Entanglement monogamy relations of qubit systems

Xue-Na Zhu[1,*] and Shao-Ming Fei[2,3]

[1]*School of Mathematics and Statistics Science, Ludong University, Yantai 264025, China*

[2]*School of Mathematical Sciences, Capital Normal University, Beijing 100048, China*

[3]*Max-Planck-Institute for Mathematics in the Sciences, 04103, Leipzig, Germany*

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We investigate the monogamy relations related to the concurrence and the entanglement of formation. General monogamy inequalities given by the αth power of concurrence and entanglement of formation are presented for N-qubit states. The monogamy relation for entanglement of assistance is also established. Based on these general monogamy relations, the residual entanglement of concurrence and entanglement of formation are studied. Some relations among the residual entanglement, entanglement of assistance, and three tangle are also presented.

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I. INTRODUCTION

Quantum entanglement [1–6] is an essential feature of quantum mechanics, which distinguishes the quantum from the classical world. As one of the fundamental differences between quantum entanglement and classical correlations, a key property of entanglement is that a quantum system entangled with one of the other systems limits its entanglement with the remaining others. The monogamy relations give rise to the structures of entanglement in the multipartite setting. Monogamy is also an essential feature allowing for security in quantum key distribution [7].

For a tripartite system A, B, and C, the monogamy of an entanglement measure ε implies that [8] the entanglement between A and BC satisfies $\varepsilon_{A|BC} \geqslant \varepsilon_{AB} + \varepsilon_{AC}$. Such monogamy relations are not always satisfied by entanglement measures. Although the concurrence C and entanglement of formation E do not satisfy such monogamy inequality, it has been shown that the squared concurrence C^2 [9,10] and the squared entanglement of formation E^2 [11] do satisfy the monogamy relations.

In this Brief Report, we study the general monogamy inequalities satisfied by the αth power of concurrence C^α and the αth power of entanglement of formation E^α. We show that C^α and E^α satisfy the monogamy inequalities for $\alpha \geqslant 2$ and $\alpha \geqslant \sqrt{2}$, respectively. The monogamy relations for the entanglement of assistance are also established. Correspondingly, the residual entanglement of concurrence and entanglement of formation are also investigated.

II. MONOGAMY RELATION OF CONCURRENCE

For a bipartite pure state $|\psi\rangle_{AB}$ in vector space $H_A \otimes H_B$, the concurrence is given by [12–14]

$$C(|\psi\rangle_{AB}) = \sqrt{2\left[1 - \mathrm{Tr}\left(\rho_A^2\right)\right]}, \tag{1}$$

where ρ_A is the reduced density matrix by tracing over the subsystem B, $\rho_A = \mathrm{Tr}_B(|\psi\rangle_{AB}\langle\psi|)$. The concurrence for a tripartite mixed state ρ_{AB} is defined by the convex roof

$$C(\rho_{AB}) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle), \tag{2}$$

where the minimum (infimum) is taken over all possible decompositions of $\rho_{AB} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, with $p_i \geqslant 0$ and $\sum_i p_i = 1$ and $|\psi_i\rangle \in H_A \otimes H_B$.

For a tripartite state $|\psi\rangle_{ABC}$, the concurrence of assistance (CoA) is defined by [15,16]

$$C_a(|\psi\rangle_{ABC}) \equiv C_a(\rho_{AB}) = \max_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle), \tag{3}$$

where the maximum (supremum) is taken over all possible decompositions of $\rho_{AB} = Tr_C(|\psi\rangle_{ABC}\langle\psi|) = \sum_i p_i |\psi_i\rangle_{AB}\langle\psi_i|$. When $\rho_{AB} = |\psi\rangle_{AB}\langle\psi|$ is a pure state, then one has $C(|\psi\rangle_{AB}) = C_a(\rho_{AB})$.

For an N-qubit state $|\psi\rangle_{AB_1,\ldots,B_{N-1}} \in H_A \otimes H_{B_1} \otimes \cdots \otimes H_{B_{N-1}}$, the concurrence $C(|\psi\rangle_{A|B_1,\ldots,B_{N-1}})$ of the state $|\psi\rangle_{A|B_1,\ldots,B_{N-1}}$, viewed as a bipartite with partitions A and $B_1B_2,\ldots,B_{N-1}$, satisfies the Coffman-Kundu-Wootters (CKW) inequality [9,10]

$$C^2_{A|B_1B_2,\ldots,B_{N-1}} \geqslant C^2_{AB_1} + C^2_{AB_2} + \cdots + C^2_{AB_{N-1}}, \tag{4}$$

where $C_{AB_i} = C(\rho_{AB_i})$ is the concurrence of $\rho_{AB_i} = \mathrm{Tr}_{B_1,\ldots,B_{i-1}B_{i+1},\ldots,B_{N-1}}(\rho)$, $C_{A|B_1B_2,\ldots,B_{N-1}} = C(|\psi\rangle_{A|B_1,\ldots,B_{N-1}})$.

Dual to the CKW inequality, the generalized monogamy relation based on the concurrence of assistance was proved in Ref. [17],

$$C^2\left(|\psi\rangle_{A|B_1,\ldots,B_{N-1}}\right) \leqslant \sum_{i=1}^{N-1} C_a^2(\rho_{AB_i}). \tag{5}$$

The inequalities (4) and (5) are valid because, instead of the concurrence and CoA, the squared concurrence and CoA are used. In fact, besides the squared concurrence, one can get the following general monogamy inequalities.

Theorem 1. For any $2 \otimes 2 \otimes \cdots \otimes 2$ mixed state $\rho \in H_A \otimes H_{B_1} \otimes \cdots \otimes H_{B_{N-1}}$, we have

$$C^\alpha_{A|B_1B_2,\ldots,B_{N-1}} \geqslant C^\alpha_{AB_1} + \cdots + C^\alpha_{AB_{N-1}} \tag{6}$$

for all $\alpha \geqslant 2$.

Proof. For arbitrary $2 \otimes 2 \otimes 2^{n-2}$ tripartite state ρ_{ABC}, one has [9,18]

$$C^2_{A|BC} \geqslant C^2_{AB} + C^2_{AC}.$$

*Corresponding author: jing_feng1986@126.com

If $\min\{C_{AB},C_{AC}\}=0$, obviously we have $C^{\alpha}_{A|BC}\geqslant C^{\alpha}_{AB}+C^{\alpha}_{AC}$. If $\min\{C_{AB},C_{AC}\}>0$, assuming $C_{AB}\geqslant C_{AC}$, we have

$$C^{\alpha}_{A|BC}\geqslant\left(C^2_{AB}+C^2_{AC}\right)^{\frac{\alpha}{2}}=C^{\alpha}_{AB}\left(1+\frac{C^2_{AC}}{C^2_{AB}}\right)^{\frac{\alpha}{2}}$$
$$\geqslant C^{\alpha}_{AB}\left[1+\left(\frac{C^2_{AC}}{C^2_{AB}}\right)^{\frac{\alpha}{2}}\right]=C^{\alpha}_{AB}+C^{\alpha}_{AC}, \quad (7)$$

where the second inequality is due to the inequality $(1+x)^t\geqslant 1+x^t$ for $x\leqslant 1$ and $t\geqslant 1$.

By partitioning the last qudit system C into two subsystems (a qubit system C_1 and a 2^{n-3}-dimensional qudit system C_2), and using the above inequality repeatedly, one gets (6). ■

Theorem 2. For any $2\otimes 2\otimes\cdots\otimes 2$ mixed state $\rho\in H_A\otimes H_{B_1}\otimes\cdots\otimes H_{B_{N-1}}$ with $C_{AB_i}\neq 0, i=1,\ldots,N-1$, we have

$$C^{\alpha}_{A|B_1,\ldots,B_{N-1}}<C^{\alpha}_{AB_1}+\cdots+C^{\alpha}_{AB_{N-1}} \quad (8)$$

for $\alpha\leqslant 0$.

Proof. Similar to the proof of Theorem 1, we only need to prove the inequality is true for arbitrary $2\otimes 2\otimes 2^{n-2}$ tripartite states

$$C^{\alpha}_{A|BC}\leqslant\left(C^2_{AB}+C^2_{AC}\right)^{\frac{\alpha}{2}}=C^{\alpha}_{AB}\left(1+\frac{C^2_{AC}}{C^2_{AB}}\right)^{\frac{\alpha}{2}}$$
$$<C^{\alpha}_{AB}\left[1+\left(\frac{C^2_{AC}}{C^2_{AB}}\right)^{\frac{\alpha}{2}}\right]=C^{\alpha}_{AB}+C^{\alpha}_{AC}, \quad (9)$$

where the first inequality is due to $\alpha\leqslant 0$ and the second inequality is due to $C^{\alpha}_{AB}>0$ and the inequality $(1+x)^t<1+x^t$ for $x>0$ and $t\leqslant 0$. ■

In (8) we have assumed that all $C_{AB_i}, i=1,\ldots,N-1$ are nonzero. In fact, if one of them is zero, the inequality still holds if one removes this term from the inequality. Namely, if $C_{AB_i}=0$, then one has $C^{\alpha}_{A|B_1,\ldots,B_{N-1}}<C^{\alpha}_{AB_1}+\cdots+C^{\alpha}_{AB_{i-1}}+C^{\alpha}_{AB_{i+1}}+\cdots+C^{\alpha}_{AB_{N-1}}$ for $\alpha\leqslant 0$.

Theorem 1 shows that the αth power of concurrence C^{α} satisfies the monogamy inequality (6) for $\alpha\geqslant 2$. While Theorem 2 shows that for $\alpha\leqslant 0$, the inequality is reversed. However, for $0<\alpha<2$, the situation is not clear. Let us consider the three-qubit case. Any three-qubit state $|\psi\rangle$ can be written in the generalized Schmidt decomposition [19,20]

$$|\psi\rangle=\lambda_0|000\rangle+\lambda_1e^{i\varphi}|100\rangle+\lambda_2|101\rangle$$
$$+\lambda_3|110\rangle+\lambda_4|111\rangle, \quad (10)$$

where $\lambda_i\geqslant 0$, $i=0,\ldots,4$, and $\sum_{i=0}^{4}\lambda_i^2=1$. From (1) and (2), we have $C_{A|BC}=2\lambda_0\sqrt{\lambda_2^2+\lambda_3^2+\lambda_4^2}, C_{AB}=2\lambda_0\lambda_2$, and $C_{AC}=2\lambda_0\lambda_3$. Without loss of generality, we set $\lambda_0=\cos\theta_0,\lambda_1=\sin\theta_0\cos\theta_1$, $\lambda_2=\sin\theta_0\sin\theta_1\cos\theta_2$, $\lambda_3=\sin\theta_0\sin\theta_1\sin\theta_2\cos\theta_3$, and $\lambda_4=\sin\theta_0\sin\theta_1\sin\theta_2\sin\theta_3$, $\theta_i\in[0,\frac{\pi}{2}]$. Then we have

$$C^{\alpha}_{A|BC}-C^{\alpha}_{AB}-C^{\alpha}_{AC}$$
$$=(2\lambda_0)^{\alpha}\left[\left(\lambda_2^2+\lambda_3^2+\lambda_4^2\right)^{\frac{\alpha}{2}}-\lambda_2^{\alpha}-\lambda_3^{\alpha}\right]$$
$$=(2\lambda_0)^{\alpha}\sin^{\alpha}\theta_0\sin^{\alpha}\theta_1[1-\cos^{\alpha}\theta_2-\sin^{\alpha}\theta_2\cos^{\alpha}\theta_3]. \quad (11)$$

From (11) we have $C^{\alpha}_{A|BC}\geqslant C^{\alpha}_{AB}+C^{\alpha}_{AC}$ for $\alpha\geqslant 2$. While for $\alpha\leqslant 0$ one has $C^{\alpha}_{A|BC}<C^{\alpha}_{AB}+C^{\alpha}_{AC}$. However, for $0<\alpha<2$, one can see that the sign of $(C^{\alpha}_{A|BC}-C^{\alpha}_{AB}-C^{\alpha}_{AC})$ is not certain.

III. RESIDUAL ENTANGLEMENT OF CONCURRENCE

Similar to the three tangle of concurrence, for the three-qubit state $|\psi\rangle_{ABC}\in H_A\otimes H_B\otimes H_C$, we can define the residual entanglement

$$\tau^C_{\alpha}(|\psi\rangle_{ABC})=C^{\alpha}_{A|BC}-C^{\alpha}_{AB}-C^{\alpha}_{AC}, \quad (12)$$

where $\alpha\geqslant 2$.

Theorem 3. For any three-qubit pure state $|\psi\rangle\in H_A\otimes H_B\otimes H_C$:

(1) $|\psi\rangle$ is bipartite separable state if and only if for any $\alpha\geqslant 2$,

$$\tau^C_{\alpha}(|\psi\rangle)=0;$$

(2) $|\psi\rangle$ is genuine entangled if and only if there is an $\alpha\geqslant 2$ such that

$$\tau^C_{\alpha}(|\psi\rangle)>0.$$

Proof. (1) If $|\psi\rangle$ is bipartite separable state, without loss of generality, we assume that $|\psi\rangle$ is a $B|AC$ bipartite separable state, then we have $C_{B|AC}=C_{BA}=C_{BC}=0$. From (10) we have $\lambda_0=0$ and $|\lambda_1\lambda_4e^{i\varphi}-\lambda_2\lambda_3|=0$, or $\lambda_3=\lambda_4=0$ and $|\lambda_1\lambda_4e^{i\varphi}-\lambda_2\lambda_3|=0$. For both the above two cases, we have $C_{A|BC}=C_{AC}$ and $C_{AB}=0$. Therefore $\tau^C_{\alpha}(|\psi\rangle)=0$ for all α.

If $\tau^C_{\alpha}(|\psi\rangle)=0$ for any $\alpha\geqslant 2$, then we obtain $C^{2\alpha}_{A|BC}-C^{2\alpha}_{AB}-C^{2\alpha}_{AC}=(C^{\alpha}_{A|BC})^2-C^{2\alpha}_{AB}-C^{2\alpha}_{AC}=(C^{\alpha}_{AB}+C^{\alpha}_{AC})^2-C^{2\alpha}_{AB}-C^{2\alpha}_{AC}=2C^{\alpha}_{AB}C^{\alpha}_{AC}=0$, i.e., either C^{α}_{AB} or C^{α}_{AC} is zero. Without loss of generality, assuming $C_{AB}=0$, we have $\lambda_0=0$ or $\lambda_3=0$. If $\lambda_0=0$, then $C_{A|BC}=0$. Hence $|\psi\rangle$ is $A|BC$ a bipartite separable state. If $\lambda_0\neq 0$ and $\lambda_3=0$ since $\tau^C_2=0$, we have $\lambda_4=0$. Hence $C_{B|AC}=0$, i.e., $|\psi\rangle$ is $B|AC$ bipartite separable.

(2) By using the above proof and the fact that $\tau^C_{\alpha}(|\psi\rangle)\geqslant 0$ for all $\alpha\geqslant 2$ from Theorem 1, one gets the result directly. ■

Example 1. Let us consider the W state

$$|W\rangle=\frac{1}{\sqrt{3}}(|100\rangle+|010\rangle+|001\rangle). \quad (13)$$

We have $\tau^C_{\alpha}(|W\rangle)=(\frac{2}{\sqrt{3}})^{\alpha}[\sqrt{(\frac{2}{3})^{\alpha}}-2(\frac{1}{\sqrt{3}})^{\alpha}]$. For $\alpha=2$, τ^C_{α} is just the three tangle of concurrence. As $\tau^C_2(|W\rangle)=0$, the three tangle of concurrence cannot capture the genuine

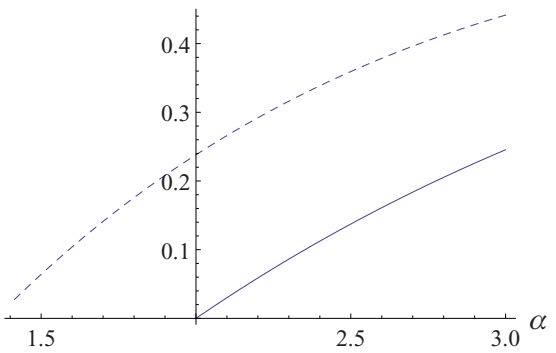

FIG. 1. (Color online) Solid line: $\tau^C_{\alpha}(|W\rangle)$ as a function of α ($\alpha\geqslant 2$); dashed line: $\tau^F_{\alpha}(|W\rangle)$ as a function of α ($\alpha\geqslant\sqrt{2}$).

entanglement of the W state. Nevertheless, for $\alpha > 2$, our residual entanglement of concurrence $\tau_\alpha^C(|W\rangle) > 0$, see Fig. 1.

IV. MONOGAMY INEQUALITY FOR EOF

The entanglement of formation (EoF) [21,22] is a well defined important measure of entanglement for bipartite systems. Let H_A and H_B be m- and n-dimensional ($m \leqslant n$) vector spaces, respectively. The EoF of a pure state $|\psi\rangle \in H_A \otimes H_B$ is defined by

$$E(|\psi\rangle) = S(\rho_A), \tag{14}$$

where $\rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|)$ and $S(\rho) = -\mathrm{Tr}(\rho \log_2 \rho)$. For a bipartite mixed state $\rho_{AB} \in H_A \otimes H_B$, the entanglement of formation is given by

$$E(\rho_{AB}) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i E(|\psi_i\rangle), \tag{15}$$

with the minimum (infimum) taking over all possible decompositions of ρ_{AB} in a mixture of pure states $\rho_{AB} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, where $p_i \geqslant 0$ and $\sum_i p_i = 1$. The corresponding entanglement of assistance (EoA) [23] is defined in terms of the entropy of entanglement [24] for a tripartite pure state $|\psi\rangle_{ABC}$

$$E_a(|\psi\rangle_{ABC}) \equiv E_a(\rho_{AB}) = \max_{\{p_i, |\psi_i\rangle\}} \sum_i p_i E(|\psi_i\rangle), \tag{16}$$

where the maximum (supremum) is taken over all possible decompositions of $\rho_{AB} = Tr_C(|\psi\rangle_{ABC}) = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, with $p_i \geqslant 0$ and $\sum_i p_i = 1$.

Denote $f(x) = H(\frac{1+\sqrt{1-x}}{2})$, where $H(x) = -x \log_2(x) - (1-x)\log_2(1-x)$. From (14) and (15), one has $E(|\varphi\rangle) = f(C^2(|\varphi\rangle))$ for $2 \otimes m$ ($m \geqslant 2$) pure state $|\varphi\rangle$, and $E(\rho) = f(C^2(\rho))$ for two-qubit mixed state ρ [25]. It is obvious that $f(x)$ is a monotonically increasing function for $0 \leqslant x \leqslant 1$. $f(x)$ satisfies the following relations:

$$f^{\sqrt{2}}(x^2 + y^2) \geqslant f^{\sqrt{2}}(x^2) + f^{\sqrt{2}}(y^2), \tag{17}$$

$$f(x^2 + y^2) \leqslant f(x^2) + f(y^2), \tag{18}$$

where $f^{\sqrt{2}}(x^2 + y^2) = [f(x^2 + y^2)]^{\sqrt{2}}$.

It has been shown that the entanglement of formation does not satisfy the inequality $E_{AB} + E_{AC} \leqslant E_{A|BC}$ [26]. In Ref. [27] the authors showed that EoF is a monotonic function $E^2(C^2_{A|B_1B_2,\ldots,B_{N-1}}) \geqslant E^2(\sum_{i=1}^{N-1} C^2_{AB_i})$. It is further proved that for N-qubit systems, one has [11],

$$E^2_{A|B_1B_2,\ldots,B_{N-1}} \geqslant E^2_{AB_1} + E^2_{AB_2} + \cdots + E^2_{AB_{N-1}}.$$

In fact, generally we can prove the following results.

Theorem 4. For any N-qubit mixed state $\rho \in H_A \otimes H_{B_1} \otimes \cdots \otimes H_{B_{N-1}}$, the entanglement of formation $E(\rho)$ satisfies

$$E^\alpha_{A|B_1B_2,\ldots,B_{N-1}} \geqslant E^\alpha_{AB_1} + E^\alpha_{AB_2} + \cdots + E^\alpha_{AB_{N-1}}, \tag{19}$$

where $\alpha \geqslant \sqrt{2}$, $E_{A|B_1B_2,\ldots,B_{N-1}}$ is the entanglement of formation of ρ in bipartite partition $A|B_1B_2,\ldots,B_{N-1}$, and E_{AB_i}, $i = 1,2,\ldots,N-1$, is the entanglement of formation of the mixed states $\rho_{AB_i} = \mathrm{Tr}_{B_1B_2,\ldots,B_{i-1}B_{i+1},\ldots,B_{N-1}}(\rho)$.

Proof. Denote $t = \alpha/\sqrt{2}$. For $\alpha \geqslant \sqrt{2}$, we have

$$\begin{aligned} f^\alpha(x^2 + y^2) &= (f^{\sqrt{2}}(x^2 + y^2))^t \\ &\geqslant (f^{\sqrt{2}}(x^2) + f^{\sqrt{2}}(y^2))^t \\ &\geqslant (f^{\sqrt{2}}(x^2))^t + (f^{\sqrt{2}}(y^2))^t \\ &= f^\alpha(x^2) + f^\alpha(y^2), \end{aligned} \tag{20}$$

where the first inequality is due to the inequality (17), and the second inequality is obtained from a similar consideration in the proof of the second inequality in (7).

Let $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \in H_A \otimes H_{B_1} \otimes H_{B_2} \otimes \cdots \otimes H_{B_{N-1}}$ be the optimal decomposition of $E_{A|B_1B_2,\ldots,B_{N-1}}(\rho)$ for the N-qubit mixed state ρ, we have

$$\begin{aligned} &E_{A|B_1B_2,\ldots,B_{N-1}}(\rho) \\ &= \sum_i p_i E_{A|B_1B_2,\ldots,B_{N-1}}(|\psi_i\rangle) \\ &= \sum_i p_i f\big(C^2_{A|B_1B_2,\ldots,B_{N-1}}(|\psi_i\rangle)\big) \\ &\geqslant f\Big(\sum_i p_i C^2_{A|B_1B_2,\ldots,B_{N-1}}(|\psi_i\rangle)\Big) \\ &\geqslant f\left(\left[\sum_i p_i C_{A|B_1B_2,\ldots,B_{N-1}}(|\psi_i\rangle)\right]^2\right) \\ &\geqslant f\big(C^2_{A|B_1B_2,\ldots,B_{N-1}}(\rho)\big) \\ &\geqslant f\big(C^2_{AB_1} + C^2_{AB_2} + \cdots + C^2_{AB_{N-1}}\big) \\ &\geqslant \sqrt[\alpha]{f^\alpha\big(C^2_{AB_1}\big) + f^\alpha\big(C^2_{AB_2}\big) + \cdots + f^\alpha\big(C^2_{AB_{N-1}}\big)} \\ &= \sqrt[\alpha]{E^\alpha_{AB_1} + E^\alpha_{AB_2} + \cdots + E^\alpha_{AB_{N-1}}}, \end{aligned}$$

where the first inequality is due to that $f(x^2)$ is a convex function. The second inequality is due to the Cauchy-Schwarz inequality: $(\sum_i x_i^2)^{\frac{1}{2}}(\sum_i y_i^2)^{\frac{1}{2}} \geqslant \sum_i x_i y_i$, with $x_i = \sqrt{p_i}$ and $y_i = \sqrt{p_i}\, C_{A|B_1B_2\ldots B_{N-1}}(|\psi_i\rangle)$. Due to the definition of concurrence (2) and that $f(x)$ is a monotonically increasing function, we obtain the third inequality. We have used the monogamy inequality (4) for N-qubit states ρ to obtain the forth inequality. The last inequality is due to the inequality (20). Since for any $2 \otimes 2$ quantum state ρ_{AB_i}, $E(\rho_{AB_i})$ satisfies $E(\rho_{AB_i}) = H(\frac{1+\sqrt{1-C^2(\rho_{AB_i})}}{2}) = f[C^2(\rho_{AB_i})]$, one gets the last equality.

The inequality (19) in Theorem 4 shows that the αth power of EoF satisfies the monogamy inequality for any $\alpha \geqslant \sqrt{2}$, which is a little different from the case of concurrence in which $\alpha \geqslant 2$. As for the entanglement of assistance, we have the following conclusion.

Theorem 5. For any N-qubit pure state $|\psi\rangle \in H_A \otimes H_{B_1} \otimes \cdots \otimes H_{B_{N-1}}$, the entanglement of assistance satisfies

$$E(|\psi\rangle_{A|B_1B_2,\ldots,B_{N-1}}) \leqslant \sum_{i=1}^{N-1} E_a(\rho_{AB_i}), \tag{21}$$

where $E(|\psi\rangle_{A|B_1B_2,\ldots,B_{N-1}})$ is the entanglement of formation of $|\psi\rangle$ in bipartite partition $A|B_1B_2,\ldots,B_{N-1}$, and $\rho_{AB_i} = \mathrm{Tr}_{B_1,\ldots,B_{i-1}B_{i+1},\ldots,B_{N-1}}(|\psi\rangle\langle\psi|)$.

Proof. Let $\rho_{AB} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ be the optimal decomposition of $C_a(\rho_{AB})$. We have

$$\begin{aligned} E_a(\rho_{AB}) &\geqslant \sum_i p_i E(|\psi_i\rangle) = \sum_i p_i f\left(C_a^2(|\psi_i\rangle)\right) \\ &\geqslant f\left[\sum_i p_i C_a^2(|\psi_i\rangle)\right] \geqslant f\left(\left[\sum_i p_i C_a(|\psi_i\rangle)\right]^2\right) \\ &= f\left[C_a^2(\rho_{AB})\right], \end{aligned} \tag{22}$$

where the first equality is due to that, for a pure state, one has $\rho_{AB} = \rho_{AB}^2$ and $C(\rho_{AB}) = C_a(\rho_{AB})$ [17]. The second inequality is due to that $f(x^2)$ is a convex function. The last inequality is due to the Cauchy-Schwarz inequality with $x_i = \sqrt{p_i}$ and $y_i = \sqrt{p_i}\, C_a(|\psi_i\rangle)$.

Therefore for an N-qubit pure state $|\psi\rangle_{AB_1,\ldots,B_{N-1}}$, we have

$$\begin{aligned} E(|\psi\rangle_{A|B_1,\ldots,B_{N-1}}) &= f[C^2(|\psi\rangle_{A|B_1,\ldots,B_{N-1}})] \\ &\leqslant f\left(C_a^2(\rho_{AB_1}) + \cdots + C_a^2(\rho_{AB_{N-1}})\right) \\ &\leqslant f\left(C_a^2(\rho_{AB_1})\right) + \cdots + f\left(C_a^2(\rho_{AB_{N-1}})\right) \\ &\leqslant E_a(\rho_{AB_1}) + \cdots + E_a(\rho_{AB_{N-1}}). \end{aligned}$$

The first inequality is due to (5). We have used the inequality (18) to get the second inequality. The last inequality is due to (22). ■

V. RESIDUAL ENTANGLEMENT OF EOF

Similar to the residual entanglement (12) defined by the αth ($\alpha \geqslant 2$) power of concurrence, we can define the residual entanglement by the αth ($\alpha \geqslant \sqrt{2}$) power of EoF for a three-qubit pure state $|\psi\rangle_{ABC}$

$$\tau_\alpha^E(|\psi\rangle_{ABC}) = E_{A|BC}^\alpha - E_{AB}^\alpha - E_{AC}^\alpha \geqslant 0. \tag{23}$$

As an example, let us consider again the W state (13). We have $E_{AB} = E_{AC} = 0.550048$ and $E_{A|BC} = 0.918296$. Therefore $\tau_\alpha^E = 0.918296^\alpha - 2(0.550048)^\alpha$, see Fig. 1.

Here it should be noted that, different from the residual entanglement of concurrence, the residual entanglement of EoF depends on which qubit is chosen to be A.

In the following we give some relations among the residual entanglement of EoF, entanglement of assistance, and three tangle.

Theorem 6. For a three-qubit pure state $|\psi\rangle_{ABC}$, we have

$$\tau_\alpha^E(|\psi\rangle_{ABC}) \geqslant f^2\left[\tau_2^C(|\psi\rangle_{ABC})\right], \tag{24}$$

and

$$E_a^\alpha(\rho_{AB}) \geqslant E^\alpha(\rho_{AB}) + f^\alpha\left[\tau_2^C(|\psi\rangle_{ABC})\right], \tag{25}$$

where $\alpha \geqslant \sqrt{2}$, $\rho_{AB} = \mathrm{Tr}_C(|\psi\rangle_{ABC}\langle\psi|)$ and $\tau_2^C(|\psi\rangle_{ABC})$ is the three tangle of concurrence.

Proof. According to the definition of $\tau_\alpha^E(|\psi\rangle_{ABC})$, we have

$$\begin{aligned} \tau_\alpha^E(|\psi\rangle_{ABC}) &= E_{A|BC}^\alpha - E_{AB}^\alpha - E_{AC}^\alpha \\ &= f^\alpha\left(C_{A|BC}^2\right) - f^\alpha\left(C_{AB}^2\right) - f^\alpha\left(C_{AC}^2\right) \\ &= f^\alpha\left[C_{AB}^2 + C_{AC}^2 + \tau_2^C(|\psi\rangle_{ABC})\right] \\ &\quad - f^\alpha\left(C_{AB}^2\right) - f^\alpha\left(C_{AC}^2\right) \\ &\geqslant f^\alpha\left[\tau_2^C(|\psi\rangle_{ABC})\right], \end{aligned}$$

where the third equality is due to the definition of the three tangle τ_2^C. We have used (20) to obtain the last inequality.

Accounting to that for a $2 \otimes 2 \otimes m$ quantum pure state $|\psi\rangle_{ABC}$, $C_a^2(\rho_{AB}) = C^2(\rho_{AB}) + \tau_2^C(|\psi\rangle_{ABC})$ [28], we have

$$\begin{aligned} E_a(\rho_{AB}) &\geqslant f\left[C_a^2(\rho_{AB})\right] \\ &= f\left[C^2(\rho_{AB}) + \tau_2^C(|\psi\rangle_{ABC})\right] \\ &\geqslant \sqrt[\alpha]{f^\alpha[C^2(\rho_{AB})] + f^\alpha\left[\tau_2^C(|\psi\rangle_{ABC})\right]} \\ &= \sqrt[\alpha]{E^\alpha(\rho_{AB}) + f^\alpha\left[\tau_2^C(|\psi\rangle_{ABC})\right]}, \end{aligned}$$

where we have used the inequality (22) to obtain the first inequality and (17) to get the last inequality. ■

The relations among entanglement of formation, entanglement of assistance, and three tangle given in Theorem 6 can be used to obtain a lower bound of EoA. Let us consider the following example.

Example 2. Superpositions of the Greenberger-Horne-Zeilinger (GHZ) state and the W state (13)

$$|\Psi\rangle_{ABC} = \sqrt{\frac{1}{2}}|\mathrm{GHZ}\rangle - \sqrt{\frac{1}{2}}|W\rangle, \tag{26}$$

where $|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$. According to Theorem 6 we obtain the lower bound of $E_a(\rho_{AB})$, $E_a(\rho_{AB}) \geqslant \{E^\alpha(\rho_{AB}) + f^\alpha[\tau_2^C(|\psi\rangle_{ABC})]\}^{\frac{1}{\alpha}}$, where $\rho_{AB} = \mathrm{Tr}_C(|\Psi\rangle_{ABC}\langle\Psi|)$, see Fig. 2. From Fig. 2, one gets that the optimal lower bound of $E_a(\rho_{AB})$ is 0.623 at $\alpha = \sqrt{2}$.

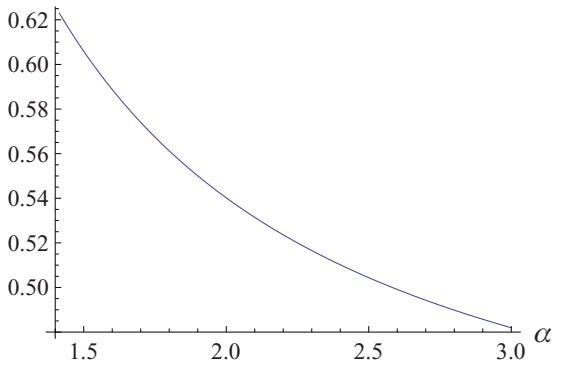

FIG. 2. (Color online) The lower bound of $E_a(\rho_{AB})$ for state (26) with $\alpha \geqslant \sqrt{2}$.

VI. CONCLUSION

Entanglement monogamy is a fundamental property of multipartite entangled states. We have investigated the monogamy relations related to the concurrence and the entanglement of formation generally for N-qubit states. We also proved that the entanglement of assistance satisfies the monogamy inequality $E(|\psi\rangle_{A|B_1B_2,\ldots,B_{N-1}}) \leqslant \sum_{i=1}^{N-1} E_a(\rho_{AB_i})$. To study the genuine

tripartite entanglement, we investigated the residual entanglement of concurrence $\tau_\alpha^C(|\psi\rangle_{ABC})$ and the residual entanglement of entanglement of formation $\tau_\alpha^E(|\psi\rangle_{ABC})$. By exploring the relations among the residual entanglement, entanglement of assistance, and three tangle, we have presented a bound of $E_a(\rho)$. Our approach may be used to study further the monogamy properties related to other quantum entanglement measures such as negativity and to quantum correlations such as quantum discord.

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