Discrete-Gauss states and the generation of focusing dark beams

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Discrete-Gauss states are a new class of Gaussian solutions of the free Schrödinger equation owning discrete rotational symmetry. They are obtained by acting with a discrete deformation operator onto Laguerre-Gauss modes. We present a general analytical construction of these states and show the necessary and sufficient condition for them to host embedded dark beam structures. We unveil the intimate connection between discrete rotational symmetry, orbital angular momentum, and the generation of focusing dark beams. The distinguishing features of focusing dark beams are discussed. The potential applications of discrete-Gauss states in advanced optical trapping and quantum information processing are also briefly discussed.

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I. INTRODUCTION

In quantum mechanics, Gaussian pure states are represented by Gaussian wave functions in position or momentum variables [1]. Pure Gaussian states play an important role in nonrelativistic quantum mechanics because their evolution is also described by simple Gaussian wave packets, which, in addition, minimize the Heisenberg uncertainty relation [2]. In classical optics, the concept of Gaussian beam is ubiquitous, since it represents an excellent approximation for the spatial distribution of many realistic light beams in the so-called paraxial approximation [3]. Gaussian beams are solutions of the paraxial scalar wave equation (PWE) for the optical field, formally identical to the two-dimensional (2D) linear Schrödinger equation (2DLSE) for the wave function in quantum mechanics [4]. For this reason, the free spatial propagation of these beams is described by means of Gaussian wave functions with identical properties of Gaussian wave packets in quantum mechanics [4]. Besides, the fact that Gaussian beams present simple transformation rules under the action of arbitrary optical elements makes them a very convenient tool for the description of a wide variety of optical systems [3].

It is also known in quantum mechanics that the 2DLSE admits vortex line solutions with quantized orbital angular momentum (OAM) presenting nontrivial dynamics [5]. In classical and quantum optics, solutions of the PWE with well-defined OAM play also an important role [6,7]. In the free propagation case, the PWE supports Gaussian solutions with well-defined OAM, which are the optical counterparts of vortex lines in quantum mechanics. Mathematically, these solutions are given by the so-called Laguerre-Gauss (LG) modes [3]. LG modes are eigenstates of the OAM operator and, consequently, also of the $O(2)$ continuous rotation group operator. They present a phase singularity located at the axis of symmetry. Since the field intensity is zero at the singularity, the associated vortex line forms a *dark ray* propagating in a straight line. The flux around the singularity forms an optical vortex and it is quantized in such a way that the associated topological charge equals the OAM of the LG mode [6,7].

Nevertheless, in more recent years it has been proven that the 2DLSE (from now on, we use this notation to refer also to the PWE) admits more complex solutions with more intricate phase profiles. In this way, multisingular solutions forming dark ray bundles, or dark beams, in $(2+1)$D have been reported in the context of quantum mechanics [5] and in optics. In the latter case, a considerable variety of dark beam solutions, based on Gaussian LG modes have been reported [8–19]. Solutions of the 2DLSE with an intricate dark ray structure forming knots and loops can also be obtained by superposition of LG modes [20–23]. Even nontrivial dark ray solutions of full Maxwell's equations can be approximated by superpositions of LG modes [24–27].

Closely related to the appearance of multisingular solutions is the phenomenon of the breaking of continuous rotational symmetry. The breaking of $O(2)$ symmetry into discrete rotational symmetry in the *nonlinear* 2D Schrödinger equation is responsible for the so-called vortex transmutation rule [28,29], which is univocally linked to the generation of multisingular solutions [30,31]. However, the generation of these multisingular solutions—in the form of symmetric dark ray bundles—using media with discrete rotational symmetry has proven to be an essentially *linear* property of the 2D Schrödinger equation [32,33]. Recently, both the vortex transmutation rule and the generation of off-axis singularities forming straight dark rays have been experimentally demonstrated in optics for free linear propagation using discrete diffractive optical elements (DOE) [34,35]. Remarkably, these experimental techniques based on discrete diffractive elements represent a simple form of generation of multisingular Gaussian beams.

Multisingular solutions, particularly those forming part of a Gaussian beam, are excellent candidates for applications in optical trapping. This is so because it is known since long ago that the momentum of light can be used for the acceleration, trapping, and levitation of particles by means of radiation pressure [36–38]. Gradient forces generated by single beams exhibiting adequate intensity gradients constitute the physical mechanism on which optical tweezers are based [39,40]. Furthermore, it has been recently proven that not only intensity but phase gradients can provide useful optical forces for optical trapping, including the transfer of the OAM of light to particles [41]. Moreover, optical forces arising from phase gradients can be used complementarily to intensity-gradient traps to design force profiles for improved optical trapping [42]. In this way, the control of the properties of both the phase and the intensity profiles of an optical beam turns out to be an essential ingredient for advanced optical

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trapping [43]. The possibility of generating optical beams with a controllable and rich phase and intensity structure is then a source for potential applications in this field. Easily manipulable multisingular Gaussian beams can thus play an important role in this context.

Multisingular Gaussian states with an embedded nontrivial dark beam structure can also play a relevant and complementary role in quantum information experiments. OAM states of light, physically implemented as Gaussian LG modes, have been proposed to realize high-dimensional quantum spaces for quantum information applications [44]. In a similar manner to OAM states, other bases of the Hilbert space of the solutions of the 2DLSE can be also used for similar purposes. In this context, recent experiments in which photons associated to a particular type of multisingular Gaussian modes (Ince-Gauss modes) have been entangled demonstrate the feasibility and the potentiality of this approach [45].

In this paper we show an explicit construction of multisingular Gaussian solutions of the 2DLSE owning discrete rotational symmetry of *any* order, the so-called discrete-Gauss (DG) states. The construction of these new DG states is general and systematic thus permitting one to unequivocally elucidate the intimate relation between discrete rotational symmetry and the generation of multisingular solutions. We shall show the necessary and sufficient conditions to generate multisingular Gaussian solutions by the action of an operator that breaks continuous rotational symmetry and which can be easily implemented using diffractive optical elements. Likewise, we will analyze the nature of the rich focusing dark beam structures, i.e., symmetric dark ray bundles with a focusing point, which can be embedded within DG states. Finally, we shall also demonstrate that this set constitutes a basis of the Hilbert space of solutions of the 2DLSE by showing that they verify a biorthogonal relation, completely analogous to that fulfilled by LG modes.

II. DISCRETE DEFORMATION OPERATOR

We start by writing the free 2DLSE in complex variables ($w = x + iy, \overline{w} = x - iy$) and use τ as a evolution parameter:

$$-i\frac{\partial \phi}{\partial \tau} + \frac{\partial^2 \phi}{\partial w \partial \overline{w}} = 0. \tag{1}$$

In optics, $\tau = \lambda z/\pi$, λ being the wavelength of light and z the axial coordinate. In quantum mechanics, $\tau = (2\hbar/M)t$, where M is the particle mass, $\hbar$ is the Planck's constant, and t is time. The equation above admits a rotationally symmetric Gaussian wave-packet solution [3]:

$$\phi_{00}(w,\overline{w},\tau) = \frac{i\tau_R}{q(\tau)} \exp\left(-\frac{iw\overline{w}}{q(\tau)}\right), \tag{2}$$

where $q(\tau) = \tau + i\tau_R$ (in optics $\tau_R = \lambda z_R/\pi$, where z_R is the Rayleigh length). The complex-argument "elegant" Laguerre-Gauss (LG) modes can be constructed by reiteratively applying the differential operators $\partial/\partial w$ and $\partial/\partial \overline{w}$ to the fundamental solution ϕ_{00} [46,47]. For interested readers, the relation between elegant and "standard" Hermite-Gauss and Laguerre-Gauss modes is extensively discussed in Siegman's book [3]. In this sense, all the LG modes entering our subsequent analyses will be considered in their elegant version.

LG modes are eigenfunctions of the third component of the orbital angular momentum operator (OAM), $\hat{L}_z \phi_{lp}^{\rm LG} = l \phi_{lp}^{\rm LG}$. LG modes with $l \neq 0$ present a single phase singularity at the origin, where the axis of rotational symmetry is also located. The topological charge of this single phase singularity [$q = (2\pi)^{-1} \oint d\mathbf{l} \cdot \nabla \arg \phi_{lp}^{\rm LG}$ calculated along a circuit enclosing it] equals its OAM, $q = l$. The value of the field at the $\tau = 0$ plane determines completely the solution for every value of τ. For LG modes, this plane is particularly characteristic since all modes reach here, like the generating ϕ_{00} function, their minimal width (in optics, this plane is known as the waist). In addition, at $\tau = 0$ LG modes with $p = 0$ take the simple form of a Gaussian vortex $\phi_{l0}^{\rm LG}(0) \sim \Omega_w^l \phi_{00}(0)$, where $\Omega_w^l \equiv \{w^{|l|}\,(l > 0), \overline{w}^{|l|}\,(l < 0)\}$. The Gaussian vortex is obtained thus by simply multiplying $\phi_{00}(0)$ by a transfer function t, given in this case by Ω_w^l. Interestingly enough, for LG modes with $p = 0$ the difference between their standard and elegant versions does not exist. The reason is that this difference stems from the real (for standard) or complex (for elegant) nature of the argument of the generalized Laguerre polynomials appearing in the expression of LG modes, so that since for $p = 0$ all these polynomials equal just 1, this distinction completely disappears [3].

We analyze now a related but different problem. The $\tau = 0$ condition can be chosen differently to change the rotational symmetry properties of the solution. This can be done by properly selecting the t function. Let us consider the most general case given by a condition of the form

$$\phi(w,\overline{w},0) = t(w,\overline{w})\phi_{00}(w,\overline{w},0), \tag{3}$$

in which now $t(w,\overline{w})$ is an arbitrary nonsingular analytical function in w and $\overline{w}$. We write this function as $t = a \exp iV$, where a is an *arbitrary* complex function and V is a real analytical function. The $\tau = 0$ condition is then

$$\phi(0) = \exp iV(w,\overline{w})\psi(0), \tag{4}$$

where $\psi = a\phi_{00}$ is now an arbitrary complex function. Although we are finding solutions of the Schrödinger equation with no potential, the particular form of the $\tau = 0$ condition is equivalent to the action of the quasi-instantaneous potential V at $\tau = 0$ on the wave function ψ [33]. We consider then a real local potential owning purely discrete rotational symmetry of order N (i.e., invariance under the $\mathcal{C}_N$ point symmetry group) with respect to the origin at leading order:

$$V(w,\overline{w}) = v(w^N + \overline{w}^N), \tag{5}$$

where $v = \varepsilon\nu$, ε being a small interval $\tau = \varepsilon > 0$ indicating the extension of the quasi-instantaneous action of the potential in the τ domain, whereas ν is the leading-order symmetry parameter characterizing the $N_{\rm th}$-fold symmetry of the local potential V. The $\mathcal{C}_N$ point symmetry group of discrete rotations of Nth order is cyclic, which means that all the N elements of this group can be obtained by repeatedly applying the group operation on a single element of the group, called its generator G_N. Under the action of this elementary transformation G_N, the complex coordinates change as

$$w \xrightarrow{G_N} \epsilon_N w, \quad \overline{w} \xrightarrow{G_N} \epsilon_N^* \overline{w}, \tag{6}$$

where $\epsilon_N = \exp(2\pi i/N)$ is the complex elementary finite rotation of Nth order. Thus w^N and $\overline{w}^N$ are $\mathcal{C}_N$ invariants and, therefore, V is an explicitly invariant potential. The potential V is real if we choose v to be real, as we shall do. Moreover, V is the most general form (up to an irrelevant constant) of the leading term in a local Taylor expansion of a purely invariant $\mathcal{C}_N$ real potential around $w = 0$. It is easy to prove that the alternative combination $vw^N + v^*\overline{w}^N$ with complex v is equivalent to Eq. (5) up to a global rotation.

A. Construction of the discrete deformation operator

We consider now that the function ψ in Eq. (4) is a Gaussian vortex, i.e., an elegant LG mode with $p = 0$, and use V as a way to change the rotational properties of the solution at the waist of this mode. For this reason, we define a new modified function at $\tau = 0$ given by

$$\begin{aligned}\phi_{lNv}(w,\overline{w},0) &= e^{iV(w,\overline{w})}\phi^{\rm LG}_{l0}(w,\overline{w},0)\\ &\overset{O(v^2)}{\sim} [1 + iv(w^N + \overline{w}^N)]\phi^{\rm LG}_{l0}(w,\overline{w},0).\end{aligned} \tag{7}$$

This function gives the first-order correction in the small interval $\tau = \varepsilon$ for the output field scattered by the quasi-instantaneous discrete potential V. A solution of the 2DLSE (1) fulfilling the condition (7) is obtained simply by formally replacing w and $\overline{w}$ by the differential operators:

$$\hat{l}_+ \equiv \hat{w} + \tau\hat{\overline{p}}, \quad \hat{l}_- \equiv \hat{\overline{w}} + \tau\hat{p}, \tag{8}$$

where $\hat{w}$ and $\hat{\overline{w}}$ are the complex position operators and $\hat{p} = -i\partial/\partial w$ and $\hat{\overline{p}} = -i\partial/\partial\overline{w}$ are their corresponding momentum ones. These operators belong to the Lie algebra of symmetries of the free (2 + 1)D Schrödinger equation [48,49]. In this way, the modified field by the presence of the discrete potential takes the form

$$\phi^{\rm DG}_{lNv}(w,\overline{w},\tau) \overset{O(v^2)}{\sim} [1 + iv(\hat{l}_+^N + \hat{l}_-^N)]\phi^{\rm LG}_{l0}(w,\overline{w},\tau). \tag{9}$$

The previous replacement rule can be proven as follows. Since complex position and momentum operators fulfill standard commutation relations, $[\hat{w},\hat{p}] = i = [\hat{\overline{w}},\hat{\overline{p}}]$, the commutation relations with respect the evolution operator of Eq. (1) $U(\tau) = \exp(i\tau\hat{H})$, where $\hat{H} = \hat{p}\hat{\overline{p}}$, are

$$[\hat{w},U(\tau)] = -\tau\hat{\overline{p}}U(\tau), \quad [\hat{\overline{w}},U(\tau)] = -\tau\hat{p}U(\tau).$$

These relations determine that the position operators $\hat{w}$ and $\hat{\overline{w}}$ transform into $\hat{l}_+$ and $\hat{l}_-$, respectively, under the action of the evolution operator. So, we have

$$U(\tau)\hat{w} = \hat{l}_+U(\tau), \quad U(\tau)\hat{\overline{w}} = \hat{l}_-U(\tau). \tag{10}$$

This property justifies the step from Eq. (7) to Eq. (9) since $\phi^{\rm DG}_{lNv}(\tau) = U(\tau)\phi_{lNv}(0)$.

We can see the important role played by the operator

$$\hat{D}_v(N) \equiv \exp[iv(\hat{l}_+^N + \hat{l}_-^N)], \tag{11}$$

which transforms an LG mode into a new Gaussian state, which in *bra-ket* notation can be written as

$$|{\rm DG}(l,p,N)\rangle_v = \hat{D}_v(N)|{\rm LG}(l,p)\rangle. \tag{12}$$

Inasmuch as $|{\rm LG}(l,p)\rangle$ is a solution of the free 2DLSE and $U(\tau)e^{iV} = \hat{D}_vU(\tau)$, we immediately show that the new state $|{\rm DG}(l,p;N)\rangle_v$ also verifies this equation.

According to the complex transformations (6) and to their definition (8), the operators $\hat{l}_\pm$ transform tensorially under $\mathcal{C}_N$ rotations as $\hat{l}_+ \overset{G_N}{\to} \epsilon_N\hat{l}_+$ and $\hat{l}_- \overset{G_N}{\to} \epsilon_N^*\hat{l}_-$, where $\epsilon_N = \exp i2\pi/N$. As a consequence, $\hat{D}_v$ is an invariant operator under the $\mathcal{C}_N$ group. Moreover, the operator $\hat{D}_v$ is *unitary*: $\hat{D}_v^\dagger\hat{D}_v = 1$. This is a consequence of the relation $\hat{l}_+ = \hat{l}_-^\dagger$, which makes the operator $v(\hat{l}_+^N + \hat{l}_-^N)$ self-adjoint for v real. Since $\hat{D}_v$ changes the rotational symmetry properties of the LG mode from continuous to discrete, we refer to it as the *discrete deformation* operator.

B. Physical implementation of the discrete deformation operator

In optics, it is well established that the experimental generation of LG modes carrying nonzero OAM can be realized by means of diffractive optical elements (DOE) acting on fundamental Gaussian beams [50,51]. DOE's are thin optical elements that modify the phase of an optical field at the plane where they are located in the way given by Eq. (3), thus transforming the subsequent diffraction of the beam. A spiral phase plate is a DOE consisting of a thin layer of variable width in the form of a spiral surface with a step discontinuity [50,51]. Since the spiral surface forms the period of a helix, the width of this DOE is proportional to the polar angle. Thus an optical field traversing the spiral phase plate experiences a phase shift proportional to this angle at the output. In this way a spiral phase plate with the appropriate design can be used to convert a fundamental $|{\rm LG}(0,0)\rangle$ mode into a higher-order $|{\rm LG}(l,0)\rangle$ mode with topological charge $q = l$. Mathematically, the action of a spiral phase plate can be represented by a transfer function of the form $t = \exp(il\theta) = w/|w|$ in Eq. (3). The transfer function t defined at $\tau = 0$ acts then as a mode converter transforming the fundamental $|{\rm LG}(0,0)\rangle$ mode for $\tau < 0$ into an $|{\rm LG}(l,0)\rangle$ mode for $\tau > 0$. This standard conversion is represented in Fig. 1(a). In a completely analogous way, we can find a DOE acting at $\tau = 0$ whose mathematical representation is a transfer function $t = \exp(iV)$—like in Eq. (4)—generated now by the real local potential with discrete rotational symmetry (5). In this case, a higher-order LG mode for $\tau < 0$ is converted into a DG state for $\tau > 0$ according to the construction in terms of the discrete deformation operator (9) or (12), as represented in Fig. 1(b). Different designs of DOE's can be used to generate this type of discrete-symmetric transfer function and we will refer to them generically as discrete symmetry diffractive elements (DSDE) [33]. Remarkably, the mode conversion presented in Fig. 1(b) has been recently experimentally demonstrated using as a DSDE a square matrix of black dots impressed in a transparent substrate [35], in which the discrete modulation introduced by the dots is responsible for the discrete symmetry of the transfer function. Analogous experiments using instead polygonal lenses as DSDE's—in which discrete symmetry is implemented in this case by means of polygonal apertures—leads to qualitatively identical results [34].

In this sense we can consider a DSDE as the physical implementation of a discrete deformation operator in optics. However, it is true that, assuming that the DSDE is located

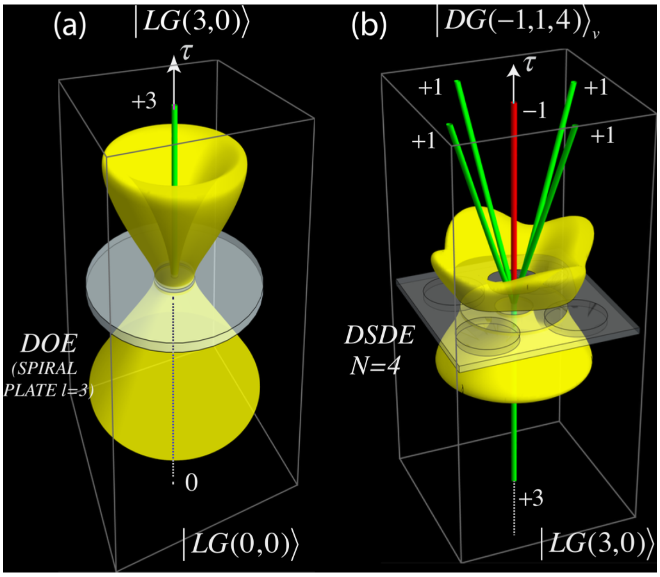

FIG. 1. (Color online) Diffractive optical elements as mode converters: (a) a spiral phase plate transforms the fundamental $|\mathrm{LG}(0,0)\rangle$ mode into a higher-order $|\mathrm{LG}(3,0)\rangle$ mode with $q=+3$; (b) a DSDE with rotational order $N=4$ converts a $|\mathrm{LG}(3,0)\rangle$ mode into a multisingular $|\mathrm{DG}(+3,0,4)\rangle_v = |\mathrm{DG}(-1,1,4)\rangle_v$ state (see Sec. III for the definition of the later notation in terms of m and k). The amplitude of the field is represented as a yellow 3D surface (in this case, the one corresponding to 1/6 maximum). Trajectories of positively (negatively) charged singularities are represented in green (red).

at $\tau=0$, the conversion of an LG mode into a DG state is produced only for positive values of τ and not for the entire space. Strictly speaking, we only have a faithful physical representation of the discrete deformation operator for $\tau>0$. On the other hand, we can conceive of the use of an appropriate combination of different DSDE's to generate DG states also for negative values of τ.

In quantum mechanics the role of the DOE is played by the instantaneous potential V. The relation between the two elements is provided by the definition of the complex transfer function t in its polar form:

$$t(w,\bar{w}) = b(w,\bar{w})e^{iV(w,\bar{w})}. \quad (13)$$

In optics the DOE is completely defined mathematically by its transfer function t. So that, if we deal with DOEs that only act on the phase of the field and do not change its amplitude—the so-called phase DOEs, t is given by a pure phase transfer function $|t|=1$. According to Eq. (13), the quantum mechanics counterpart of a phase DOE is an instantaneous *real* potential V. For instance, the counterpart of a spiral phase plate would be an instantaneous potential of the form $V(\theta)=l\theta$. However, in the general case in which both the amplitude and phase of the input field are modified by the DOE, we no longer can characterize the system quantum mechanically by a real instantaneous potential. Nevertheless, we still can use the polar form of t (13) to define a *complex* "instantaneous" potential $V_c = V - i\ \ln b$ in such a way that $t=\exp(iV_c)$. In conclusion, in quantum mechanics it is possible to emulate all kinds of transfer functions t by using both dissipative and nondissipative instantaneous potentials. In terms of operators, it is easy to convince oneself by following the construction in Sec. II A that only real potentials generate unitary discrete deformation operators, whereas in the general case the unitarity property is not preserved.

III. DISCRETE-GAUSS STATES AND DISCRETE ROTATIONAL SYMMETRY

The states generated by the discrete deformation operator $\hat{D}_v$ by means of the deformation equation (12) are *discrete-Gauss states*. We recognize that Eq. (9) is, up to $O(v^2)$ terms, nothing but the deformation equation (12) for $p=0$ states. Consequently, ϕ^{DG}_{lNv} is a discrete-Gauss (DG) state with $p=0$. Although it is not strictly necessary from the formal point of view, we will restrict ourselves from now on to discrete deformations of LG modes with $p=0$, so the index p will be ignored unless explicitly mentioned.

By construction, the rotational symmetry group of a $|\mathrm{DG}(l,N)\rangle_v$ is no longer $O(2)$ but $\mathcal{C}_N$. Examples of two DG states $O(v^2)$ corresponding to two different discrete deformations of an LG mode, which clearly reflect this feature, are given in Fig. 2.

According to Eq. (12), $|\mathrm{DG}(l,N)\rangle_v$ states are eigenfunctions of the *discrete* rotation operator $\hat{G}_N$:

$$\hat{G}_N|\mathrm{DG}(l,N)\rangle_v = \epsilon_N^l|\mathrm{DG}(l,N)\rangle_v, \quad (14)$$

where $\epsilon_N = \exp i2\pi/N$. However, they are not eigenfunctions of the OAM operator $\hat{L}_z$. Nevertheless, despite the fact that l is not the OAM of the state, it is still a good quantum number for a DG state. Now, it is interpreted as the "unfolded" value of the discrete angular momentum (or angular pseudomomentum) m, which is the real conserved quantity associated to the $\mathcal{C}_N$ symmetry of the DG state [29]. According to group theory, since the real rotational symmetry of the solution is $\mathcal{C}_N$ and this is a cyclic group, the eigenvalues of a DG state are given by the roots of unity of Nth order. In other words, the $|\mathrm{DG}(l,N)\rangle_v$ also satisfy

$$\hat{G}_N|\mathrm{DG}(l,N)\rangle_v = \epsilon_N^m|\mathrm{DG}(l,N)\rangle_v, \quad (15)$$

where $-N/2 \leqslant m \leqslant N/2$ in such a way that $\epsilon_N^m = \exp(i2\pi m/N)$ constitute all the Nth roots of unity.

The discrete angular momentum m is the corresponding folded value of l, since it is bounded $|m| \leqslant N/2$ and $l = m + kN$, where $k \in \mathbb{Z}$. The relation between l and m is forced by the uniqueness of the $\hat{G}_N$ eigenvalues $\epsilon_N^l = \exp(2\pi i l/N) = \epsilon_N^m$. This feature is an important characteristic of DG states since it clearly unveils the relation between l (OAM) and m (discrete angular momentum), i.e., between the two conserved quantities associated to spatial rotations in the $O(2)$ (continuous) and $\mathcal{C}_N$ (discrete) cases.

Since k plays an essential role in our discussion, we will provide more physical insight about the relation between l, m, and k. For this purpose, we will follow the line of reasoning presented in Ref. [52]. The characterization of any discrete-symmetry potential V of Nth order in terms of the angular variable θ is simply $V(r,\theta+2\pi/N) = V(r,\theta)$. We can map the unit circle where the angular variable is defined $\theta \in [-\pi,\pi]$ into the finite interval of the real axis $[-D/2,D/2]$ by means of the mapping $\theta \to s = \theta D/(2\pi)$.

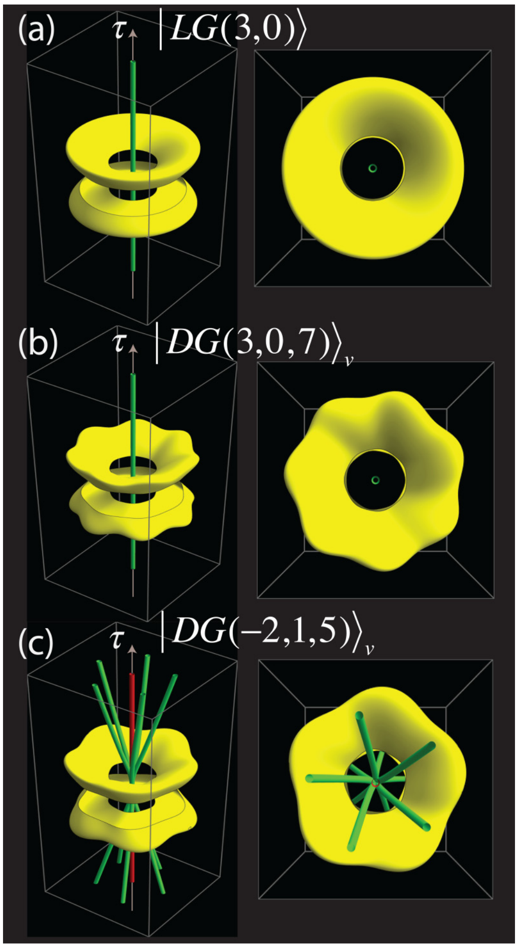

FIG. 2. (Color online) Amplitude (3D surface at half-maximum) and singularity structure of a Laguerre-Gauss mode $|\mathrm{LG}(l,0)\rangle$ and two of its discrete deformations $|\mathrm{DG}(m,k,N)\rangle$ in the $xy\tau$ space [right column, top view; green (red) color indicates positive (negative) q; $\tau_R = 1$; $v = 0.1$]. (a) $|\mathrm{LG}(3,0)\rangle$ mode; single dark ray trajectory is shown. (b) DG state with $k = 0$ and its single associated dark ray with $q_{\rm ax} = +3$. (c) DG state with $k = 1$ generating a *focusing* dark beam with $q_{\rm ax} = -2$ and $N = 5$ off-axis $q = +1$ singularities.

In this way the invariance under finite rotations of the original potential becomes a periodicity property in terms of the noncompact coordinate s: $V(r,\theta) = \tilde{V}(r,s) = \tilde{V}(r,s + D/N)$. We can apply now our knowledge on 1D periodic potentials to our "flattened" potential $\tilde{V}(r,s)$, for which the period is $a = D/N$. The eigenstates of the Hamiltonian for such a potential are the well-known Bloch waves characterized by the so-called Bloch momentum q: $\phi_q(r,s) = e^{iqs}\tilde{u}_q(r,s)$, where $\tilde{u}_q(r,s + a) = \tilde{u}_q(r,s)$ has the same periodicity of the potential. These Bloch waves are also eigenstates of the finite translation operator T_a so that $T_a\phi_q(r,s) = \phi_q(r,s + a) = e^{iqa}\phi_q(r,s)$. Unlike ordinary linear momentum p, which can take all possible real values, the Bloch momentum q is, in fact, restricted to lie in the interval $[-\pi/a,\pi/a]$, the so-called first Brillouin zone, inasmuch as the eigenfunctions $\phi_q(r,s)$ and their corresponding eigenvalues are periodic in q with a periodicity given by a vector (1D in this case) called the reciprocal lattice vector $Q = 2\pi/a$. Therefore, two Bloch momenta q and p are equivalent if they differ in an integer multiple of the reciprocal lattice vector Q: $p - q = kQ$. Using this relation any Bloch momentum p can be mapped into their univocally defined counterpart q within the first Brillouin zone $[-\pi/a,\pi/a]$, which is characterized by the index $k = 0$. On top of that, since the size of the spatial interval $[-D/2,D/2]$ is finite, the Bloch momentum is also discretized according to the condition $e^{iq_m D} = 1$, which implies that $q_m = m2\pi/D$, where $m \in \mathbb{Z}$.

All these arguments apply to the functions written in terms of the noncompact variable s, but we can undo the mapping and analyze these results for our original functions in terms of the angular variable θ. Since DG states will transform under finite rotations exactly as these Bloch modes, these arguments necessarily apply to them as well. From the relation $q_m = m2\pi/D$, the Bloch waves in terms of the angular variable become simply $\varphi_m(r,\theta) = u_m(r,\theta)e^{im\theta}$. We see that the Bloch momentum q becomes the integer m, which for obvious reasons we refer to as the Bloch angular momentum. The Bloch, or discrete, angular momentum m is restricted to lie in the first *angular* Brillouin zone $[-N/2,N/2]$ in the same way its counterpart q is in the first linear Brillouin zone $[-\pi/a,\pi/a]$. When we write the relation $p - q = kQ$ for two equivalent discretized Bloch momenta, one outside the first Brillouin zone $p_l = l(2\pi/D)$ and the other restricted to the first Brillouin zone $q_m = m(2\pi/D)$, we immediately find the relation

$$l - m = kN, \tag{16}$$

where the integer k can be naturally interpreted as the index of the angular Brillouin zone for l. We can see here clearly the physical interpretation of l as the "unfolded" value of the "folded" discrete angular momentum m lying in the first angular Brillouin zone, characterized by $k = 0$. The previous relation along with the first angular Brillouin-zone restriction, $m \in [-N/2,N/2]$, establishes a univocal relationship between the unfolded value l and the folded one m. Both represent the same angular Bloch wave and, therefore, the characterization in terms of l or of the pair (m,k) is completely equivalent.

IV. MATHEMATICAL CONSTRUCTION OF DISCRETE-GAUSS STATES

A. Scattering modes

An LG mode with $p = 0$ takes the form of a Gaussian vortex at $\tau = 0$, so that $\phi_{l0}^{\rm LG}(0) \sim \Omega_w^l\phi_{00}(0)$, where $\Omega_w^l \equiv \{w^{|l|}\ (l > 0),\overline{w}^{|l|}\ (l < 0)\}$. Using the replacement rule (10), this implies that its value for $\tau \neq 0$ can be obtained simply by replacing the complex function Ω_w^l by the operators $\hat{l}^{|l|}_{\rm sgn(l)}$. This means that we can also write the LG mode in Eq. (9) as $\phi_{l0}^{\rm LG} \sim \hat{l}^{|l|}_{\rm sgn(l)}\phi_{00}$. Consequently, the DG state in Eq. (9) appears as a linear combination of modes of the form

$$\Phi_{n\overline{n}} = \hat{l}_+^n\hat{l}_-^{\overline{n}}\phi_{00}. \tag{17}$$

The operators $\hat{l}_\pm$ belong to the Lie algebra of symmetries of the free 2DLSE (1). They commute with the Schrödinger differential operator $L_0 = i\partial/\partial\tau + H$ that defines this equation. Since ϕ_{00} is a solution, and thus $L_0\phi_{00} = 0$, and, on the other hand, $[\hat{l}_\pm,L_0] = 0$, it is automatically guaranteed that the $\Phi_{n\overline{n}}$ modes are also solutions of Eq. (1) [48,49]. Due to the linearity of the

2DLSE, another consequence of this property is that any linear combination of $\Phi_{n\bar{n}}$ modes is also a solution. This provides another alternative proof that DG states are solutions of Eq. (1).

In fact, these modes solve a more general problem, namely, that of the scattering of a Gaussian wave packet by an *arbitrary* instantaneous potential $V(w,\overline{w})$ acting at $\tau = 0$. Let us consider the most general form for an analytical quasi-instantaneous potential at $\tau = 0$ acting on a fundamental Gaussian wave packet. Since V is analytical we take its Taylor expansion in w and $\bar{w}$ around $w = 0$ in Eq. (4) with $\psi(0) = \phi_{00}(0)$. We would have

$$\phi(w,\bar{w},0) \overset{O(v^2)}{\sim} \left[1 + iv\sum_{n,\bar{n}} V_{n\bar{n}} w^n \bar{w}^{\bar{n}}\right]\phi_{00}(w,\bar{w},0). \quad (18)$$

According to the replacement rule (8), the solution for the field scattered by the quasi-instantaneous potential V would be given by

$$\phi(w,\bar{w},\tau) \overset{O(v^2)}{\sim} \left[1 + iv\sum_{n,\bar{n}} V_{n\bar{n}} \hat{l}_+^n \hat{l}_-^{\bar{n}}\right]\phi_{00}(w,\bar{w},\tau)$$

$$= \phi_{00}(w,\bar{w},\tau) + iv\sum_{n,\bar{n}} V_{n\bar{n}}\Phi_{n\bar{n}}(w,\bar{w},\tau). \quad (19)$$

For this reason, we refer to the $\Phi_{n\bar{n}}$ modes as *scattering* modes (SM).

We determine next the functional structure of SM by analyzing their symmetry properties. Due to its form in terms of the complex position and momentum operators (8), the operators $\hat{l}_\pm$ transform under continuous rotations as

$$\hat{l}_+ \overset{G(\alpha)}{\to} \epsilon\hat{l}_+, \quad \hat{l}_- \overset{G(\alpha)}{\to} \epsilon^*\hat{l}_-, \quad (20)$$

where $\epsilon = \exp i\alpha$ is a complex $O(2)$ rotation of angle α. In this way, under a $O(2)$ rotation $\hat{G}(\alpha)$, SM in Eq. (17) transform as

$$\hat{G}(\alpha)\Phi_{n\overline{n}} = \exp i\alpha(n-\overline{n})\Phi_{n\overline{n}} = \epsilon^l\Phi_{n\overline{n}}. \quad (21)$$

Consequently, SM are eigenfunctions of the continuous rotation operator $\hat{G}(\alpha)$ with eigenvalue ϵ^l. Therefore, $l = n - \overline{n}$ is the OAM of the SM. In terms of their OAM, we can rearrange operators in Eq. (17), so that SM can also be written as

$$\Phi_{lp} = \hat{l}_{\mathrm{sgn(l)}}^{|l|}\hat{\Delta}^p\phi_{00}, \quad (22)$$

where $\hat{\Delta} \equiv \hat{l}_+\hat{l}_-$ is an $O(2)$-invariant operator and $p \equiv \min(n,\overline{n})$. Since the function $\Omega_w^l \equiv \{w^{|l|}\,(l \geqslant 0), \overline{w}^{|l|}\,(l < 0)\}$ transforms under continuous rotations as $\Omega_w^l \to \epsilon^l\Omega_w^l$, i.e., exactly as Φ_{lp} does, rotational symmetry determines that the functional form of a generic SM can always be given by

$$\Phi_{lp}(w,\overline{w},\tau) = \Omega_w^l f_{lp}(|w|^2,\tau)\phi_{00}, \quad (23)$$

where f_{lp} is an $O(2)$-invariant function depending exclusively on the modulus of w (as ϕ_{00}.) However, the $O(2)$-invariant functions f_{lp} can be *explicitly* constructed by successive application of the operators $\hat{l}_\pm$ on ϕ_{00} according to Eq. (22). Note that, due to the form of the differential $\hat{l}_\pm$ operators (8), their action on the Gaussian function ϕ_{00} (2) always provides products of polynomials in w and $\overline{w}$ times the original function ϕ_{00}. According to symmetry considerations, these products can always be rearranged as in Eq. (23) in a systematic manner,

TABLE I. Coefficients of the lower-order $F_p^{|l|}$ polynomials.

	$p=1$		$p=2$			$p=3$			
$c_{pi}^{\|l\|}$	$c_{10}^{\|l\|}$	$c_{11}^{\|l\|}$	$c_{20}^{\|l\|}$	$c_{21}^{\|l\|}$	$c_{22}^{\|l\|}$	$c_{30}^{\|l\|}$	$c_{31}^{\|l\|}$	$c_{32}^{\|l\|}$	$c_{33}^{\|l\|}$
$l=0$	1	−1	2	−4	1	6	−18	9	−1
$l=\pm1$	2	−1	6	−6	1	24	−36	12	−1
$l=\pm2$	3	−1	12	−8	1	60	−60	15	−1
$l=\pm3$	4	−1	20	−10	1	120	−90	18	−1
$l=\pm4$	5	−1	30	−12	1	210	−126	21	−1

so that an analytical procedure to obtain any function f_{lp} is established. An accurate analysis of this construction of f_{lp} functions permits one to identify a general structure for them, given by

$$f_{lp}(|w|^2,\tau) = \alpha^{|l|}\beta^p F_p^{|l|}(x), \quad (24)$$

where $\alpha = i\tau_R/q(\tau)$, $\beta = \tau\tau_R/q(\tau)$, and $F_p^{|l|}(x)$ is a polynomial of pth order in $x = [q(\tau)\tau]^{-1}\tau_R|w|^2$:

$$F_p^{|l|}(x) = \sum_{i=0}^{p} c_{pi}^{|l|}x^i. \quad (25)$$

All coefficients in $F_p^{|l|}(x)$ are known in this construction. As an example, the value of the coefficients for the lower-order $F_p^{|l|}$ polynomials is given in Table I. In addition, $F_0^{|l|}(x) = 1$ for all l.

The SM modes Φ_{lp} constitute a nonorthogonal basis in which any analytical solution of the 2DLSE (1) can be expanded. In particular, they can be used to expand both elegant and standard LG modes. The general form of an LG mode (up to a normalization constant) can be written in the $\tau = 0$ plane as

$$\phi_{lp}^{\mathrm{LG}}(w,\bar{w},0) \sim \Omega_w^l L_p^{|l|}[a(0)w\bar{w}]\phi_{00}(w,\bar{w},0), \quad (26)$$

where $L_p^{|l|}$ is the generalized Laguerre polynomial of order p and ϕ_{00} is the fundamental Gaussian mode (2). Let us note here that the particular form of a differs in the elegant and standard cases [3], although, nevertheless, our argument will remain valid for both versions of LG modes. If we write the generalized Laguerre polynomial explicitly—$L_p^{|l|}(x) = \sum_{i=0}^p b_i^{|l|p}x^i$—we have

$$\phi_{lp}^{\mathrm{LG}}(w,\bar{w},0) \sim \sum_{i=0}^{p} b_i^{|l|p}a(0)^i\left[\Omega_w^l w^i\bar{w}^i\phi_{00}(w,\bar{w},0)\right]. \quad (27)$$

As before, the value at an arbitrary τ is obtained by means of the replacement rule (8), which, in this case, takes the form $\Omega_w^l \to \hat{l}_{\mathrm{sgn(l)}}^{|l|}$ and $w^i\bar{w}^i \to \hat{\Delta}^i$. After this replacement is made, we immediately recognize the definition of a SM mode (22) between the square brackets in the previous expression. Thus, indeed an arbitrary LG mode carrying OAM l can be written as a linear combination of $p+1$ SM modes with the same OAM:

$$\phi_{lp}^{\mathrm{LG}}(w,\bar{w},\tau) \sim \sum_{i=0}^{p} b_i^{|l|p}a(\tau)^i\Phi_{li}(w,\bar{w},\tau). \quad (28)$$

The expressions of both the $a(\tau)$ and $b_i^{|l|p}$ coefficients are perfectly known from the particular form of the LG mode, so that the previous linear combination is unique (up to the normalization constant of the LG mode). Note that, for $p=0$, LG and SM modes are identical up to a normalization constant. When $p>0$, however, we need $p+1$ SM modes to describe a given LG mode, and vice versa.

B. Constructing discrete-Gauss states from scattering modes

Taking into account that we can write $\phi_{l0}^{\rm LG}\sim\hat{l}_{\rm sgn(l)}^{|l|}\phi_{00}$, it is immediate to obtain an analytical expression for a $|{\rm DG}(l,N)\rangle_v$ state in terms of SM using the definition of SM (22) in Eq. (9):

$$\phi_{lNv}^{\rm DG}\sim\begin{cases}\Phi_{l0}+iv\Phi_{l+N,0}+iv\Phi_{l-N,\overline{N}}, & l\geqslant 0,\\ \Phi_{l0}+iv\Phi_{l+N,\overline{N}}+iv\Phi_{l-N,0}, & l\leqslant 0,\end{cases}\tag{29}$$

where $\overline{N}=\min(|l|,N)$. Because of the univocal relationship between the unfolded and folded values l and (m,k), an alternative notation for a DG state is $|{\rm DG}(l,N)\rangle_v=|{\rm DG}(m,k,N)\rangle_v$. As we will see next, the singularity structure of a DG state crucially depends on the value of its "folding" parameter k.

Since SM are found analytically so are DG states. This property can be made explicit by substituting the expression for SM (23) into Eq. (29). However, as just mentioned, the role of k is essential, so that we want to transform the conditions for l in Eq. (29) in conditions for k. We need to distinguish between the $k=0$ and $k\neq 0$ case. In the latter case ($k\neq 0$), which is the one we will study first, there is a biunivocal relation between the values of k and l. Due to the fact that $l=m+kN$ and $|m|\leqslant N/2$, it is easy to check that the conditions $l\geqslant 0$ and $k\geqslant 1$ are equivalent provided $k\neq 0$. In the same way, it is proven that $l\leqslant 0$ is equivalent to $k\leqslant -1$. For this reason and following Eq. (29), we distinguish the $k\geqslant 1$ ($l\geqslant 0$) and $k\leqslant -1$ ($l\leqslant 0$) cases in our first analysis for $k\neq 0$.

We start by considering that $k\geqslant 1$. We use a symmetry argument to find the general structure of DG states. This is the counterpart of the argument we used to determine the structure of SM (23) by means of their transformation properties under $O(2)$. Now, the transformation property of DG functions under the $\mathcal{C}_N$ symmetry $\phi_{mkNv}^{\rm DG}\to\epsilon_N^m\phi_{mkNv}^{\rm DG}$ tells us that they have to be proportional to the Ω_w^m function (which transforms as $\Omega_w^m\to\epsilon_N^m\Omega_w^m$) times a $\mathcal{C}_N$-invariant function. We write $\phi_{mkNv}^{\rm DG}$ then in the following way:

$$\phi_{mkNv}^{\rm DG}(w,\overline{w},\tau)=\Omega_w^m w^{(k-1)N}\mathcal{F}_{mkNv}(w^N,\overline{w}^N,|w|^2,\tau)\phi_{00},\tag{30}$$

where $\mathcal{F}_{mkNv}$ is an explicitly invariant $\mathcal{C}_N$ function, in the same way as $w^{(k-1)N}$ and ϕ_{00}. The comparison of the result obtained after substituting Eq. (23) into Eq. (29) with the general expression for the DG state (30) provides us with an explicit construction for the $\mathcal{F}_{mkNv}$ functions in terms of the analytical $\mathcal{C}_N$-invariant functions f_{lp} found previously.

Thus, for $k\geqslant 1$ ($l\geqslant 0$), we have

$$\mathcal{F}_{mkNv}=\begin{cases}f_{m+kN,0}w^N+ivf_{m+(k+1)N,0}w^{2N}+ivf_{m+(k-1)N,N}, & m\geqslant 0\ (l=l_1),\\ |w|^{-2|m|}(f_{m+kN,0}w^N+ivf_{m+(k+1)N,0}w^{2N}+ivd_{mk}f_{m+(k-1)N,l}), & m\leqslant 0\ (l=l_2),\end{cases}\tag{31}$$

where $d_{mk}=|w|^{2|m|}$ if $k=1$ and 1 if $k\geqslant 2$. We see that for a given value of $k\geqslant 1$ and $|m|$, there are two possible values for l, $l_1=|m|+|k|N$ and $l_2=-|m|+|k|N$, depending on whether $m=|m|$ or $m=-|m|$. Since $|m|\leqslant N/2$, these two values are positive and verify that $l_1,l_2>N$, except when $k=1$, in which $l_2<N$. Note that, due to this fact, $\overline{N}=\min(|l|,N)$ equals N in the former case and $|l_2|$ in the latter. As we will see next, these different values give rise to different types of solutions.

For $k\leqslant -1$, one can still use the previous expressions by invoking an important $w\leftrightarrow\overline{w}$ duality symmetry of DG states. Under the exchange between w and $\overline{w}$, the fundamental Gaussian function ϕ_{00} (2) is invariant. On the other hand, we immediately see from their definition (8) that this transformation exchange the $\hat{l}_\pm$ operators: $\hat{l}_+\overset{w\leftrightarrow\overline{w}}{\to}\hat{l}_-$. For this reason, the LG functions with $p=0$ ($\phi_{l0}^{\rm LG}\sim\hat{l}_{\rm sgn(l)}^{|l|}\phi_{00}$) change under this duality transformation between w and $\overline{w}$ simply as $\phi_{l0}^{\rm LG}\overset{w\leftrightarrow\overline{w}}{\to}\phi_{-l0}^{\rm LG}$. Since the discrete deformation operator (11) is invariant under the exchange of $\hat{l}_+$ and $\hat{l}_-$, the very definition of DG states (12) imply that DG states transform as LG modes under the duality transformation $w\leftrightarrow\overline{w}$:

$$\phi_{-l,N,v}^{\rm DG}(w,\overline{w},\tau)=\phi_{l,N,v}^{\rm DG}(\overline{w},w,\tau)\tag{32}$$

or, equivalently,

$$\phi_{-m,-k,N,v}^{\rm DG}(w,\overline{w},\tau)=\phi_{m,k,N,v}^{\rm DG}(\overline{w},w,\tau).\tag{33}$$

In terms of the $\mathcal{F}_{mkNv}$ functions, this property reads

$$\mathcal{F}_{-l,Nv}(w,\overline{w},\tau)=\mathcal{F}_{l,N,v}(\overline{w},w,\tau)\tag{34}$$

or, in terms of (m,k),

$$\mathcal{F}_{-m,-k,N,v}(w,\overline{w},\tau)=\mathcal{F}_{m,k,N,v}(\overline{w},w,\tau).\tag{35}$$

For $k\leqslant -1$ ($l\leqslant 0$), by applying this duality symmetry to Eq. (31), we have

$$\mathcal{F}_{mkNv}=\begin{cases}|\overline{w}|^{-2|m|}(f_{-m+|k|N,0}\overline{w}^N+ivf_{-m+(|k|+1)N,0}\overline{w}^{2N}+ivd_{m|k|}f_{-m+(|k|-1)N,|l|}), & m\geqslant 0\ (l=-l_2),\\ f_{|m|+|k|N,0}\overline{w}^N+ivf_{|m|+(|k|+1)N,0}\overline{w}^{2N}+ivf_{|m|+(|k|-1)N,N}, & m\leqslant 0\ (l=-l_1).\end{cases}\tag{36}$$

In this way, we have extended the expression of the $\mathcal{F}_{mkNv}$ functions to all nonzero values of k.

However, for $k \leqslant -1$ the form of the DG functions also changes according to Eq. (33). Instead of Eq. (30), we have

$$\phi^{\rm DG}_{mkNv}(w,\overline{w},\tau) = \Omega^m_w \overline{w}^{(|k|-1)N} \mathcal{F}_{mkNv}(w^N,\overline{w}^N,|w|^2,\tau)\phi_{00}, \tag{37}$$

where $\mathcal{F}_{mkNv}$ is now the extended function defined for all $k \neq 0$.

For $k=0$, we have that $l=m$ and the previous construction has to be changed accordingly. Although the symmetry arguments used to construct the DG state functions still hold, we need to define a new set of $\mathcal{C}_N$-invariant functions. Therefore, instead of Eqs. (30) and (37), we have

$$\phi^{\rm DG}_{m0Nv}(w,\overline{w},\tau) = \Omega^m_w \mathcal{G}_{mNv}(w^N,\overline{w}^N,|w|^2,\tau)\phi_{00}, \tag{38}$$

where the function $\mathcal{G}_{mNv}$ has a similar, but not identical, structure than that of $\mathcal{F}_{mkNv}$ in terms of the f_{lp} functions:

$$\mathcal{G}_{mNv} = \begin{cases} f_{m0} + iv f_{m+N,0} w^N + iv|w|^{-2m} f_{N-m,N-m}\overline{w}^N, & l=m \geqslant 0, \\ f_{|m|0} + iv f_{|m|+N,0}\overline{w}^N + iv|w|^{-2|m|} f_{N-|m|,N-|m|} w^N, & l=m \leqslant 0. \end{cases} \tag{39}$$

The final property of DG states we want to deal with in this section is biorthogonality. As complex-argument elegant LG modes, DG states are not orthogonal but *biorthogonal*. The complex-argument elegant LG modes $|{\rm LG}(l,p)\rangle$ form a biorthogonal set, which means that there exists a *different* set of states, known as adjoint states $|\overline{\rm LG}(l',p')\rangle$, which satisfy $\langle\overline{\rm LG}(l',p')|{\rm LG}(l,p)\rangle = \delta_{ll'}\delta_{pp'}$ [53]. By defining a discrete deformation of these states using the operator $\hat{D}_v$ analogously as we did in Eq. (11),

$$|\overline{\rm DG}(l',p',N)\rangle_v = \hat{D}_v(N)|\overline{\rm LG}(l',p')\rangle, \tag{40}$$

we immediately realize that the $\{{\rm DG}(l,p,N),\overline{\rm DG}(l',p',N)\}$ set is also biorthogonal. Indeed, inasmuch as the deformation operator is unitary, as proven in the previous section, it is true that $\hat{D}^\dagger_v\hat{D}_v = 1$, and therefore the scalar product is preserved, so that

$$\begin{aligned}\langle\overline{\rm DG}(l',p',N)_v|{\rm DG}(l,p,N)_v\rangle &= \langle\overline{\rm LG}(l',p')|{\rm LG}(l,p)\rangle \\ &= \delta_{ll'}\delta_{pp'}.\end{aligned} \tag{41}$$

C. Amplitude and phase of DG states

Despite the apparent complexity of the previous construction, the main qualitative properties of a DG state are given by the set of numbers that characterize it in a rather simple way. Therefore, the amplitude of a $|{\rm DG}(m,k,N)\rangle_v$ state explicitly exhibits the discrete rotational symmetry of Nth order for all values of the evolution parameter τ. Besides, since this DG state is a discrete deformation of a standard LG mode, its amplitude behaves like a Gaussian vortex modulated now by a discrete rotational symmetry. We can clearly visualize this fact in Figs. 3 and 4, in which the evolution of two different DG states with the same rotational order $N=3$ are presented. In the left column of these two figures we can appreciate the vortex-Gaussian-like evolution of the amplitudes of a $|{\rm DG}(-1,1,3)\rangle_v$ (Fig. 3) and a $|{\rm DG}(1,1,3)\rangle_v$ (Fig. 4) state. Like a Gaussian beam, both states focus for $\tau<0$ and defocus for $\tau>0$ reaching its minimum width at $\tau=0$ (the beam "waist," in optical notation). This is a general feature for all DG states constructed by means of the mechanism presented previously. However, besides the obvious discrete modulation, we also observe a crucial difference with respect to ordinary LG modes. Both DG states exhibit a multisingular structure visible in amplitude and phase (right column). This a direct consequence of the fact that $k=1$ for both states. As we will prove in the next section, the multisingular structure appears whenever a DG state presents $k \neq 0$.

The qualitative structure of the phase of a $|{\rm DG}(m,k,N)\rangle_v$ state is also easily understood in terms of its values of m, k, and N. By analyzing the phase of the $|{\rm DG}(-1,1,3)\rangle_v$ and $|{\rm DG}(1,1,3)\rangle_v$ states in Figs. 3 and 4, we immediately recognize that the total charge of all off-axis singularities is given by the order of symmetry or, to be more precise, by the product ${\rm sgn}(k)N$, which in this particular case is $+3$ for both states. We can also appreciate that in this case the value of m provides the topological charge at the axis of symmetry for all values of τ except for $\tau=0$. As we will see later, this is a particular case, valid for $|k|=1$ DG states, of a general rule stating that this axial charge equals $m+{\rm sgn}(k)(|k|-1)N$ for $\tau \neq 0$. Therefore, the $|{\rm DG}(-1,1,3)\rangle_v$ presents an antivortex with $q=-1$ at the rotation axis for $\tau \neq 0$, whereas $|{\rm DG}(1,1,3)\rangle_v$ exhibits a single $q=+1$ at the same position.

Interestingly enough, the properties of the phase at the $\tau=0$ plane are closely linked to the value of the unfolded discrete angular momentum $l=m+kN$ as given in Eq. (16). The $|{\rm DG}(-1,1,3)\rangle_v$ and $|{\rm DG}(1,1,3)\rangle_v$ states differ only in the sign of their index m. However, this difference provides two different values for l: $l=+2$ for the former and $l=+4$ for the latter. This is precisely the topological charge at the $\tau=0$ plane, as one can clearly appreciate in the phase representation at the $\tau=0$ plane in Figs. 3 and 4. These two values of l—$l=l_1$ and $l=l_2$—correspond to the two different functional structures presented in Eq. (31) for $k \geqslant 1$. The value of the unfolded discrete angular momentum l provides then the topological charge of the phase singularity located at the symmetry axis at the beam waist, i.e., at $\tau=0$.

We will prove all these properties in the next section by introducing the concept of *focusing dark beam*, which is just a natural consequence of the symmetry properties of the analytical $\mathcal{C}_N$-invariant function $\mathcal{F}_{mkNv}$.

V. FOCUSING DARK BEAMS

As already mentioned, the singularity structure of LG modes is simple. In the case that the mode carries OAM, i.e., when $l \neq 0$, there exists a single phase singularity located at the symmetry axis of the mode. The topological charge of this singularity coincides with the OAM of the mode $q=l$. In a

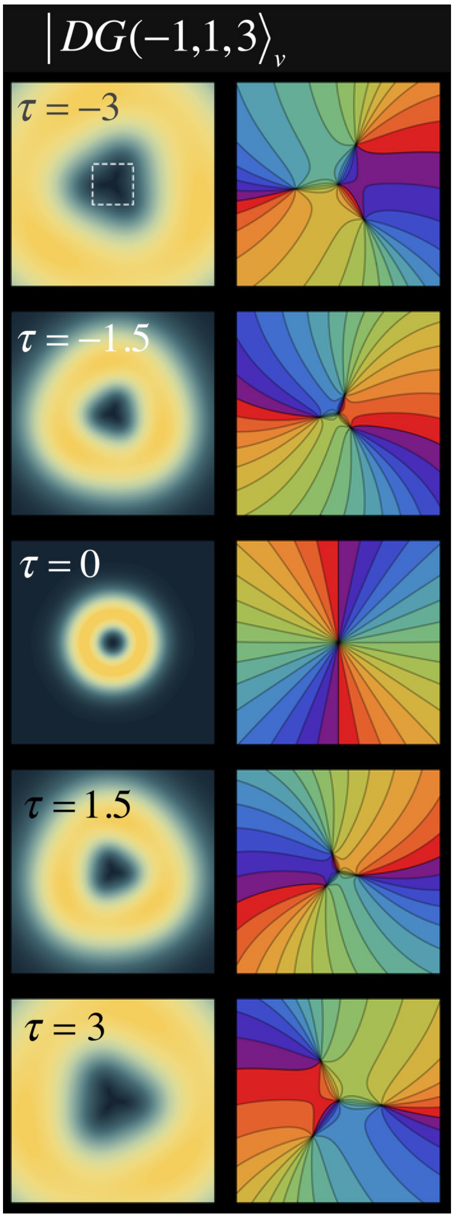

FIG. 3. (Color online) Evolution in τ of the amplitude [left column] and phase [right column] of a $|\mathrm{DG}(-1,1,3)\rangle_v$ state. Using the equivalent notation $|\mathrm{DG}(l,N)_v\rangle$ in terms of the "unfolded" discrete angular momentum l, this state is identical to the $|\mathrm{DG}(+2,3)\rangle_v$ state. The dashed white square in the $\tau=-3$ snapshot indicates the region that is enlarged for the representation of the phase.

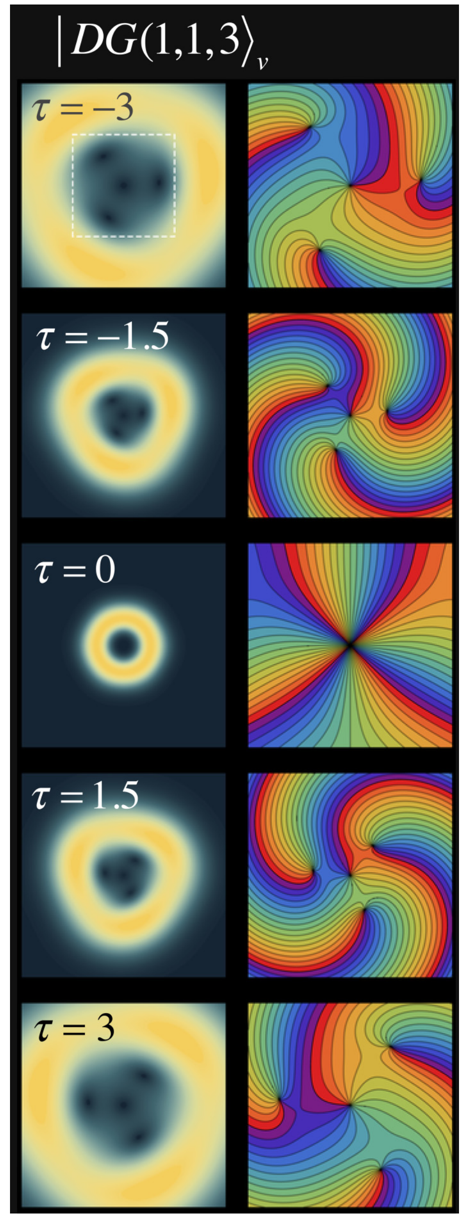

FIG. 4. (Color online) Same as in Fig. 3 but for the $|\mathrm{DG}(-1,1,3)\rangle_v$ state, or $|\mathrm{DG}(+4,3)\rangle_v$ state, using equivalent $|\mathrm{DG}(l,N)_v\rangle$ notation.

representation of the field amplitude, the trajectory followed by this singularity in the $xy\tau$ space is just a straight line. This straight line constitutes a dark ray where the intensity of the field vanishes. A characteristic example of such a dark ray is shown in Fig. 2(a), where we can see a view of the mode from two different viewpoints. We also observe in this figure that the amplitude profile of the mode hosting this dark ray exhibits at the same time the perfect $O(2)$ symmetry of LG modes. It is clear that the action of the discrete deformation operator changes the continuous rotational properties of the LG mode by transforming it into a DG state with discrete rotational symmetry. However, as Figs. 2(b) and 2(c) unveil, this transformation can occur in two completely different ways. We can see that two different discrete deformations of the same LG mode—Fig. 2(a)—can produce either a simple modulation of the amplitude without changing the original dark ray—Fig. 2(b)—or give rise to a completely new multisingular structure formed by a bundle of rays converging at the waist plane—Fig. 2(c). We shall refer to this bundle of converging dark rays as the *focusing dark beam* associated to the DG state. We will show next how the previous mechanism of generation of focusing dark beams is linked to the properties of the DG states analyzed in the previous section. In particular, we will learn the key role played by the folding parameter k to understand the generation or not of dark beams embedded in a given DG state.

A. Singularity structure of discrete-Gauss states

The singularity structure of DG states arises from the condition $\phi^{\rm DG}_{mkNv}=0$. According to Eqs. (30), (37), and (38), the zeros of $\phi^{\rm DG}_{mkNv}$ occur when one of the following conditions are met: (i) $\Omega^m_w w^{(k-1)N}=0$ $(k\geqslant 1)$, (ii) $\Omega^m_w \overline{w}^{(|k|-1)N}=0$ $(k\leqslant -1)$, (iii) $\mathcal{F}_{mkNv}=0$ $(k\neq 0)$, (iv) $\Omega^m_w=0$ $(k=0)$, or (v) $\mathcal{G}_{mNv}=0$ $(k=0)$.

In the quest for singularities of DG states, it is important to know what occurs with singularities on axis. In this context, an important property of the $\mathcal{F}_{mkNv}$ and $\mathcal{G}_{mNv}$ functions is that they do not show zeros at $w=0$ when $\tau\neq 0$ (provided $|m|<N/2$; the case $|m|=N/2$ should be analyzed separately.) This property can be proven by taking the limit $w\to 0$ in Eqs. (31), (36), and (39). In all cases, both functions tend to a quantity proportional to some f_{lp} function evaluated at $w=0$, its corresponding proportionality constant being always different from zero if $\tau\neq 0$. In addition, according to the form of f_{lp} functions (24), when $\tau\neq 0$, $f_{lp}(0)$ is nonvanishing since the polynomial $F^{|l|}_p(x)$ always shows nonzero values for its zero-order terms.

The fact that both $\mathcal{F}_{mkNv}$ and $\mathcal{G}_{mNv}$ have no zeros at $w=0$ determines that axial singularities at $w=0$ and $\tau\neq 0$ are given by one of the three previously presented conditions: (i) or (ii), if $k\neq 0$; or, alternatively, (iv) if $k=0$. A simple analysis of these expressions permits one to establish that a generic $|{\rm DG}(m,k,N)\rangle_v=|{\rm DG}(l,N)\rangle_v$ state with $l\neq 0$ necessarily presents a singularity located at the axis $w=0$ with topological charge:

$$q_{\rm ax}=m+{\rm sgn}(k)(|k|-1)N=l-{\rm sgn}(k)N\quad(\tau\neq 0). \tag{42}$$

On the other hand, the value of the axial topological charge at the waist ($\tau=0$) is always l. This is due to the unitary nature of the deformation operator. According to the modified waist condition (7), the action of this operator at $\tau=0$ on the LG mode is simply a multiplication by the unimodular complex function $\exp iV$ (recall V is a real function.) Inasmuch as $\exp iV$ cannot be zero, the zero of the DG state at $\tau=0$ is the same as that of the LG mode, i.e., it is located at $w=0$ and it has topological charge $q_{\rm ax}=l$.

We immediately recognize an important qualitative difference in the axial topological function $q_{\rm ax}(\tau)$ when comparing the $k\neq 0$ and $k=0$ cases. For $k=0$, Eq. (42) tells us that the axial charge function $q_{\rm ax}$ is continuous for all values of the evolution parameter τ. Moreover, it is a constant function that takes always the value $q_{\rm ax}=l$, exactly as occurs in the LG mode from which it is derived. However, when $k\neq 0$, the axial charge function experiences two qualitative changes: first, it develops a discontinuity at $\tau=0$, and, second, its value for $\tau\neq 0$ is no longer l but $l-{\rm sgn}(k)N$. We can understand now better the results already presented in Fig. 2, which provide a neat visualization of this analysis. We see that the DG state in Fig. 2(b) is a $k=0$ state. Consequently, in agreement with our previous argument, the axial topological charge function is continuous and constant and it physically corresponds to a single dark ray with charge $q_{\rm ax}=l$, identical to the one of the LG mode in Fig. 2(a). In Fig. 2(c) we present a discrete deformation of the same LG mode in Fig. 2(a), but now with $k\neq 0$. We see that the axial topological charge at $\tau=0$ is still l. However, for the rest of the values of τ, $q_{\rm ax}=l-N$. The physical process associated to this discontinuity in the axial charge is clearly visualized in Fig. 2(c). We see how this discontinuity is produced by the presence of N off-axis singularities focusing at $\tau=0$ and symmetrically distributed around the symmetry axis. So, the discontinuity in the $q_{\rm ax}(\tau)$ function is intimately related to the generation of a *focusing* dark beam. Since the discontinuity in the axial charge function occurs only for $k\neq 0$ DG solutions, we have here a clear signal that the generation of a focusing dark beam is determined by the nonzero value of the unfolding parameter k.

We can rigorously prove our last statement by analyzing the $w\to 0$ and $\tau\to 0$ limits of the $\mathcal{F}_{mkNv}=0$ and $\mathcal{G}_{mNv}=0$ conditions. In this way, we can unveil the off-axis singularity structure of a given DG state. We first go to Eqs. (31) and (36) and find the form of $\mathcal{F}_{mkNv}$ functions by taking into account that in this regime we can neglect the $O(w^{2N})$ terms and that f_{lp} functions in Eq. (24) can be approximated as $f_{lp}\sim(-i\tau)^pF^{|l|}_p(0)$.

For $k\geqslant 1$ $(l\geqslant 0)$, we find that for a given DG state characterized by the indices $(l,N)\Leftrightarrow(m,k,N)$, the $\mathcal{F}_{mkNv}=0$ condition becomes, near the origin,

$$\begin{aligned}
&w^N+iv(-i\tau)^N\gamma_{lN}\approx 0\\
&(\text{for}\quad l=l_1\quad\text{or}\quad l=l_2\quad\text{with}\quad k\geqslant 2)\Leftrightarrow l>N,\\
&w^N+iv|w|^{2|m|}(-i\tau)^l\gamma'_{lN}\approx 0\\
&(\text{for}\quad l=l_2\quad\text{with}\quad k=1)\Leftrightarrow 0<l<N,
\end{aligned}\tag{43}$$

where $\gamma_{lN}\equiv F^{|l|-N}_N(0)$ and $\gamma'_{lN}\equiv F^{|l|-N}_l(0)$. This property shows that, indeed, N off-axis zeros of $\mathcal{F}_{mkNv}$ occur at the same radial position $r_0=|w_0|$ given by

$$\begin{aligned}
&r_0\approx v^{1/N}\gamma^{1/N}_{lN}\tau\\
&(\text{for}\quad l=l_1\quad\text{or}\quad l=l_2\quad\text{with}\quad k\geqslant 2)\Leftrightarrow l>N,\\
&r_0\approx v^{1/N}\gamma'^{1/N}_{lN}\tau^{\frac{l}{2l-N}}\\
&(\text{for}\quad l=l_2\quad\text{with}\quad k=1)\Leftrightarrow 0<l<N.
\end{aligned}\tag{44}$$

Both types of phase singularities tend to zero in the $\tau\to 0$ limit. In the second case, let us emphasize that the exponent of τ in this expression is always finite and positive since $l=l_2>0$ and $2l_2-N=N-2|m|>0$ because we are excluding explicitly the $|m|=N/2$ case and, therefore, our constraint on m is $|m|<N/2$. Therefore, there is no singularity at $\tau=0$. Consequently, we have shown that there exist N singularities with charge $q=+1$ approaching symmetrically the axis when $\tau\to 0$. For $k\leqslant -1$ $(l\leqslant 0)$, we would obtain an equivalent property but for $q=-1$ charges corresponding to the $w\leftrightarrow\overline{w}$ duality symmetry of the $\mathcal{F}_{mkNv}$ functions. The two different behaviors in Eqs. (43) and (44) would correspond then to $|l|>N$ in the first case and to $|l|<N$ in the second. In all cases we conclude that any DG state with $k\neq 0$ will generate a focusing dark beam.

However, the situation is completely different for $k=0$ since $\mathcal{G}_{mNv}$ does not show any zero approaching $w=0$ when $\tau\to 0$. We can see this property by writing the condition $\mathcal{G}_{mNv}=0$ close to the origin in a similar way as we did before—we use now Eq. (39). We have

$$1 + iv|w|^{-2|m|}(-i\tau)^N \gamma_{mN}\overline{w}^N \approx 0 \quad (m \geqslant 0),$$
$$1 + iv|w|^{-2|m|}(-i\tau)^N \gamma_{mN} w^N \approx 0 \quad (m \leqslant 0).$$

The zeros of these complex equations are located at the radial position $r_0 \sim \tau^{-|m|/(N-2|m|)}$, so that they diverge when $\tau \to 0$ since $|m| < N/2$. Hence the zeros of $\mathcal{G}_{mNv}$ cannot connect to the point $w = 0$ in the $\tau \to 0$ limit.

In summary, we have demonstrated that DG states with $k = 0$ do not show any bifurcation at $\tau = 0$. On the contrary, all DG states with $k \neq 0$ exhibit a *focusing* point at $(w,\tau) = (0,0)$, where all trajectories of phase singularities converge. Therefore, a *focusing* dark beam structure is embedded in a DG state *if and only if* $k \neq 0$—see Fig. 2(c). For $k = 0$, a single axial dark ray is present when $l \neq 0$ and only a modulation in the amplitude reveals the discrete nature of the DG state as compared to an LG mode with the same value of l—compare Figs. 2(a) and 2(b). Because all dark rays of a DG state with $k \neq 0$ focus at $\tau = 0$, we call the $w = 0$ point in this plane the *dark focus* of the DG state.

B. Geometrical structure of focusing dark beams

The paradigmatic structure of characteristic focusing dark beams corresponding to DG states with $k \neq 0$ are given in Fig. 5. For a given value of $|m|$ and $|k|$ there are four possible DG states generated by combining the signs of $m = \pm|m|$ and $k = \pm|k|$. As we have seen in the previous section, these four states are related by the $w \leftrightarrow \overline{w}$ duality symmetry—see Eqs. (32) and (33). These four states are $|\mathrm{DG}(\pm l_1,N)\rangle_v$ and $|\mathrm{DG}(\pm l_2,N)\rangle_v$—where $l_1 = |m| + |k|N$ and $l_2 = -|m| + |k|N$—or, equivalently using (m,k) indices, $|\mathrm{DG}(\pm|m|, \pm|k|,N)\rangle_v$ and $|\mathrm{DG}(\mp|m|, \pm|k|,N)\rangle_v$. Our previous analysis of the trajectories of phase singularities close to the origin reflected in Eq. (44) points out a qualitatively different behavior for focusing dark beams generated by the states $|\mathrm{DG}(\pm l_2,N)\rangle_v$ when $|k| = 1$ as compared to their quadruplet counterparts $|\mathrm{DG}(\pm l_1,N)\rangle_v$. In order to explicitly visualize this difference, we present in Fig. 5 a quadruplet of states corresponding to the $|k| = 1$ and $|m| = 1$ case, in which the pair of states $|\mathrm{DG}(\pm l_2,N)\rangle_v$ fulfilling the condition $|l| < N$ are shown in the left column and the other pair $|\mathrm{DG}(\pm l_2,N)\rangle_v$ fulfilling $|l| > N$ is shown in the right. According to Eq. (44), when $|k| = 1$, the τ dependence of the radial coordinate of phase singularities is different for $|\mathrm{DG}(\pm l_2,N)\rangle_v$ states, as compared to the rest of cases. This feature can be clearly appreciated in Fig. 5 by comparing the different behavior of dark beams near the origin in the left and right columns.

The dark focus is one of the most distinguishing features of a DG state. A general property of the dark focus is apparent in the examples provided in Fig. 5. The axial charge, as dictated by Eq. (42), is $l - \mathrm{sgn}(k)N$ for $\tau \neq 0$. However, the convergence of the N dark rays of the DG state in $\tau = 0$ determines the topological charge of the dark focus to be precisely l: $q_{\mathrm{df}} = l$. In this way, Eq. (42) can be understood now as a conservation law for the topological charge: $q_{\mathrm{df}} = q_{\mathrm{ax}} + \mathrm{sgn}(k)N$. For $k > 0$ ($k < 0$), off-axis dark rays correspond to $+1$ (-1) charges. The conservation of l, despite being no longer the OAM of the state, can be interpreted as the topological conservation law

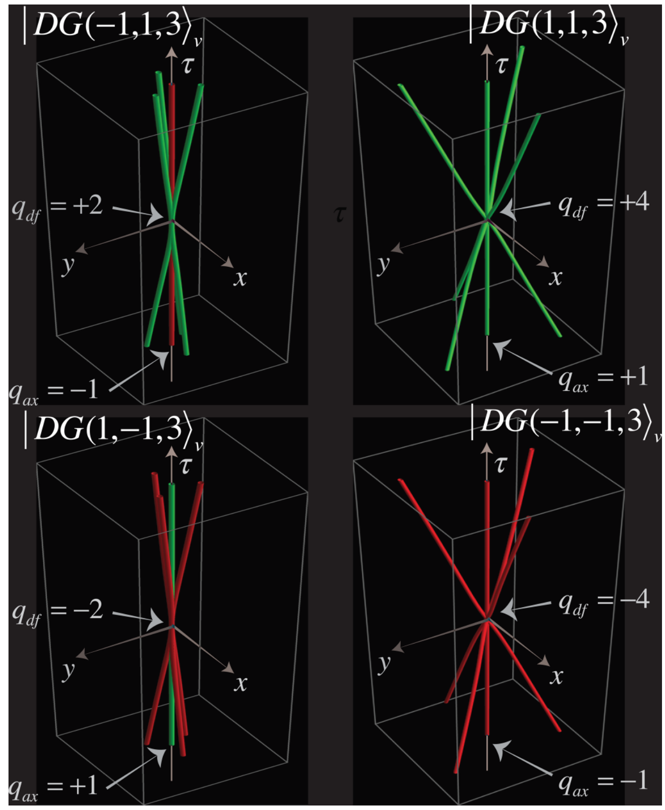

FIG. 5. (Color online) Structure of focusing dark beams embedded in lower-order states near the *dark focus* for a quadruplet with $|k| = 1$ and $|m| = 1$ for $N = 3$. The two states on the left column correspond to $l = \pm l_2 = \pm 2$, whereas the ones in the right column correspond to $l = \pm l_1 = \pm 4$. [$\tau_R = 1$; $v = 0.1$; $\tau_{\min} = -3$; $\tau_{\max} = 3$; transverse range: left $L = 0.75$ and right $L = 1.5$.]

associated to the $\mathcal{C}_N$ discrete rotational symmetry of the DG state.

Besides, DG states with $q_{\mathrm{df}} = \pm|l|$ present a dark beam structure that is related by the $w \leftrightarrow \overline{w}$ duality symmetry. In Cartesian coordinates this symmetry is equivalent to the mirror reflection:

$$R_x : (x,y) \xrightarrow{R_x} (x,-y), \tag{45}$$

together with a *simultaneous* charge conjugation $q \to -q$ of all topological charges. We can check this symmetry in Fig. 5 as well. Position of dark rays for the states in the lower row can be obtained, respectively, by properly mirror reflecting with respect to the x axis the dark beams of the upper row along with charge conjugation (red-green color exchange in Fig. 5).

DG states with $k \neq 0$ exhibit a rich diversity of dark beam structures embedded in their Gaussian-like amplitudes. The form of a generic solution of a DG state, such as given in Eqs. (30) and Eqs. (37), indicates that the properties of the dark beam, encoded in the $\mathcal{F}_{mkNv}$ function, and of the bright part of the beam, encoded in the Gaussian function ϕ_{00}, present a certain degree of independence. We see that the v parameter only affects the dark beam function $\mathcal{F}_{mkNv}$, whereas τ_R and the beam parameter $q(\tau)$ appear in both functions, but in completely different functional ways. Thus we expect some

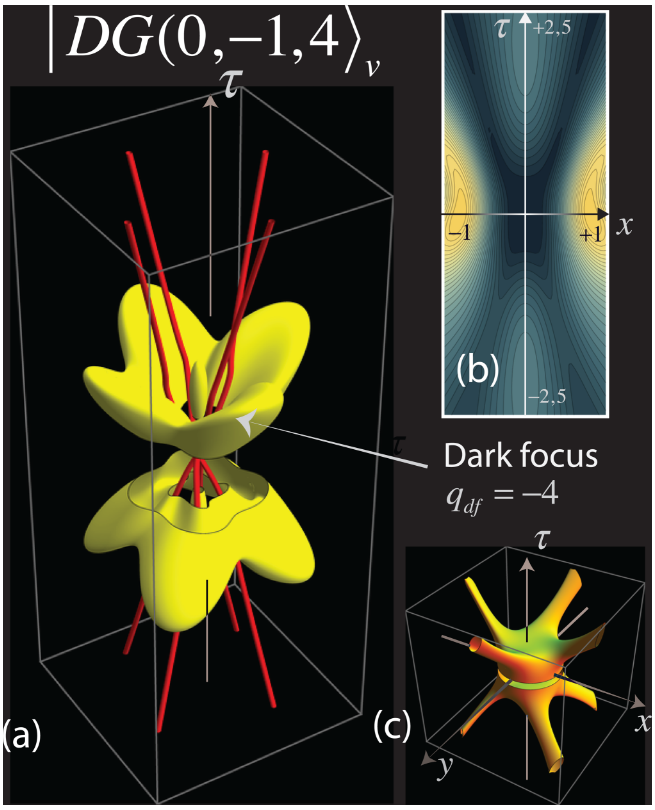

FIG. 6. (Color online) Dark focus structure of a DG state with $m = 0$ and no axial dark ray [$\tau_R = 1$; $v = 0.25$]. (a) Amplitude at 1/4 maximum and dark beam with $q_{\rm ax} = 0$ and $q_{\rm df} = -4$; 3D box is $8 \times 8 \times 20$. (b) Dark focus xz section. (c) 3D representation of the dark focus region (1/30 maximum); 3D box is $1\times1\times2.5$.

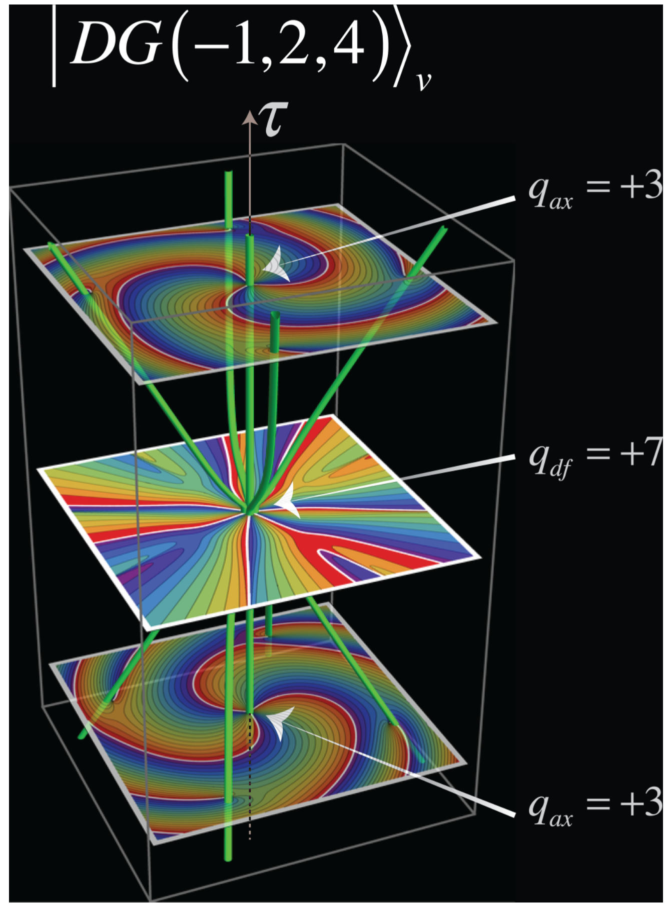

FIG. 7. (Color online) DG state with $k = 2$ generating a focusing dark beam with a highly charged singularity on axis [$\tau_R = 1$; $v = 0.1$]. Phase profiles are represented at $\tau = -5$, 0, and +5. 3D box is $7 \times 7 \times 12$.

type of interplay between dark beams and bright amplitudes in terms of these parameters. In Fig. 6 we present an interesting case of the DG state characterized by $m = 0$ and $k = -1$. In such a state, the dark focus does not exhibit a dark ray on axis since, according to Eq. (42), its axial charge is $q_{\rm ax} = 0$ for all $\tau \neq 0$. The only on-axis singularity is located at $\tau = 0$ being absent for $\tau \neq 0$. Besides, the interplay between the bright part of the beam and the dark beam here is strong. This fact is reflected in the remarkable modulation of the amplitude near the dark rays, visible in Fig. 6(a). In this case, as mentioned before, a strong interplay between the bright and dark parts of the beam is achieved by increasing the value of v (larger than in previous cases.) The 2D and 3D representations of the dark focus region near the origin in Figs. 6(b) and 6(c) reveals a combination of high intensity gradients with high phase contrasts (note that the topological charge at the focus is $q_{\rm df} = 4$.)

In discrete symmetry media, the presence of a $\mathcal{C}_N$-invariant potential owning discrete rotational symmetry and extending infinitely in τ forces the axial charge of a vortex to be constrained by the rule $|q_{\rm max}| < N/2$ [30,31]. On the contrary, DG states with highly charged singularities on axis are allowed by the topological law (42) beyond this cutoff provided $|k| \geqslant 2$. In Fig. 7 we present an example of such a state with $k = 2$. The dark beam pattern is, nevertheless, the same as for any other DG state. N single off-axis phase singularities merge with the axial singularity at the dark focus once and then diverge. The difference now is that the axial charge $q_{\rm ax} = 3$ exceeds the maximum value for the axial charge allowed by the previous rule for discrete potentials (in this case, $|q_{\rm max}| = 1$, for $N = 4$.) Note that the later rule applies to potentials that act during an infinitely long period in τ, whereas DG states are associated to the action of instantaneous potentials. In this way, the seeming contradiction is removed.

VI. CONCLUSIONS

Generation of DG states is possible because the discrete deformation operator, generated by the instantaneous $\mathcal{C}_N$ potential, acts as a "state converter" changing an LG mode for $\tau < 0$ into a DG state for $\tau > 0$. Since the form of the potential appearing in the discrete deformation operator $\hat{D}_v$ is valid for general real discrete potentials for small w (up to a global rotation), approximated DG states are expected to appear in scattering or diffraction experiments in which $O(2)$ symmetry is broken. Recent experiments of vortex diffraction in optics using discrete diffractive optical elements (DOE) show, in fact, output states that can be assimilated to DG or quasi-DG states [34,35,54–57]. These experiments can be reinterpreted as examples of discrete deformation operators generating DG states in $\tau > 0$ out of LG modes in $\tau < 0$.

The general framework here presented opens the door to the control of DG states and dark beams beyond previously proposed strategies [33]. It is feasible to find other types of instantaneous potentials—not necessarily real—leading to different deformation operators acting as generalized state converters between arbitrary DG states. As an example, it is possible to design potentials transforming a DG state with a dark focus on a given value of τ into a DG state with a dark focus in a different position. This designed potential would act as a lens for dark rays imaging one dark focus onto the other. Its experimental feasibility in optics is realistic using current encoding techniques to design DOE with arbitrary phase profiles [58]. A similar strategy including reflecting optical elements [4] would permit the design of DG resonators acting as dark beam cavities. A novel geometrical optics for dark rays can be then envisaged in analogy to the classical geometrical optics used for the manipulation of ordinary bright rays [33].

Besides the control of dark rays, DG states present a rich and versatile structure for the gradients of both the phase and amplitude of the field. Thus the present formalism can be of help to design adequate optical forces for optical trapping of small neutral particles, atoms, and molecules [38,41–43]. We have seen that DG states present the possibility to manipulate their dark (i.e., phase) and bright (i.e., intensity) profiles with a certain degree of independence. This feature combined with the potential control of dark beams using DOE, which are also standard tools for manipulating Gaussian beams, permits one to foresee interesting applications in optical trapping. It is remarkable here that DG states are experimentally obtained by simple diffraction using discrete DOE [34,35], instead of by multiple interference of LG modes, as required in the generation of other multisingular solutions [8–19].

Finally, the potential application of DG states in quantum optics and quantum mechanics should not be ignored, which is based in the fact that DG states form a biorthogonal set in the same way as the complex-argument LG modes from which they are derived [53]. So, DG states can be legitimately used as a basis for operator expansions of the quantum field in the same way as plane waves (momentum expansion) or LG modes (angular momentum expansion). The advantage here is that they present a richer phase singularity structure than other Gaussian modes. Besides, they provide an expansion in a different quantum number, namely, the discrete angular momentum m. Quantum states based on the discrete angular momentum m can provide an alternative to high-dimensional quantum spaces based on OAM [44]. Additionally, they present a different quantum operational algebra and a more complex spatial mode structure that can bring a new perspective for quantum information processing [45].

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