langle m|z_a\rangle = e^{-\frac{1}{2}(|z_b|^2+|z_a|^2)}\sum_n \frac{(z_b^* z_a)^n}{n!} e^{iT\mu n-\frac{iU}{2}n(n-1)}. \quad (38)$$

Then, using the identity

$$e^{-i\frac{TU}{2}n(n-1)} = e^{i\frac{UT}{8}}e^{-i\frac{TU}{2}(n-1/2)^2} = e^{i\frac{UT}{8}}\sqrt{\frac{T}{2\pi iU}}\int_{-\infty}^{\infty} d\omega e^{i\frac{T}{2U}\omega^2+iT\omega(n-1/2)}, \quad (39)$$

we can rewrite the propagator into the following exact form [14]:

$$K_{ba} = e^{i\frac{UT}{8}}\sqrt{\frac{T}{2\pi iU}}\int_{-\infty}^{\infty} d\omega\exp\left\{i\frac{T}{2U}\omega^2 - \frac{i\omega T}{2} + z_b^* z_a e^{i(\omega+\mu)T} - \frac{1}{2}(|z_b|^2+|z_a|^2)\right\}. \quad (40)$$

We can arrive at the same result starting from the functional integral

$$K_{ba}=\int_{\substack{z^*(T)=z_b^*\\ z(0)=z_a}}\mathcal{D}^2z\ e^{-\Gamma_{ba}}e^{i\int_0^T dt[\frac{i}{2}(z^*\dot{z}-\dot{z}^*z)-H^F_{\rm BH}(z^*,z)]},\tag{41}$$

in which the Hamiltonian has already be defined in Eq. (19).

Once again, the Hubbard-Stratonovich transformation can be used to recast the integral (41) into the following form:

$$K_{ba}=e^{-iT(\frac{\mu}{2}+\frac{3U}{8})}e^{-\frac{1}{2}(|z_b|^2+|z_a|^2)}\int\mathcal{D}\zeta\int\mathcal{D}\sigma\times e^{-i\int_0^T d\tau\sigma\zeta-i\frac{U}{2}\int_0^T dt\zeta^2+i(\mu+U)\int_0^T dt\zeta}\tilde{K}_{ba},\tag{42}$$

where the kernel reads

$$\tilde{K}_{ba}=\int_{z_b^*,z_a}\mathcal{D}^2z\ e^{i\int_0^T dt[\frac{i}{2}(z^*\dot{z}-\dot{z}^*z)+\sigma|z|^2]+\frac{1}{2}(z_b^*z(T)+z^*(0)z_a)}=\exp\left\{\frac{i}{2}\int_0^T d\tau\sigma+z_b^*z_ae^{i\int_0^T d\tau\sigma}\right\}.\tag{43}$$

Note that in order to arrive at the result indicated in the second line of the above expression we have made the replacements $z\to z+z_{\rm cl}$ and $z^*\to z^*+z^*_{\rm cl}$, where

$$z_{\rm cl}=z_ae^{-\frac{i}{\hbar}\int_0^t dt'\sigma},\quad z^*_{\rm cl}=z_b^*e^{-\frac{i}{\hbar}\int_t^T dt'\sigma}\tag{44}$$

are the solutions of the classical equations of motion, and at the same time we have used the vacuum expectation value of a harmonic oscillator with a time-dependent frequency [3]. In order to proceed further we expand the second term that appears in the exponential factor (43) and insert the result into Eq. (42), where the integration over σ yields the constraint $\zeta=n+1/2$. Thus the propagator now reads

$$\begin{aligned}K_{ba}&=e^{-iT\frac{3U}{8}}e^{-\frac{1}{2}(|z_b|^2+|z_a|^2)}\times\sum_{n=0}^{\infty}\frac{(z_b^*z_a)^n}{n!}e^{iT\mu n-i\frac{U}{2}T(n+\frac{1}{2})^2+iUT(n+\frac{1}{2})}\\&=e^{-iT\frac{3U}{8}}e^{-\frac{1}{2}(|z_b|^2+|z_a|^2)}\int_{-\infty}^{\infty}dxe^{-i\frac{UT}{2}x^2+iTUx}\times\sum_{n=0}^{\infty}\frac{(z_b^*z_ae^{iT\mu})^n}{n!}\delta\left(x-n-\frac{1}{2}\right).\end{aligned}\tag{45}$$

Moreover, by inserting into this expression the identity

$$\delta(x-n-1/2)=T\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega T(x-n-1/2)},\tag{46}$$

we arrive at the exact result

$$\begin{aligned}K_{ba}&=e^{-iT\frac{3U}{8}}e^{-\frac{1}{2}(|z_b|^2+|z_a|^2)}\frac{T}{2\pi}\int_{-\infty}^{\infty}d\omega e^{\frac{i\omega T}{2}+z_b^*z_ae^{iT(\mu+\omega)}}\times\int_{-\infty}^{\infty}dxe^{-i\frac{UT}{2}x^2+iTUx-i\omega Tx}\\&=e^{i\frac{UT}{8}}e^{-\frac{1}{2}(|z_b|^2+|z_a|^2)}\times\sqrt{\frac{T}{2\pi iU}}\int_{-\infty}^{\infty}\omega e^{i\frac{T}{2U}\omega^2-\frac{i\omega T}{2}+z_b^*z_ae^{i(\omega+\mu)T}}.\end{aligned}\tag{47}$$

V. SEMICLASSICAL CALCULATIONS

To probe in a transparent way the classical limit, one can introduce [12,14] the dimensionless parameter h through the rescaling $(\hat{a},\hat{a}^\dagger)\to(\hat{a},\hat{a}^\dagger)/\sqrt{h}$. In this notation $[\hat{a},\hat{a}^\dagger]=h$ and $|z\rangle=e^{-|z|^2/2h}\sum_{n=0}^{\infty}\frac{(z/h)^n}{\sqrt{n!}}|n\rangle$ while the classical limit is achieved at the limit $h\to0$. The quantum BH Hamiltonian (13) is written as $\hat{H}_{\rm BH}=-\mu\hat{n}+\frac{U}{2}\hat{n}(\hat{n}-h)$ and the exact propagator (47) assumes the form

$$K_{ba}=e^{i\frac{hUT}{8}}\sqrt{\frac{T}{2\pi ihU}}\int_{-\infty}^{\infty}d\omega\exp\left\{\frac{1}{h}\Phi_\omega-\frac{i\omega T}{2}\right\},\tag{48}$$

where

$$\Phi_\omega=i\frac{T}{2U}\omega^2+z_b^*z_ae^{i(\omega+\mu)T}-\frac{1}{2}(|z_b|^2+|z_a|^2).\tag{49}$$

At the limit $h\to0$ the integral (48) can be evaluated [14] by finding the stationary points of Φ_ω . At the same result one can arrive starting from the path integral (41) expressed in terms of the rescaled variables:

$$K_{ba}=\int_{\substack{z^*(T)=z_b^*/\sqrt{h}\\ z(0)=z_a/\sqrt{h}}}\mathcal{D}^2z\quad e^{-\Gamma_{ba}/h}\times\exp\left\{\frac{i}{h}\int_0^T dt\left[\frac{i}{2}(z^*\dot{z}-\dot{z}^*z)-H^F_{\rm BH}(|z|^2;h)\right]\right\}.\tag{50}$$

In the above integral the Hamiltonian is the rescaled version of the function appearing in Eq. (19):

$$H^F_{\rm BH}(|z|^2;h)=-(\mu+hU)h|z|^2+h^2\frac{U}{2}|z|^4+\frac{h\mu}{2}+h^2\frac{3U}{8}.\tag{51}$$

We shall consider here the case of an arbitrary Hamiltonian as long as it has the form $\hat{H}=H(\hat{n})$. In this case $H^F=H^F(|z|^2;h)$ and the correlation function (42) can be written as follows:

$$K_{ba}=e^{-\frac{1}{2h}(|z_b|^2+|z_a|^2)}\times\int\mathcal{D}\zeta\int\mathcal{D}\sigma e^{-\frac{i}{h}\int_0^T d\tau\sigma\zeta-\frac{i}{h}\int_0^T dtH^F(\zeta;h)}\tilde{K}_{ba}(h).\tag{52}$$

The factor $\tilde{K}_{ba}$ in the last expression is the rescaled version of Eq. (43):

$$\tilde{K}_{ba}(h)=\exp\left\{\frac{i}{2h}\int_0^T d\tau\sigma+\frac{z_b^*z_a}{h}e^{\frac{i}{h}\int_0^T d\tau\sigma}\right\}.\tag{53}$$

Inserting Eq. (53) into Eq. (52) and repeating the steps of the previous section we arrive at the following result:

$$K_{ba}=e^{-\frac{1}{2h}(|z_b|^2+|z_a|^2)}\times\sum_{n=0}^{\infty}\frac{(z_b^*z_a/h)^n}{n!}e^{-\frac{i}{h}TH^F(n+\frac{1}{2};h)}.\tag{54}$$

This expression can be compared with the standard semiclassical analysis where one has to solve the classical equations $\dot{z}_{\rm cl}=-i\partial H_F/\partial z^*_{\rm cl}$ and $\dot{z}^*_{\rm cl}=i\partial H_F/\partial z_{\rm cl}$ with boundary conditions $z_{\rm cl}(0)=z_a$ and $z^*_{\rm cl}(T)=z_b^*$, respectively. This task can be considerably simplified by the introduction of an auxiliary field σ that serves as a functional Lagrange multiplier:

$$H^F \to H^F(\zeta) + \frac{\sigma}{h}(\zeta - |z|^2). \tag{55}$$

In this treatment, which is obviously equivalent to the Hubbard-Stratonovich transformation we adopted in the previous sections, the result indicated in Eq. (53) coincides with the careful calculation of the fluctuation determinant presented in [13]. Minimizing the final result with respect to σ one arrives at the expression indicated in Eq. (54).

Supposing that the Hamiltonian we are dealing with is an analytic function of h we write

$$\begin{aligned} H^F(\zeta;h) &= \sum_{k=0}^{N} h^k H_F^{(k)}(\zeta;0), \\ H_F^{(k)}(\zeta;0) &= \frac{1}{k!}\frac{\partial^k}{\partial h^k} H^F(\zeta;h)\bigg|_{h=0}. \end{aligned} \tag{56}$$

If the original Hamiltonian was a polynomial in powers of $|z|^2$ the highest power appearing in each term $H_F^{(k)}(\zeta)$ is ζ^k. Thus,

$$\begin{aligned} H^F(\zeta;h) = {}& h(a_1\zeta + a_0) + h^2(b_2\zeta^2 + b_1\zeta + b_0) \\ &+ \sum_{k=3}^{N} h^k H_F^{(k)}(\zeta;0). \end{aligned} \tag{57}$$

In this equation we wrote

$$\begin{aligned} H_F^{(1)}(0;\zeta) &= a_1\zeta + a_0, \\ H_F^{(2)}(0;\zeta) &= b_2\zeta^2 + b_1\zeta + b_0. \end{aligned} \tag{58}$$

Neglecting the last term in Eq. (57) and following the algebra presented in the previous section we can find

$$\begin{aligned} K_{ba} = {}& e^{-iT(a_0+a_1/2)-iTh(b_0-b_1^2/4b_2)} \\ &\times\sqrt{\frac{T}{4\pi i h b_2}}\int_{-\infty}^{\infty} d\omega e^{\frac{1}{h}\Phi_\omega + \frac{i\omega T}{2}(1+b_1/b_2)}. \end{aligned} \tag{59}$$

Here,

$$\Phi_\omega = i\frac{T}{4b_2}\omega^2 + z_b^* z_a e^{i(\omega-a_1)T} - \frac{1}{2}(|z_b|^2 + |z_a|^2). \tag{60}$$

In the BH model the relevant parameters are

$$\begin{aligned} &a_0 = \mu/2, \quad a_1 = -\mu, \\ &b_0 = 3U/8, \quad b_1 = -U, \quad b_2 = U/2, \end{aligned} \tag{61}$$

and the result of Eq. (47) is retrieved. It is obvious that having neglected the higher-order terms in the expansion (57) the integral in Eq. (59) must be evaluated in terms of the stationary points of the function (60).

VI. CONCLUSIONS AND OUTLOOK

Second quantized Hamiltonians for bosonic systems are used in a great variety of physical problems [16–23]. Furthermore, the experimental advances call for theoretical methods that will allow the study of the many-body dynamics deep in the quantum regime. In the present work we have introduced a method for defining and handling time continuous coherent-state path integrals without facing inconsistencies. Such a path-integral formalism opens, in principle at least, new possibilities for the analytical study of a variety of second quantized models. The aim of this paper is not the presentation of new results. It is, rather, an attempt to set a solid basis for very interesting calculations which can go beyond the already known approximate methods.

In our approach, the Hamiltonian that weights the paths in the complexified phase space is produced through three simple steps. In the first step one rewrites the second quantized Hamiltonian $\hat{H}(\hat{a},\hat{a}^\dagger)$ in terms of "position" and "momentum" operators. The second step consists of constructing the Feynman phase-space integral in which the classical form of this Hamiltonian $H^F(p,q)$ enters. The third step is just a canonical change of variables that produces the final form $H^F(z,z^*)$ which enters into the time continuous form of the coherent-state path integral. We have followed this simple method for the case of the one-site BH model and we have derived the correct expressions for the partition function and the propagator of the system. We have also discussed a semiclassical calculation pertaining to a Hamiltonian that depends only on the number operator.

In a forthcoming study, we intend to use nonperturbative techniques, already known in the quantum-field-theory community, in order to study the dynamics in realistic systems and compare our results with that of already known approximate methods. Another direction would be the combination of our technique with the so-called Feynman-Vernon influence functional formalism for studying open many-body systems, like the open Bose-Hubbard chains or cavity systems [30], which are of increasing interest both theoretically [31,32] and experimentally [33–35].

[1] R. P. Feynman, Rev. Mod. Phys. **20**, 367 (1948).
[2] R. P. Feynman, Phys. Rev. **84**, 108 (1951).
[3] H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets* (World Scientific, Singapore, 2006).
[4] J. R. Klauder and B. S. Skagerstam, *Coherent States, Applications in Physics and Mathematical Physics* (World Scientific, Singapore, 1985).
[5] J. R. Klauder, in *Path Integrals*, edited by G. J. Papadopoulos and J. T. Devreese, NATO Advanced Study Institute Series B: Physics (Plenum, New York, 1978), p. 5.
[6] J. R. Klauder, Phys. Rev. D **19**, 2349 (1979).
[7] J. R. Klauder, in *Random Media, Random Media*, edited by G. Papanicolaou (Springer, New York, 1987).
[8] Y. Weissman, J. Chem. Phys. **76**, 4067 (1982).
[9] J. R. Klauder, in *A Garden of Quanta*, edited by J. Arafume *et al.* (World Scientific, Singapore, 2003).
[10] L. Ademir, Jr. Xavier and M. A. M. de Aguiar, Phys. Rev. A **54**, 1808 (1996).
[11] M. Baranger, M. A. M. de Aguiar, F. Keck, H. J. Korsch, and B. Schellhaab, J. Phys. A **34**, 7227 (2001).
[12] E. A. Kohetov, J. Phys. A **31**, 4473 (1998).

[13] M. Stone, K. Park, and A. Garg, J. Math. Phys. (NY) **41**, 8025 (2000).
[14] J. H. Wilson and V. Galitski, Phys. Rev. Lett. **106**, 110401 (2011).
[15] Y. Yanay and E. J. Mueller, arXiv:1212.4802v2 [quant-ph].
[16] D. Jaksch, C. Bruder, J. I. Cirac, C. W. Gardiner, and P. Zoller, Phys. Rev. Lett. **81**, 3108 (1998).
[17] M. Aspelmeyer, T. J. Kippenberg, and F. Marquardt, arXiv:1303.0733 [Rev. Mod. Phys. (to be published)].
[18] A. Tomadin, S. Diehl, M. D. Lukin, P. Rabl, and P. Zoller, Phys. Rev. A **86**, 033821 (2012).
[19] J. Dujardin, A. Saenz, and P. Schlagheck, Appl. Phys. B (2014), doi:10.1007/s00340-014-5804-3.
[20] A. Ivanov, G. Kordas, A. Komnik, and S. Wimberger, Eur. Phys. J. B **86**, 345 (2013).
[21] E. M. Graefe, U. Guenther, H. J. Korsch, and A. E. Niederle, J. Phys. A **41**, 255206 (2008).
[22] M.-X. Huo and D. G. Angelakis, Phys. Rev. A **85**, 023821 (2012).
[23] M. Leib and M. J. Hartmann, New J. Phys. **12**, 093031 (2010).
[24] R. Stratonovich, Sov. Phys. Dokl. **2**, 416 (1957).
[25] J. Hubbard, Phys. Rev. Lett. **3**, 77 (1959).
[26] B. Muhlshegel, J. Math. Phys. **3**, 522 (1962).
[27] M. B. Halpern, Nucl. Phys. B **173**, 504 (1980).
[28] A. Jevicki and B. Sakita, Nucl. Phys. B **185**, 89 (1981).
[29] H. Kleinert, *Multivalued Fields in Condensed Matter, Electromagnetism and Gravitation* (World Scientific, Singapore, 2008).
[30] F. Brennecke, R. Mottl, K. Baumann, R. Landig, T. Donner, and T. Esslinger, Proc. Nat. Acad. Sci. USA **110**, 11763 (2013).
[31] D. Witthaut, F. Trimborn, H. Hennig, G. Kordas, T. Geisel, and S. Wimberger, Phys. Rev. A **83**, 063608 (2011).
[32] G. Kordas, S. Wimberger, and D. Witthaut, Europhys. Lett. **100**, 30007 (2012).
[33] W. S. Bakr, A. Peng, M. E. Tai, R. Ma, J. Simon, J. Gillen, S. Foelling, L. Pollet, and M. Greiner, Science **329**, 547 (2010).
[34] G. Barontini, R. Labouvie, F. Stubenrauch, A. Vogler, V. Guarrera, and H. Ott, Phys. Rev. Lett. **110**, 035302 (2013).
[35] M. Endres, M. Cheneau, T. Fukuhara, C. Weitenberg, P. Schauß, C. Gross, L. Mazza, M. C. Bañuls, L. Pollet, I. Bloch, and S. Kuhr, Appl. Phys. B **113**, 27 (2013).