Jones and Stokes parameters for polarization in three dimensions

Colin J. R. Sheppard*
Istituto Italiano di Tecnologia, Via Morego 30, 16163 Genova, Italy
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For partially polarized three-dimensional electromagnetic fields, the polarization state is usually expressed in terms of nine generalized Stokes parameters, the coefficients of the nine Gell-Mann matrices in the expansion of the coherency matrix. We consider fully polarized fields, for which the Jones vector for the polarization state can be expressed in terms of five real parameters. Relationships between these five parameters and the nine generalized Stokes parameters are derived. For partially polarized three-dimensional fields there are four additional parameters that describe the partial polarization, two that specify the degree of polarization, and two that specify its orientation.

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I. INTRODUCTION

For highly focused or near-field electromagnetic radiation, polarization is three dimensional in nature [1–9]. Early work on three-dimensional (3D) polarization was reported in diverse application areas such as geophysical waves and radar scattering [10–16]. The extension for the Stokes parameters from the 2D case has been based on expansion of the coherency matrix in terms of Gell-Mann matrices [17], in contrast to the 2D case where the Pauli spin matrices are used. However, some of the implications are still not fully appreciated and there are some properties that are different from the 2D case, notably that a partially polarized field cannot be written simply as the sum of polarized and unpolarized components [5]. For the 2D case, the properties of the field, given by the Stokes parameters, is described in texts on polarization [18]. Here we concentrate on fully polarized three-dimensional fields. The most complete description for the 3D fully polarized case is that of Carozzi *et al.* [19]. The basic groundwork for the general polarization properties of fully polarized fields in three dimensions was presented by Nye and Hajnal [20,21]. They described the existence of special features such as T surfaces (surfaces where the direction of propagation is in the plane of the polarization ellipse), L^T lines (lines where the polarization is linear), and C^T lines (lines where the polarization is circular). Török *et al.* [22] used Stokes vectors in three dimensions to describe propagation of electromagnetic fields through an optical system. Hannay has described an interesting representation for polarization of plane waves traveling in an arbitrary direction, where a polarization state is described by two points on the Majorana sphere [23]. However, there seem surprisingly few papers on the properties of fully polarized 3D fields.

II. POLARIZATION IN THREE DIMENSIONS

For the fully polarized case, the electric field in three dimensions can be written as a 3D Jones vector

$$\mathbf{E} = \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \begin{bmatrix} E_{0x}e^{i\delta_x} \\ E_{0y}e^{i\delta_z} \\ E_{0z}e^{i\delta_z} \end{bmatrix} = e^{i\delta_x}\begin{bmatrix} E_{0x} \\ E_{0y}e^{i\delta} \\ E_{0z}e^{i\delta'} \end{bmatrix},$$
$$\delta = \delta_y - \delta_x, \quad \delta' = \delta_z - \delta_x, \tag{1}$$

so there are six independent parameters, or five discounting an absolute phase. It has been shown that the electric field can be described by a planar polarization ellipse, as in the 2D case. In three dimensions we need two parameters to fix the plane of the polarization ellipse and three more parameters to specify the ellipse within the plane: the size, ellipticity, and orientation, totaling five parameters in all.

For the partially polarized case, several papers have generalized the Stokes parameter treatment to the three-dimensional case. These generalized Stokes parameters become, for the fully polarized case,

$$\begin{aligned}
&\Lambda_0 = E_{0x}^2 + E_{0y}^2 + E_{0z}^2, \quad \Lambda_1 = 3E_{0x}E_{0y}\cos\delta,\\
&\Lambda_2 = 3E_{0x}E_{0y}\sin\delta,\\
&\Lambda_3 = \tfrac{3}{2}\left(E_{0x}^2 - E_{0y}^2\right), \quad \Lambda_4 = 3E_{0x}E_{0z}\cos\delta',\\
&\Lambda_5 = 3E_{0x}E_{0z}\sin\delta',\\
&\Lambda_6 = 3E_{0y}E_{0z}\cos(\delta'-\delta), \quad \Lambda_7 = 3E_{0y}E_{0z}\sin(\delta'-\delta),\\
&\Lambda_8 = \frac{\sqrt{3}}{2}\left(E_{0x}^2 + E_{0y}^2 - 2E_{0z}^2\right)
\end{aligned} \tag{2}$$

and the positive-semidefinite three-dimensional coherency matrix [24] is

$$\mathbf{C}_3 = \frac{1}{3}\begin{bmatrix} \Lambda_0 + \Lambda_3 + \frac{1}{\sqrt{3}}\Lambda_8 & \Lambda_1 - i\Lambda_2 & \Lambda_4 - i\Lambda_5 \\ \Lambda_1 + i\Lambda_2 & \Lambda_0 - \Lambda_3 + \frac{1}{\sqrt{3}}\Lambda_8 & \Lambda_6 - i\Lambda_7 \\ \Lambda_4 + i\Lambda_5 & \Lambda_6 + i\Lambda_7 & \Lambda_0 - \frac{2}{\sqrt{3}}\Lambda_8 \end{bmatrix}, \tag{3}$$

*colinjrsheppard@gmail.com

which can be expanded in terms of nine Gell-Mann matrices [25], which are Hermitian, traceless (except for the zeroth-order matrix), and linearly independent. Note that the ordering and signs of the Gell-Mann coefficients are taken differently in some papers. There are advantages in using the standard ordering, as has been argued for the Pauli matrices in the 2D case [26,27]. These advantages become significant when Müller matrices are considered, but we do not explore the 9×9 generalized Müller matrix (for the partially polarized case), or the 3×3 complex Jones matrix (for the fully polarized case) here.

From Eq. (2) we have the relationships

$$\begin{aligned} E_{0x}^2 &= \frac{1}{3}\left(\Lambda_0 + \Lambda_3 + \frac{\sqrt{3}}{3}\Lambda_8\right), \\ E_{0y}^2 &= \frac{1}{3}\left(\Lambda_0 - \Lambda_3 + \frac{\sqrt{3}}{3}\Lambda_8\right), \\ E_{0z}^2 &= \frac{1}{3}\left(\Lambda_0 - \frac{2\sqrt{3}}{3}\Lambda_8\right). \end{aligned} \tag{4}$$

There are nine generalized Stokes parameters, but as these can be expressed in terms of five variables, there are relationships between the Stokes parameters for the fully polarized case. We have the three relationships

$$\begin{aligned} \Lambda_1^2 + \Lambda_2^2 = 9E_{0x}^2E_{0y}^2 &= \left(\Lambda_0 + \Lambda_3 + \frac{\sqrt{3}}{3}\Lambda_8\right) \\ &\quad \times \left(\Lambda_0 - \Lambda_3 + \frac{\sqrt{3}}{3}\Lambda_8\right), \\ \Lambda_4^2 + \Lambda_5^2 = 9E_{0z}^2E_{0x}^2 &= \left(\Lambda_0 + \Lambda_3 + \frac{\sqrt{3}}{3}\Lambda_8\right) \\ &\quad \times \left(\Lambda_0 - \frac{2\sqrt{3}\Lambda_8}{3}\right), \\ \Lambda_6^2 + \Lambda_7^2 = 9E_{0y}^2E_{0z}^2 &= \left(\Lambda_0 - \Lambda_3 + \frac{\sqrt{3}}{3}\Lambda_8\right) \\ &\quad \times \left(\Lambda_0 - \frac{2\sqrt{3}\Lambda_8}{3}\right), \end{aligned} \tag{5}$$

which can be combined to give the relationship, analogous to that for the Stokes parameters in two dimensions [18],

$$\Lambda_0^2 = \tfrac{1}{3}\sum_{i=1}^{8}\Lambda_i^2. \tag{6}$$

This agrees with the condition obtained by setting the degree of polarization equal to unity [1,12,13].

For consistency of the off-diagonal terms, we require that

$$27E_{0x}^2E_{0y}^2E_{0z}^2 = (\Lambda_1 + i\Lambda_2)(\Lambda_4 - i\Lambda_5)(\Lambda_6 + i\Lambda_7) = S, \tag{7}$$

say, is purely real. So

$$\arg(\Lambda_1 + i\Lambda_2) + \arg(\Lambda_4 - i\Lambda_5) + \arg(\Lambda_6 + i\Lambda_7) = 0, \tag{8}$$

and equating real and imaginary parts of Eq. (7) gives

$$\Lambda_1\Lambda_5\Lambda_6 - \Lambda_2\Lambda_4\Lambda_6 - \Lambda_1\Lambda_4\Lambda_7 = \Lambda_2\Lambda_5\Lambda_7 \tag{9}$$

and

$$S = \Lambda_1\Lambda_4\Lambda_6 + \Lambda_1\Lambda_5\Lambda_7 + \Lambda_2\Lambda_5\Lambda_6 - \Lambda_2\Lambda_4\Lambda_7. \tag{10}$$

Equations (5) and (9) are four conditions relating the nine Stokes parameters so that they reduce to five degrees of freedom. Solving the simultaneous equations for the off-diagonal terms in Eqs. (5) and (7) gives

$$\begin{aligned} E_{0x} &= \sqrt{\frac{S}{3(\Lambda_6^2 + \Lambda_7^2)}} = \frac{1}{\sqrt{3}}\left[\frac{(\Lambda_1^2 + \Lambda_2^2)(\Lambda_4^2 + \Lambda_5^2)}{(\Lambda_6^2 + \Lambda_7^2)}\right]^{1/4}, \\ E_{0y} &= \sqrt{\frac{S}{3(\Lambda_4^2 + \Lambda_5^2)}} = \frac{1}{\sqrt{3}}\left[\frac{(\Lambda_6^2 + \Lambda_7^2)(\Lambda_1^2 + \Lambda_2^2)}{(\Lambda_4^2 + \Lambda_5^2)}\right]^{1/4}, \\ E_{0z} &= \sqrt{\frac{S}{3(\Lambda_1^2 + \Lambda_2^2)}} = \frac{1}{\sqrt{3}}\left[\frac{(\Lambda_4^2 + \Lambda_5^2)(\Lambda_6^2 + \Lambda_7^2)}{(\Lambda_1^2 + \Lambda_2^2)}\right]^{1/4} \end{aligned} \tag{11}$$

and

$$\begin{aligned} &\delta = \arg(\Lambda_1 + i\Lambda_2), \quad \delta' = \arg(\Lambda_4 + i\Lambda_5), \\ &\delta' - \delta = \arg(\Lambda_6 + i\Lambda_7) \end{aligned} \tag{12}$$

or by choosing a particular phase reference to give a cyclical form

$$\begin{aligned} \delta_x &= -\tfrac{1}{3}[\arg(\Lambda_1 + i\Lambda_2) + \arg(\Lambda_4 + i\Lambda_5)], \\ \delta_y &= -\tfrac{1}{3}[\arg(\Lambda_6 + i\Lambda_7) - \arg(\Lambda_1 + i\Lambda_2)], \\ \delta_z &= \tfrac{1}{3}[\arg(\Lambda_4 + i\Lambda_5) + \arg(\Lambda_6 + i\Lambda_7)]. \end{aligned} \tag{13}$$

The fact that the behavior of the electric field is described by a polarization ellipse can be proven very simply. There is always a direction defined by direction cosines l, m, and n such that the phasor components of the electric field lE_{0x}, $mE_{0y}e^{i\delta}$, and $nE_{0z}e^{i\delta'}$ sum to zero. By applying the sine rule to the triangle of the phasors, we can establish that, if the direction cosines of the normal to the plane of the polarization ellipse are l, m, and n,

$$\frac{lE_{0x}}{\sin(\delta' - \delta)} = -\frac{mE_{0y}}{\sin\delta'} = \frac{nE_{0z}}{\sin\delta} \tag{14}$$

and so, as $l^2 + m^2 + n^2 = 1$,

$$\begin{aligned} &l = \frac{\Lambda_7}{\sqrt{\Lambda_2^2 + \Lambda_5^2 + \Lambda_7^2}}, \quad m = -\frac{\Lambda_5}{\sqrt{\Lambda_2^2 + \Lambda_5^2 + \Lambda_7^2}}, \\ &n = \frac{\Lambda_2}{\sqrt{\Lambda_2^2 + \Lambda_5^2 + \Lambda_7^2}}. \end{aligned} \tag{15}$$

So the equation of the plane of the polarization ellipse is [19]

$$\Lambda_7 x - \Lambda_5 y + \Lambda_2 z = 0 \tag{16}$$

and its normal is

$$\frac{x}{\Lambda_7} = -\frac{y}{\Lambda_5} = \frac{z}{\Lambda_2}. \tag{17}$$

These equations can also be established by considering the projection of the polarization ellipse into the y-z, z-x, and x-y planes. The treatment of Carozzi *et al.* was based on Lie group arguments [19].

Then we have for the lengths of the semimajor and semiminor axes of the polarization ellipse

$$\begin{aligned} a^2 &= \left[\Lambda_0 + \sqrt{\Lambda_0^2 - \tfrac{4}{9}\left(\Lambda_2^2+\Lambda_5^2+\Lambda_7^2\right)}\right]/2, \\ b^2 &= \left[\Lambda_0 - \sqrt{\Lambda_0^2 - \tfrac{4}{9}\left(\Lambda_2^2+\Lambda_5^2+\Lambda_7^2\right)}\right]/2, \end{aligned} \tag{18}$$

so that $\Lambda_0 = a^2 + b^2$ and $\pi ab = (\pi/3)\sqrt{\Lambda_2^2+\Lambda_5^2+\Lambda_7^2}$ is the area of the polarization ellipse. The case when $\Lambda_2^2+\Lambda_5^2+\Lambda_7^2 = 0$ ($a^2 = \Lambda_0$ and $b^2 = 0$) corresponds to linear polarization and when $\Lambda_2^2+\Lambda_5^2+\Lambda_7^2 = \frac{9}{4}\Lambda_0^2$ ($a^2 = b^2 = \Lambda_0/2$) the field corresponds to pure circular polarization. For convenience we introduce the dimensionless linear polarization parameter L, given by

$$L = \frac{\sqrt{\Lambda_0^2 - \frac{4}{9}\left(\Lambda_2^2+\Lambda_5^2+\Lambda_7^2\right)}}{\Lambda_0}, \quad 0 \leqslant L \leqslant 1. \tag{19}$$

Then $L = 1$ corresponds to pure linear polarization and $L = 0$ to circular polarization. We have $a^2 = \Lambda_0(1+L)/2$, $b^2 = \Lambda_0(1-L)/2$, and $(\Lambda_2^2+\Lambda_5^2+\Lambda_7^2) = \frac{9}{4}\Lambda_0^2(1-L^2)$.

We now consider the orientation in the plane of the polarization ellipse. The point on the polarization ellipse corresponding to the major axis can be considered to be on the intersection of the sphere

$$x^2 + y^2 + z^2 = a^2 \tag{20}$$

and the plane

$$lx + my + nz = 0. \tag{21}$$

Its projection onto the x-y plane is then given by

$$\left(\Lambda_2^2+\Lambda_7^2\right)x^2 + \left(\Lambda_2^2+\Lambda_5^2\right)y^2 - 2\Lambda_5\Lambda_7 xy = a^2\Lambda_2^2. \tag{22}$$

This point must intersect with the projection of the polarization ellipse

$$E_{0y}^2 x^2 + E_{0x}^2 y^2 - \frac{2\Lambda_1}{3} = \frac{\Lambda_2^2}{9}. \tag{23}$$

The solution can be obtained in terms of different choices for the variables, but after some algebra we can obtain for the direction cosines of the major and minor axes

$$\begin{aligned} \lambda_a^2 &= \frac{1}{2L}\left[\frac{2E_{0x}^2}{\Lambda_0} - (1-L)(m^2+n^2)\right], \\ \mu_a^2 &= \frac{1}{2L}\left[\frac{2E_{0y}^2}{\Lambda_0} - (1-L)(n^2+l^2)\right], \\ \nu_a^2 &= \frac{1}{2L}\left[\frac{2E_{0z}^2}{\Lambda_0} - (1-L)(l^2+m^2)\right] \end{aligned} \tag{24}$$

and

$$\begin{aligned} \lambda_b^2 &= \frac{1}{2L}\left[-\frac{2E_{0x}^2}{\Lambda_0} + (1+L)(m^2+n^2)\right], \\ \mu_b^2 &= \frac{1}{2L}\left[-\frac{2E_{0y}^2}{\Lambda_0} + (1+L)(n^2+l^2)\right], \\ \nu_b^2 &= \frac{1}{2L}\left[-\frac{2E_{0z}^2}{\Lambda_0} + (1+L)(l^2+m^2)\right], \end{aligned} \tag{25}$$

respectively. We see that the sum of the three squares equals unity, as of course it must. The sum $a^2 + b^2$ is equal to Λ_0, agreeing with Eq. (18). We also have the relationships

$$\begin{aligned} \lambda_a^2 + \lambda_b^2 = m^2 + n^2 &= \frac{\Lambda_2^2+\Lambda_5^2}{\Lambda_2^2+\Lambda_5^2+\Lambda_7^2}, \\ \mu_a^2 + \mu_b^2 = n^2 + l^2 &= \frac{\Lambda_7^2+\Lambda_2^2}{\Lambda_2^2+\Lambda_5^2+\Lambda_7^2}, \\ \nu_a^2 + \nu_b^2 = l^2 + m^2 &= \frac{\Lambda_5^2+\Lambda_7^2}{\Lambda_2^2+\Lambda_5^2+\Lambda_7^2}. \end{aligned} \tag{26}$$

Equations (24) and (25) simplify for the special case when $L = 1$, corresponding to linear polarization. When $L = 0$ (circular polarization) the direction of the major and minor axes is indeterminate. We now examine these special cases in more detail.

A. Circular polarization

The special case of circular polarization $L = 0$ (on C^T lines) gives

$$\begin{aligned} E_{0x}^2 &= \frac{2}{9}\frac{\Lambda_2^2+\Lambda_5^2}{\Lambda_0}, \quad E_{0y}^2 = \frac{2}{9}\frac{\Lambda_7^2+\Lambda_2^2}{\Lambda_0}, \\ E_{0z}^2 &= \frac{2}{9}\frac{\Lambda_5^2+\Lambda_7^2}{\Lambda_0}, \end{aligned} \tag{27}$$

so that

$$\begin{aligned} \Lambda_2^2 &= \frac{9}{4}\Lambda_0\left(E_{0x}^2 + E_{0y}^2 - E_{0z}^2\right) = \frac{3}{4}\Lambda_0\left(\Lambda_0 + \frac{4\sqrt{3}}{3}\Lambda_8\right), \\ \Lambda_5^2 &= \frac{9}{4}\Lambda_0\left(E_{0x}^2 - E_{0y}^2 + E_{0z}^2\right) \\ &= \frac{3}{4}\Lambda_0\left(\Lambda_0 + 2\Lambda_3 - \frac{2\sqrt{3}}{3}\Lambda_8\right), \\ \Lambda_7^2 &= \frac{9}{4}\Lambda_0\left(-E_{0x}^2 + E_{0y}^2 + E_{0z}^2\right) \\ &= \frac{3}{4}\Lambda_0\left(\Lambda_0 - 2\Lambda_3 - \frac{2\sqrt{3}}{3}\Lambda_8\right), \end{aligned} \tag{28}$$

and

$$\Lambda_1 = \frac{2\Lambda_5\Lambda_7}{3\Lambda_0}, \quad \Lambda_4 = -\frac{2\Lambda_7\Lambda_2}{3\Lambda_0}, \quad \Lambda_6 = \frac{2\Lambda_2\Lambda_5}{3\Lambda_0}. \tag{29}$$

Thus

$$\frac{\Lambda_5\Lambda_7}{\Lambda_1} = -\frac{\Lambda_7\Lambda_2}{\Lambda_4} = \frac{\Lambda_2\Lambda_5}{\Lambda_6} = \frac{3\Lambda_0}{2} \tag{30}$$

and

$$\Lambda_1\Lambda_2 = -\Lambda_4\Lambda_5 = \Lambda_6\Lambda_7 = \frac{2}{3}\frac{\Lambda_2\Lambda_5\Lambda_7}{\Lambda_0}. \tag{31}$$

As Λ_0 is positive definite, Eq. (28) gives some requirements on the positivity or otherwise of the other parameters. The direction cosines of the normal to the polarization ellipse are given by

$$l = \frac{2\Lambda_7}{3\Lambda_0}, \quad m = -\frac{2\Lambda_5}{3\Lambda_0}, \quad n = \frac{2\Lambda_2}{3\Lambda_0}. \tag{32}$$

Note that changing the sign of all of Λ_2, Λ_5, and $\Lambda_7$changes the direction along its length of the normal to the polarization ellipse and therefore the handedness of the circular polarization.

B. Linear polarization

For linear polarization (on L^T lines) $L = 1$, $\delta = \delta' = 0$, $\Lambda_2 = \Lambda_5 = \Lambda_7 = 0$, $a^2 = \Lambda_0$, and $b^2 = 0$. Then, from Eq. (2)

$$\Lambda_1 = 3E_{0x}E_{0y}, \quad \Lambda_4 = 3E_{0x}E_{0z}, \quad \Lambda_6 = 3E_{0y}E_{0z}, \tag{33}$$

so that

$$E_{0x}^2 = \frac{\Lambda_1\Lambda_4}{3\Lambda_6}, \quad E_{0y}^2 = \frac{\Lambda_6\Lambda_1}{3\Lambda_4}, \quad E_{0z}^2 = \frac{\Lambda_4\Lambda_6}{3\Lambda_1}, \tag{34}$$

agreeing with Eq. (11). Then

$$\Lambda_3 = \frac{\Lambda_1\left(\Lambda_4^2 - \Lambda_6^2\right)}{2\Lambda_4\Lambda_6}, \quad \Lambda_8 = \frac{\sqrt{3}}{6}\frac{\Lambda_1^2\Lambda_4^2 + \Lambda_6^2\Lambda_1^2 - 2\Lambda_4^2\Lambda_6^2}{\Lambda_1\Lambda_4\Lambda_6}. \tag{35}$$

The direction cosines of the major axis of the polarization ellipse λ, μ, and ν are given, from Eq. (22), by

$$\lambda^2 = \frac{E_{0x}^2}{\Lambda_0} = \frac{\Lambda_1\Lambda_4}{3\Lambda_0\Lambda_6}, \quad \mu^2 = \frac{E_{0y}^2}{\Lambda_0} = \frac{\Lambda_6\Lambda_1}{3\Lambda_0\Lambda_4},$$
$$\nu^2 = \frac{E_{0z}^2}{\Lambda_0} = \frac{\Lambda_4\Lambda_6}{3\Lambda_0\Lambda_1}, \tag{36}$$

so that

$$\Lambda_1 = 3\lambda\mu\Lambda_0, \quad \Lambda_4 = 3\lambda\nu\Lambda_0, \quad \Lambda_6 = 3\mu\nu\Lambda_0. \tag{37}$$

From Eq. (36) Λ_1, Λ_4, and Λ_6 can only be all positive or any two of Λ_1, Λ_4, and Λ_6 can be negative. Then, from Eq. (37), only one at most of λ, μ, and ν can be negative.

III. DISCUSSION

The five parameters needed to specify the behavior for the fully polarized case can be taken in many different ways. If we know, for example, $(\Lambda_0,\Lambda_1,\Lambda_2,\Lambda_4,\Lambda_5)$ (as assumed by Samson) the remaining generalized Stokes parameters can be calculated from Eqs. (5), (6), and (8). Another possibility is to specify $(\Lambda_0,\Lambda_2,\Lambda_5,\Lambda_7)$, thus determining the intensity, the plane of the polarization ellipse, and the ellipticity directly. A further parameter is needed to specify the orientation of the polarization ellipse within its plane, but the most appropriate choice for this is not obvious. It could be (Λ_1 or Λ_4 or Λ_6) or (Λ_3 or Λ_8). Finally $(\Lambda_0,\Lambda_1,\Lambda_3,\Lambda_4,\Lambda_8)$ specifies directly the intensity, magnitude, and direction of the major axis of the polarization ellipse and the ellipticity. The plane of the polarization ellipse then follows from Eqs. (5) and (9).

For the 2D fully polarized case, the Poincaré sphere is a 2D surface in 3D space, represented by the SU(2) symmetry group. Similarly, for the 3D case, the generalization is a 7D hyperspherical surface [with the equation given by Eq. (6)], represented by SU(3), in 8D Stokes parameter space. For the fully polarized case, however, the nine parameters (including intensity) reduce to five independent parameters, or four in addition to intensity. These four parameters specify two rotations each in 3D space, one rotation giving the plane of the polarization ellipse and the other the Poincaré sphere representation of the polarization state within this plane. The polarization state is then represented by the symmetry group SU(2) $\oplus$ SU(2) rather than the SU(3) of the partially polarized case. It is interesting to note that Hannay's treatment in terms of the Majorana sphere also involves two rotations in 3D space, one for each point on the Majorana sphere [23].

The coherency matrix consists of nine elements, but the fully polarized case has five independent parameters only. This suggests that for a partially polarized field there must be more than one parameter to describe completely the degree of polarization and hence to attempt to define a single degree of polarization parameter seems futile [5]. A single scalar degree of polarization is insufficient to fully describe the degree of partial polarization. The choice of two appropriate measures requires them to be mutually independent [28,29].

The coherency matrix $\mathbf{C}_3$ can be diagonalized and separated into an incoherent sum of three purely polarized components [5,11,12,16] with strengths given by the eigenvalues (real) $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3$. Then

$$\mathbf{C}_3 = \lambda_1\mathbf{e}_1^*\mathbf{e}_1^T + \lambda_2\mathbf{e}_2^*\mathbf{e}_2^T + \lambda_3\mathbf{e}_3^*\mathbf{e}_3^T, \tag{38}$$

where $\mathbf{e}_i$, $i = 1,2,3$, are the eigenvectors and the partially polarized field is represented by the sum of three purely polarized components, the coherency matrix of each of which factorize. An explicit expression for the (real and non-negative) eigenvalues has been given and interpreted geometrically [28,29]. The field is completely defined by the eigenvalues and eigenvectors, which include nine independent parameters. Then the coherency matrix can be written [5,12,16,30]

$$\mathbf{C}_3 = (\lambda_1 - \lambda_2)\mathbf{e}_1^*\mathbf{e}_1^T + (\lambda_2 - \lambda_3)(\mathbf{e}_1^*\mathbf{e}_1^T + \mathbf{e}_2^*\mathbf{e}_2^T) + \lambda_3\mathbf{I}, \tag{39}$$

where, with $\mathbf{I}$ the identity matrix, it is decomposed into a purely polarized component, a component unpolarized within a plane, and an unpolarized component. Of the nine independent parameters needed to specify the field, five are necessary to define the polarized component, three are necessary to define the component that is unpolarized within a plane (its strength and two direction cosines to define its orientation), and one to define the strength of the unpolarized component. Apart from the intensity and orientational components, two parameters that depend on the eigenvalues are needed to define the degree of polarization. We can therefore identify the nine independent parameters for the partially polarized case as corresponding to the intensity, two to specify the partial polarization, four to define the orientation of the pure polarization component, and two to describe the orientation of the 2D unpolarized component. In the purely polarized case $\lambda_2 = \lambda_3 = 0$ and Eq. (39) reduces to Eqs. (2) and (3) with $\mathbf{E} = \sqrt{\lambda_1}\mathbf{e}_1$ and $\Lambda_0 = \lambda_1$.

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