

Theory of optomechanical interactions in superfluid He

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A general theory is presented to describe optomechanical interactions of acoustic phonons, having extremely long lifetimes in superfluid ^4He , with optical photons in the medium placed in a suitable electromagnetic cavity. The acoustic nonlinearity in the fluid motion is included to consider processes beyond the usual linear process involving the absorption or emission of one phonon at a time. We first apply our formulation to the simplest one-phonon process involving the usual resonant anti-Stokes upconversion of an incident optical mode. However, when the allowed optical cavity modes are such that there is no single-phonon mode in the superfluid, which can give rise to a resonant allowed anti-Stokes mode, we must consider the possibility of two-phonon upconversion. For such a case, we show that the two-step two-phonon process could be dominant. We present arguments for a large two-step process and negligible single-step two-phonon contribution. The two-step process also shows interesting quantum interference among different transition pathways.

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I. INTRODUCTION

In the field of optomechanics, one is always designing mechanical systems with the lowest possible friction [1,2] and highest possible optomechanical coupling constant [3–5]. This is because one would like to produce and use coherent phonons with a long coherence time. One prominent application of coherent phonons is in the storage and retrieval of light using optomechanical systems [6,7]. An attractive system which has received considerable attention is the levitated microsphere trapped in an optical cavity. Both trapping and levitation can be produced by optical fields [1,2]. Another very attractive system is superfluid He, which has zero viscosity. It is known to have acoustic phonons with almost zero friction at low temperatures, a finite value arising only from thermal three-phonon scattering processes [8]. De Lorenzo and Schwab [8] have performed initial optomechanical experiments on superfluid He by coupling it to a superconducting resonator. Flowers-Jacobs *et al.* [9] have reported progress in doing optomechanics with superfluid He using optical cavities.

In view of the current interest [8,9] in the optomechanics with superfluid He, we present in this paper theoretical foundations of optomechanics in such systems. The organization of this paper is as follows. In Sec. II, we derive the basic semiclassical equations for the optomechanical interactions in superfluid He. In Sec. III, we present a Hamiltonian formulation of the problem in terms of the canonical variables, so that this can be adopted for quantized phonon and photon fields. The theory is formulated in terms of the fields, both electromagnetic and fluid density, so that situations involving many phonons and photons of different frequencies can be handled. In Sec. IV, we present a quantized description of the optomechanical interactions. We present estimates for the strength of the optomechanical interactions. The linear optomechanical interaction (shift of the cavity resonance per photon) is quite significant in cavities like a fiber cavity [10]. We derive the canonical form [11–14] of the Hamiltonian for linear optomechanical interactions in superfluid He. Having

obtained the canonical form, we can study all the physical processes that have been studied with other optomechanical systems. An estimate of the single-step two-phonon anti-Stokes process due to acoustic nonlinearity is also given in this section. In Sec. V, we discuss two-step two-phonon processes, which are shown to be significant in superfluid He. It may be noted that the knowledge of the linear interaction Hamiltonian (51) with the strength of g estimated after Eq. (47) is sufficient to understand the processes in Sec. V. When the electromagnetic cavity is designed in such a way that allowed optical modes are such that no anti-Stokes upconversion is possible via the absorption of any single phonon in the medium, one must consider the absorption of two phonons for possible upconversion. Such a process can be controlled well when these phonons are external phonons injected in the medium. Because of the intrinsic nonlinearity of the superfluid He, we have new possibilities arising from the combination of the superfluid nonlinearity and the optomechanical interactions.

II. CLASSICAL NONLINEAR EQUATIONS FOR SUPERFLUID HELIUM OPTOMECHANICS

In this section, we start with the fundamental equations [15] for the superfluid density ρ and the velocity \vec{v} and we obtain modifications of these due to interaction with the electromagnetic fields. The basic equations for ρ and \vec{v} in the absence of the electromagnetic fields are given by

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0, \quad (1)$$

$$\frac{\partial}{\partial t}(\rho \vec{v}) + \vec{\nabla} \cdot \overleftrightarrow{T} = 0, \quad (2)$$

where the stress tensor \overleftrightarrow{T} is given by

$$T_{ij} = p\delta_{ij} + \rho v_i v_j. \quad (3)$$

Here p is the pressure in the superfluid. We have set the viscosity term zero. We note that we have a set of nonlinear equations as pressure is generally expanded [16,17] in terms of the normalized deviation $\tilde{\rho} = (\rho - \rho_0)/\rho_0$ from the equilibrium value ρ_0 :

$$p - p_0 \approx \left(\rho \frac{\partial p}{\partial \rho} \right)_0 \tilde{\rho} + \frac{1}{2} \left(\rho^2 \frac{\partial^2 p}{\partial \rho^2} \right)_0 \tilde{\rho}^2 + \dots \quad (4)$$

We now discuss the modification of Eqs. (1) and (2) due to the interaction with the electromagnetic fields. Clearly Eq. (1) remains unchanged. We need to modify Eq. (2) by the addition of the Maxwell stress contribution $\overleftrightarrow{T}^{(\text{em})}$ to (2), i.e.,

$$\frac{\partial}{\partial t}(\rho \vec{v}) + \vec{\nabla} \cdot \overleftrightarrow{T} - \vec{\nabla} \cdot \overleftrightarrow{T}^{(\text{em})} = 0. \quad (5)$$

The form of the Maxwell stress tensor depends on the nature of the medium. It is derived from the considerations of the electromagnetic force on the medium. On dropping the magnetic polarization contribution, the force on a linear medium can be written as

$$F_j^{(\text{em})} = \int d^3r \sum_i \frac{\partial}{\partial r_i} T_{ij}^{(\text{em})},$$

$$T_{ij}^{(\text{em})} = -\frac{1}{2} \delta_{ij} \epsilon(\vec{r}) \vec{E} \cdot \vec{E} + \epsilon(\vec{r}) E_i E_j. \quad (6)$$

Here \vec{E} is the electromagnetic field and $\epsilon(\vec{r})$ is the optical dielectric function of the isotropic superfluid. The electromagnetic force can also be written in an alternate form [18]

$$\vec{F}^{(\text{em})} = -\frac{1}{2} \int d^3r E^2(\vec{r}) \vec{\nabla} \epsilon(\vec{r}). \quad (7)$$

The dielectric function of the medium depends on \vec{r} through the density, i.e., $\epsilon(\vec{r}) = \epsilon[\rho(\vec{r})]$ and hence

$$\vec{\nabla} \epsilon(\vec{r}) = \frac{\partial \epsilon[\rho(\vec{r})]}{\partial \rho} \vec{\nabla} \rho, \quad (8)$$

and hence Eq. (7) reduces to

$$\vec{F}^{(\text{em})} = -\frac{1}{2} \int d^3r E^2(\vec{r}) \frac{\partial \epsilon}{\partial \rho} \vec{\nabla} \rho$$

$$= \frac{1}{2} \int d^3r \rho(\vec{r}) \vec{\nabla} \left[\frac{\partial \epsilon}{\partial \rho} E^2(\vec{r}) \right]. \quad (9)$$

Using Eq. (9), Eq. (5) becomes

$$\frac{\partial}{\partial t}(\rho \vec{v}) + \vec{\nabla} \cdot \overleftrightarrow{T} + \frac{1}{2} \rho \vec{\nabla} \left[\frac{\partial \epsilon}{\partial \rho} E^2(\vec{r}) \right] = 0. \quad (10)$$

Equations (1) and (10) are the basic equations for the optomechanical interactions in superfluid He. The only assumption that we made in deriving Eq. (10) is the linear electromagnetic response $\epsilon[\rho(\vec{r})]$ of superfluid He. Equations (1) and (10) are to be supplemented by the expansion (4). The electric field obeys the equation

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} + \frac{1}{\epsilon_0 c^2} \frac{\partial^2}{\partial t^2} (\epsilon[\rho(\vec{r})] \vec{E}) = 0, \quad (11)$$

which is obtained from the Maxwell equations.

III. HAMILTONIAN DESCRIPTION OF THE BASIC EQS. (10) AND (11)

In the Hamiltonian description, one introduces the conjugate variables $\rho(\vec{r})$ and $\Phi(\vec{r})$ and the classical velocity is related to Φ via

$$\vec{v} = -\vec{\nabla} \Phi. \quad (12)$$

The Hamiltonian description of the superfluid equations (1) and (2) is well known and for completeness we recall the main aspects. The unperturbed Hamiltonian density is

$$\mathcal{H}_0 = \frac{1}{2} \rho (\nabla \Phi)^2 \quad (13)$$

and the interaction term is

$$\mathcal{H}_1 = \rho W(\rho). \quad (14)$$

The function W is related to the pressure via the thermodynamic relation [17]

$$W(\rho) = \int_{\rho_0}^{\rho} \frac{p(\rho')}{\rho'^2} d\rho'. \quad (15)$$

Using the total Hamiltonian $H = \int d^3r (\mathcal{H}_0 + \mathcal{H}_1)$, we can see how Eqs. (13) to (15) lead to Eqs. (1) and (2). For this purpose, we use the Hamiltonian formulation for fields [19]

$$\dot{\rho} = -\frac{\delta H}{\delta \Phi} = \sum_j \frac{\partial}{\partial r_j} \left(\frac{\partial H}{\partial (\partial \Phi / \partial r_j)} \right) - \frac{\partial H}{\partial \Phi}$$

$$= \sum_j \frac{\partial}{\partial r_j} \left[\rho \frac{\partial \Phi}{\partial r_j} \right] = -\sum_j \frac{\partial}{\partial r_j} [\rho v_j]$$

$$= -\vec{\nabla} \cdot (\rho \vec{v}), \quad (16)$$

$$\dot{\Phi} = \frac{\delta H}{\delta \rho} = \frac{1}{2} (\nabla \Phi)^2 + \frac{\partial}{\partial \rho} [\rho W(\rho)]$$

$$= \frac{1}{2} v^2 + \left(W + \frac{p(\rho)}{\rho} \right). \quad (17)$$

We can convert Eq. (17) into an equation for $(\rho \vec{v})$ as follows

$$\frac{\partial}{\partial t}(\rho \vec{v}) = \frac{\partial \rho}{\partial t} \vec{v} - \rho \frac{\partial}{\partial t} \vec{\nabla} \Phi$$

$$= -[\vec{\nabla} \cdot (\rho \vec{v})] \vec{v} - \rho \vec{\nabla} \left[\frac{1}{2} v^2 + W + \frac{p(\rho)}{\rho} \right]$$

$$= -\rho (\vec{\nabla} \rho) \frac{\partial}{\partial \rho} \left[W + \frac{p(\rho)}{\rho} \right] - \vec{\nabla} \cdot [\rho \vec{v} \vec{v}]$$

$$= -\vec{\nabla} \cdot [\rho \vec{v} \vec{v}] - \vec{\nabla} \cdot (\rho \overleftrightarrow{T}), \quad (18)$$

where \overleftrightarrow{T} is the unit tensor. Equations (16) and (18) are identical to Eqs. (1) and (2), respectively.

The Hamiltonian for optomechanical interactions in superfluid He will then be

$$H = \int d^3r (\mathcal{H}_0 + \mathcal{H}_1) + \int \mathcal{H}^{(\text{em})} d^3r, \quad (19)$$

$$\mathcal{H}^{(\text{em})} = \mathcal{H}_0^{(\text{em})} + \mathcal{H}_1^{(\text{em})}, \quad (20)$$

$$\mathcal{H}_0^{(\text{em})} = \frac{1}{2} \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right), \quad (21)$$

$$\mathcal{H}_1^{(\text{em})} = -\frac{1}{2}\vec{P} \cdot \vec{E} = -\left(\frac{\epsilon[\rho] - \epsilon_0}{2}\right)E^2, \quad (22)$$

where \vec{P} is the polarization in a superfluid medium. Using Eq. (19), the equations for the canonical conjugate variables ρ and Φ are

$$\dot{\rho} - \vec{\nabla} \cdot (\rho \vec{\nabla} \Phi) = 0, \quad (23)$$

$$\dot{\Phi} = \frac{1}{2}(\vec{\nabla} \Phi)^2 + \frac{\partial}{\partial \rho}(\rho W) - \frac{1}{2}\left(\frac{\partial \epsilon}{\partial \rho}\right)E^2. \quad (24)$$

A simple exercise shows that Eq. (24) is equivalent to Eq. (10) for $\rho \vec{v}$.

The Hamiltonian (19) depends on the density ρ to all orders. To bring out some of the important physical processes, we consider an expansion of H in powers of the deviation $\tilde{\rho}$, $(\rho - \rho_0)/\rho_0$ from the equilibrium value ρ_0 . We will examine terms up to second order in $\tilde{\rho}$. The expansion of the optomechanical interaction term is straightforward

$$\epsilon[\rho] = \epsilon[\rho_0] + \rho_0 \left(\frac{\partial \epsilon}{\partial \rho}\right)_0 \tilde{\rho} + \frac{1}{2}\rho_0^2 \left(\frac{\partial^2 \epsilon}{\partial \rho^2}\right)_0 \tilde{\rho}^2 + \dots, \quad (25)$$

$$\mathcal{H}_1^{(\text{em})} = -\frac{\epsilon[\rho_0] - \epsilon_0}{2}E^2 - \frac{1}{2}g_1\epsilon_0\tilde{\rho}E^2 - \frac{1}{2}g_2\epsilon_0\tilde{\rho}^2E^2 + \dots, \quad (26)$$

$$g_1 = \frac{\rho_0}{\epsilon_0} \left(\frac{\partial \epsilon}{\partial \rho}\right)_0, \quad g_2 = \frac{\rho_0^2}{2\epsilon_0} \left(\frac{\partial^2 \epsilon}{\partial \rho^2}\right)_0.$$

Here g_1 and g_2 are the coupling constants for the linear and quadratic optomechanical interactions. A rough estimate of g_1 and g_2 can be obtained from the experimental data [20] on liquid He

$$\frac{\epsilon(\rho)}{\epsilon_0} = \frac{1 + \frac{8\pi}{3}\frac{\alpha_m}{m}\rho}{1 - \frac{4\pi}{3}\frac{\alpha_m}{m}\rho}, \quad (27)$$

where the molecular polarizability α_m is $1.23296 \times 10^{-7} \text{ m}^3/\text{mole}$, $m = 4.0026 \times 10^{-3} \text{ kg/mole}$, equilibrium density $\rho_0 = 145.1397 \text{ kg/m}^3$, and hence

$$g_1 \cong 0.05826, \quad g_2 \cong 0.00111. \quad (28)$$

Further, $\epsilon(\rho_0)/\epsilon_0 = 1.057$ and therefore the term $-\frac{\epsilon(\rho_0) - \epsilon_0}{2}E^2$ contributes to small frequency shifts of the electromagnetic fields. We will ignore such frequency shifts. The term \mathcal{H}_1 gives the nonlinearities of the superfluid in the absence of any applied electromagnetic fields. We write $p(\rho)$ as

$$p(\rho) - p_0 = \rho_0 \frac{\partial p}{\partial \rho} \tilde{\rho} + \frac{1}{2}\rho_0^2 \frac{\partial^2 p}{\partial \rho^2} \tilde{\rho}^2 + \dots$$

$$= (\rho_0 v_s^2) \tilde{\rho} + \frac{1}{2}A_2 \tilde{\rho}^2 + \dots, \quad (29)$$

and use $W(\rho_0) = 0$, to obtain

$$\mathcal{H}_1 = \frac{1}{2}(\rho_0 v_s^2) \tilde{\rho}^2 + \frac{1}{6}(A_2 - \rho_0 v_s^2) \tilde{\rho}^3 + \dots. \quad (30)$$

The parameter $A_2/(2\rho_0 v_s^2)$ is called the Gruneisen constant [17] and has the value 2.84. In Eq. (29), $v_s (=238 \text{ m/s})$ is the velocity of sound. The nonlinear conversion of phonons, as determined by the $\tilde{\rho}^3$ term in Eq. (30), has been discussed

by Wright *et al.* [16]. To simplify \mathcal{H}_0 in powers of $\tilde{\rho}$, we need to find the expansion of $\vec{\nabla} \Phi$, which can be obtained from Eq. (23), which to lowest order in density yields

$$\nabla^2 \Phi = \tilde{\rho}. \quad (31)$$

IV. QUANTIZATION OF THE HAMILTONIAN FOR OPTOMECHANICAL INTERACTIONS

To do the quantization, we invoke the space-time structure of the electromagnetic and density (acoustic) fields. We would be studying optomechanical interactions in a cavity which could be an optical one like a fiber cavity [10] or a superconducting one [8]. The electromagnetic field can be written as a superposition of orthogonal and orthonormal transverse modes $\vec{u}^{(i)}$, i.e.,

$$\vec{E}(\vec{r}, t) = \sum_i \vec{u}^{(i)}(\vec{r}) \mathcal{E}^{(i)} e^{-i\omega_i t} + \text{c.c.}, \quad (32)$$

where the mode function $\vec{u}^{(i)}$ has frequency Ω_i and is a solution of $\nabla^2 \vec{u}^{(i)} + (\Omega_i^2/c^2)\vec{u}^{(i)} = 0$. The Hamiltonian (21) for the electromagnetic field leads to

$$\mathcal{H}_0^{(\text{em})} = 2\epsilon_0 \sum_i |\mathcal{E}^{(i)}|^2, \quad (33)$$

where we used the orthogonality of the mode functions $\int [u^{(i)}(\vec{r}) \cdot u^{(j)*}(\vec{r})] d^3r = \delta_{ij}$. To do the quantization, we identify $2\epsilon_0 |\mathcal{E}|^2$ with $\hbar\omega a^\dagger a$. Thus the amplitude \mathcal{E} is to be replaced by the annihilation operator a via

$$\mathcal{E} \rightarrow \sqrt{\frac{\hbar\omega}{2\epsilon_0}} a. \quad (34)$$

Therefore, the quantized form of the electric field is

$$\vec{E} = \sum_i \sqrt{\frac{\hbar\omega_i}{2\epsilon_0}} \vec{u}^{(i)}(\vec{r}) a_i e^{-i\omega_i t} + \text{c.c.}, \quad (35)$$

and the unperturbed Hamiltonian is

$$H_0^{(\text{em})} = \sum_i \hbar\omega_i a_i^\dagger a_i. \quad (36)$$

We expand the phonon field in terms of the normalized mode functions ψ_i with frequency f_i ,

$$\tilde{\rho} = \sum_i \psi_i(\vec{r}) e^{-if_i t} \sigma_i + \text{c.c.}, \quad (37)$$

$$\nabla^2 \psi_i + (f_i^2/v_s^2)\psi_i = 0. \quad (38)$$

Note that ψ_i has the dimension $1/\sqrt{\text{Volume}}$ and hence σ has the dimension $\sqrt{\text{Volume}}$. The quantization of the free phonon field is more complicated due to the nonlinear nature of the interaction term (14). To do the quantization, we look at the harmonic version of (14), i.e., we use Eq. (30) up to order $\tilde{\rho}^2$.

We next find H_0 [Eq. (13)] to lowest order, i.e., up to second order in density. From Eqs. (31), (37), and (38), we can easily obtain

$$\Phi = \sum_i i \frac{v_s^2}{f_i} \psi_i(\vec{r}) e^{-if_i t} \sigma_i + \text{c.c.}, \quad (39)$$

and hence for a given mode

$$\begin{aligned} \int \mathcal{H}_0 d^3r &= -\frac{1}{2}\rho_0 \int \Phi \nabla^2 \Phi d^3r \\ &= \frac{f_i^2}{2v_s^2} \rho_0 \int \Phi^2 d^3r = \rho_0 v_s^2 |\sigma|^2. \end{aligned} \quad (40)$$

We thus quantize the phonon field via

$$\sigma_i \rightarrow \sqrt{\frac{\hbar f_i}{2\rho_0 v_s^2}} b_i, \quad (41)$$

where b_i is the Bosonic annihilation operator for the phonon with frequency f_i . The unperturbed Hamiltonian for the phonon field

$$\tilde{\rho} = \sum \psi_i(\vec{r}) \sqrt{\frac{\hbar f_i}{2\rho_0 v_s^2}} b_i e^{-if_i t} + \text{c.c.}, \quad (42)$$

is

$$H_0 = \sum_i \hbar f_i b_i^\dagger b_i + O(\tilde{\rho}^3). \quad (43)$$

The terms of the order $\tilde{\rho}^3$ can be obtained by using the expansion (30). These correspond to three-phonon scattering processes. The quantized form of the interaction Hamiltonian (26) can now be obtained by using Eqs. (35) and (42). The final result is dependent on the different modes involved in the optomechanical interactions and their overlap integrals. We write

$$H_1^{(\text{em})} = v_L + v_{NL}, \quad (44)$$

where v_L and v_{NL} are the linear and nonlinear optomechanical interactions, respectively. In what follows, we drop all rapidly oscillating terms at two times the cavity frequencies.

The linear part then can be written as

$$\begin{aligned} v_L &= -\hbar \sum_{ijl} \{g'_{ijl} e^{i(\omega_i - \omega_j - f_l)t} a_i^\dagger a_j b_l \\ &\quad + g''_{ijl} e^{i(\omega_i - \omega_j + f_l)t} a_i^\dagger a_j b_l^\dagger\}, \end{aligned} \quad (45)$$

$$g'_{ijl} = \sqrt{\frac{\hbar f_l \omega_i \omega_j}{8\rho_0 v_s^2}} g_1 \int \psi_l(\vec{r}) [\vec{u}^{(i)*}(\vec{r}) \cdot \vec{u}^{(j)}(\vec{r})] d^3r, \quad (46)$$

and g'' is obtained from g' by replacing $\psi(\vec{r})$ by $\psi^*(\vec{r})$. For simplicity, we choose ψ to be real and then we can drop the distinction between g' and g'' . The quantity g'_{iil} is the frequency shift of the cavity mode for one phonon. We can get an approximate estimate of g'_{ijl} by using $\psi(\vec{r}) = 1/\sqrt{V}$,

$$g'_{ijl} \approx \sqrt{\frac{\hbar f_l}{8\rho_0 v_s^2}} \frac{\omega g_1}{\sqrt{V}}. \quad (47)$$

For the fiber cavity [10] taking the mode volume about $V \sim 10^{-14} \text{ m}^3$, ω corresponding to $1 \mu\text{m}$, $f_1 \sim 2\pi \times 10 \text{ MHz}$, $\rho_0 v_s^2 \approx 8213380 \text{ J/m}^3$, we find $g'_{iil}/g_1 \approx 2\pi \times 30 \text{ kHz}$ and hence $g'_{iil} \approx 2\pi \times 1.8 \text{ kHz}$. Thus, linear optomechanical interaction is quite significant and is comparable to that obtained with mechanical elements [3,4].

The nonlinear part v_{NL} has several contributions. We do not write all the terms but make an estimate. A term corresponding to two-phonon absorption has the form

$$-\hbar \sum_{ijl_1 l_2} p_{ijl_1 l_2} e^{i(\omega_i - \omega_j - f_{l_1} - f_{l_2})t} a_i^\dagger a_j b_{l_1} b_{l_2},$$

where $p_{ijl_1 l_2}$ has the form

$$\begin{aligned} p_{ijl_1 l_2} &= \sqrt{\frac{\hbar f_{l_1} \omega_i \omega_j}{8\rho_0 v_s^2}} g_2 \sqrt{\frac{\hbar f_{l_2}}{2\rho_0 v_s^2}} \\ &\quad \times \int d^3r \psi_{l_1}(\vec{r}) \psi_{l_2}(\vec{r}) \vec{u}^{(i)*} \cdot \vec{u}^{(j)}. \end{aligned} \quad (48)$$

Let us take $\vec{u}^{(i)}$ and $\vec{u}^{(j)}$ to be the same mode, then

$$p_{ijl_1 l_2} \approx \sqrt{\frac{\hbar f_{l_1}}{8\rho_0 v_s^2 V}} \sqrt{\frac{\hbar f_{l_2}}{2\rho_0 v_s^2 V}} \omega g_2. \quad (49)$$

Thus compared to the first-order optomechanical coupling, the second order is smaller by a factor

$$\sqrt{\frac{\hbar f_l}{2\rho_0 v_s^2 V}} \left(\frac{g_2}{g_1}\right) \approx 1 \times 10^{-10} \left(\frac{g_2}{g_1}\right) \sim 5 \times 10^{-12},$$

for parameters that were used in the estimation of linear optomechanical coupling. The second-order contribution has been generally found to be unimportant in most mechanical systems. However, considerable progress has been reported in achieving higher second-order coupling by placing the mechanical element at the crossing of two modes [4,5]. One possibility to enhance p would be to use the nanometric volume for He. In view of the smallness of $p_{ijl_1 l_2}$ compared to g_{ijl} , we drop the contribution v_{NL} and work with

$$v_L = -\hbar \sum_{ijl} \{g_{ijl} a_i^\dagger a_j e^{i(\omega_i - \omega_j)t} (b_l e^{-if_l t} + b_l^\dagger e^{if_l t}) + \text{H.c.}\}. \quad (50)$$

For two field modes and one acoustic mode and assuming that $\vec{u}^{(1)} = \vec{u}^{(2)}$, we can simplify Eq. (50) to

$$\begin{aligned} v_L &= -\hbar g (a_1^\dagger a_1 + a_2^\dagger a_2 + a_1^\dagger a_2 e^{i(\omega_1 - \omega_2)t} \\ &\quad + a_1 a_2^\dagger e^{-i(\omega_1 - \omega_2)t}) (b_l e^{-if_l t} + b_l^\dagger e^{if_l t}). \end{aligned} \quad (51)$$

This is the standard form of the optomechanical interaction. The terms like $(a_1^\dagger a_2 b_l e^{i(\omega_1 - \omega_2 - f_l)t} + \text{H.c.})$ describe the upconversion process where a phonon and a photon ω_2 combine to produce a photon ω_1 if $\omega_1 > \omega_2$. The terms like $(a_1 a_2^\dagger b_l^\dagger e^{-i(\omega_1 - \omega_2 - f_l)t} + \text{H.c.})$ describe a downconversion process where a photon ω_2 and a phonon f_l are produced from a photon of frequency ω_1 . The Hamiltonian (51) also consists of nonresonant terms like $a_1^\dagger a_1 b_l$. These terms play a significant role in two-step two-phonon processes as discussed in the next section. Clearly, the previously discussed processes [11–14] in other optomechanical systems would also apply to optomechanics in superfluid He since the linear coupling constant g in Eq. (51) is quite significant. The advantage of superfluid He is its very large coherence time of the phonon,

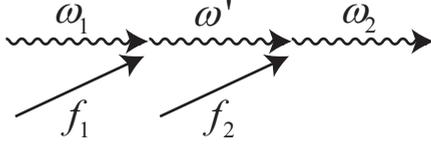


FIG. 1. The two-step two-phonon anti-Stokes process.

which is especially useful in quantum processing applications like state transfer [21] and quantum memories [6,7,22].

V. OPTOMECHANICAL INTERACTIONS IN SUPERFLUID He INVOLVING TWO-STEP TWO-PHONON PROCESSES

We next consider the very interesting possibility of two-phonon absorption [23] in optomechanical interactions in superfluid He. This, in a sense, is the analog of two-photon absorption in atomic systems. Let us consider the following two steps:

$$\begin{aligned} \text{phonon}(f_1) + \text{photon}(\omega_1) &\rightarrow \text{intermediate photon}(\omega'), \\ \text{intermediate photon}(\omega') + \text{phonon}(f_2) &\rightarrow \text{photon}(\omega_2). \end{aligned}$$

This two-step process shown in Fig. 1 is resonant if

$$\omega_1 + f_1 + f_2 = \omega_2, \quad (52)$$

and is a combination of two upconversion processes. The two-step process is mediated by an intermediate photon of frequency ω' . Such a two-step process can be significant as the intermediate photon can have the same frequency as the strong input photon ω_1 . Such a contribution comes from the term $a_1^\dagger a_1 b_1$ in the interaction (50). For the present problem, the interaction v_L can be written as

$$\begin{aligned} v_L = &-\hbar g(a_1 + a_2 e^{-i(\omega_2 - \omega_1)t})^\dagger (a_1 + a_2 e^{-i(\omega_2 - \omega_1)t}) \\ &(b_1 e^{-if_1 t} + b_2 e^{-if_2 t} + \text{H.c.}). \end{aligned} \quad (53)$$

Let the initial state be $|n_1, n_2, \mu_1, \mu_2\rangle$ which has $n_1(n_2)$ photons of frequency $\omega_1(\omega_2)$ and $\mu_1(\mu_2)$ phonons of frequency $f_1(f_2)$. The final state is $|n_1 - 1, n_2 + 1, \mu_1 - 1, \mu_2 - 1\rangle$. The transition probability for this process can be obtained by using second-order Fermi golden rule

$$R^{[2]} = \frac{2\pi}{\hbar} \delta(E_f - E_i) \left| \sum_j \frac{\langle f | v_L | j \rangle \langle j | v_L | i \rangle}{E_j - E_i} \right|^2, \quad (54)$$

where $|j\rangle$ is the allowed intermediate state. The four important intermediate states with the corresponding energies are

$$\begin{aligned} |n_1 - 1, n_2 + 1, \mu_1 - 1, \mu_2\rangle, & \quad E_j - E_i = \hbar(\omega_2 - \omega_1 - f_1); \\ |n_1, n_2, \mu_1 - 1, \mu_2\rangle, & \quad E_j - E_i = -\hbar f_1; \\ |n_1, n_2, \mu_1, \mu_2 - 1\rangle, & \quad E_j - E_i = -\hbar f_2; \\ |n_1 - 1, n_2 + 1, \mu_1, \mu_2 - 1\rangle, & \quad E_j - E_i = \hbar(\omega_2 - \omega_1 - f_2). \end{aligned}$$

Using these intermediate states and Eq. (53), the two-phonon absorption rate is calculated to be

$$\begin{aligned} R^{[2]} = &\frac{2\pi g^4 (2\gamma/\pi)}{[(2\gamma)^2 + (\omega_1 - \omega_2 + f_1 + f_2)^2]} \mu_1 \mu_2 n_1 (n_2 + 1) (n_1 + n_2)^2 \\ &\times \left| \frac{1}{(-\omega_1 + \omega_2 - f_1)} - \frac{1}{f_1} - \frac{1}{f_2} + \frac{1}{(-\omega_1 + \omega_2 - f_2)} \right|^2, \end{aligned} \quad (55)$$

where we introduced the width γ for the phonon distribution. Note that the sum over the intermediate states vanishes. Thus there is interference between different quantum pathways. Such interference effects are well known in atomic physics in the context of two-photon processes (see Sec.7.5 in Ref. [11]). It is also known that relaxation effects generally make perfect interference imperfect leading to nonzero transition amplitudes. For our system in a cavity, the cavity line width κ is an important factor and it makes $R^{[2]}$ nonzero. The denominators like $1/f$ need to be modified by inclusion of the phonon line width γ , which is much smaller than κ . The denominators depending on ω_1 and ω_2 get modified by the inclusion of κ . A simple argument then modifies (55) to

$$\begin{aligned} R^{[2]} = &\frac{2\pi g^4 (2\gamma/\pi)}{(2\gamma)^2 + (\omega_1 - \omega_2 + f_1 + f_2)^2} \\ &\times \left(\frac{\kappa}{f}\right)^2 \frac{1}{f} \mu_1 \mu_2 n_1 (n_2 + 1) (n_1 + n_2)^2. \end{aligned} \quad (56)$$

This should be compared to the corresponding result $R^{[1]}$ for one-phonon absorption

$$R^{[1]} = \frac{2\pi g^2 (\gamma/\pi)}{\gamma^2 + (\omega_1 - \omega_2 + f_1)^2} \mu_1 n_1 (n_2 + 1), \quad (57)$$

which is easily obtained from Eq. (50). Let us compare the strength of $R^{[2]}$ with $R^{[1]}$ at resonance

$$\begin{aligned} \frac{R^{[2]}}{R^{[1]}} &\approx \frac{g^2}{2} \left(\frac{\kappa^2}{f^4}\right) \cdot (n_1 + n_2)^2 \mu_2 \\ &\sim \frac{g^2 \kappa^2}{2f^4} n_1^2 \mu_2 \quad \text{as } n_1 \gg n_2. \end{aligned} \quad (58)$$

For $g \sim 2\pi \times 20$ Hz, $f \sim 2\pi \times 10$ MHz, $\kappa/f \sim 1/10$,

$$\begin{aligned} \frac{R^{[2]}}{R^{[1]}} &\approx 2 \times 10^{-16} n_1^2 \mu_2 \\ &= 2 \times 10^{-4} \mu_2 \quad \text{for } n_1 \sim 10^6. \end{aligned} \quad (59)$$

For temperatures of the order of 10 mK, $\mu_2 \approx 10$ and hence $R^{[2]}/R^{[1]} \sim 10^{-3}$, leading to a substantial probability for two-step two-phonon absorption. Note that, instead of using thermal phonons, we can inject phonons from an external source [16].

VI. CONCLUSION

In conclusion, we have developed a first-principles theory of the optomechanical interactions in superfluid He. The theory is formulated in terms of the superfluid density field, so that multimode phonon optomechanics can be studied. The intrinsic nonlinearities of the superfluid are included. We

presented estimates of the strength of the optomechanical interactions and derived the canonical form of the Hamiltonian for linear optomechanical interactions. Using such canonical Hamiltonian standard effects like normal mode splitting, electromagnetically induced transparency in superfluid optomechanics can be studied. We also showed the importance of the two-step two-phonon process in superfluid He. The superfluid also has the possibility of nonlinear phonon processes, which one can integrate with the optomechanical processes. For example, two phonons f_1 and f_2 can combine via the cubic

nonlinearity in Eq. (30) and the generated phonon can be used for optomechanical interactions.

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