Simulation of single-qubit open quantum systems

Ryan Sweke[*]
Quantum Research Group, School of Chemistry and Physics, University of KwaZulu-Natal, Durban 4001, South Africa

Ilya Sinayskiy[†] and Francesco Petruccione[‡]
Quantum Research Group, School of Chemistry and Physics, University of KwaZulu-Natal, Durban 4001, South Africa
and National Institute for Theoretical Physics (NITheP), KwaZulu-Natal, South Africa

(Received 26 May 2014; published 27 August 2014)

A quantum algorithm is presented for the simulation of arbitrary Markovian dynamics of a qubit, described by a semigroup of single-qubit quantum channels $\{T_t\}$ specified by a generator $\mathcal{L}$. This algorithm requires only single-qubit and controlled-NOT gates and approximates the channel $T_t = e^{t\mathcal{L}}$ up to the chosen accuracy ϵ, with a slightly superlinear cost $O((\|\mathcal{L}\|_{(1\to 1)}t)^{1+1/2k}/\epsilon^{1/2k})$ for any integer k. Inspired by developments in Hamiltonian simulation, a decomposition and recombination technique is utilized which allows for the exploitation of recently developed methods for the approximation of arbitrary single-qubit channels. In particular, as a result of these methods the algorithm requires only a single ancilla qubit, the minimal possible dilation for a nonunitary single-qubit quantum channel.

DOI: 10.1103/PhysRevA.90.022331 PACS number(s): 03.67.Ac, 03.65.Yz, 89.70.Eg

I. INTRODUCTION

One of the primary motivations for the development of quantum computation is the possibility of efficiently simulating quantum systems [1–3], as suggested in Feynman's seminal paper on the topic [4]. The natural first step towards this vision is the simulation of closed quantum systems undergoing Hamiltonian-generated unitary evolution, and over the past two decades consistent progress has been made in this field. Initially, Lloyd demonstrated a technique for the efficient simulation of Hamiltonians constructed as a tensor product of simpler Hamiltonians [5], and over time new methods and techniques have been introduced which have generalized the class of Hamiltonians which can be efficiently simulated while simultaneously tightening the relevant cost and error bounds [6–14].

However, equally important is the development of methods for the simulation of open quantum systems [15,16], crucial for enhancing our understanding of nonequilibrium dynamics and thermalization in a wide range of systems, from damped-driven spin-boson models to complex many fermion-boson models [2,3]. In particular, one would like to begin by simulating quantum channels, representing the most general quantum dynamics possible, and dynamical semigroups of quantum channels, which describe Markovian dynamics: continuous-time processes resulting from interactions with a Markovian environment in the Born approximation [17]. A straightforward methodology for the simulation of these systems is instantly suggested by the Stinespring dilation theorem [18], in which one introduces an initially pure state environment, with a size the square of the system size in the general case, such that one may simulate the open system dynamics of the system via Hamiltonian dynamics of the larger system-environment combination. Initially, Lloyd [5] conjectured that this approach may be improved by utilizing environments initialized in a mixed state, but this conjecture was quickly falsified by Terhal *et al.* [19], who proved that in the worst case an environment of dimension n^2 is necessary for the simulation of n-dimensional quantum channels via the Stinespring dilation.

An important early contribution was also made by Bacon *et al.* [20], who provide a method for decomposing the generators of Markovian evolution into simpler "primitive" generators. In particular, they demonstrate that for the single-qubit case universal simulation of Markovian dynamics requires only the ability to simulate a specific continuous one-parameter family of generators, as well as the ability to implement the recombination methods of linear combination and unitary conjugation. The development of collision models [21,22] for understanding quantum decoherence processes also suggests a constructive approach for the simulation of open quantum systems, and combining these insights with the results of Bacon *et al.* allowed for the development of collision-model-based methods for the simulation of single-qubit unital semigroups [23], generalized phase-damping processes [24], and indivisible qubit channels [25].

More recently the notion of dissipative quantum computation and state preparation [26] has been introduced, in which, under the assumption of Markovian dynamics described by a Lindblad master equation, the interactions of a system with its environment are no longer considered destructive but are, instead, utilized to drive a desired computational process. This formalism offers a natural setting for the simulation of open quantum systems, and research in this direction has resulted in successful experimental demonstrations of the dissipative simulation of complex many-body spin models [27,28]. In addition, dissipative quantum computation has allowed for alternative approaches to state preparation [29–38] and universal quantum computation [39,40]. Importantly, however, it has recently been shown that dissipative quantum computing is no more powerful than the traditional circuit model: the so-called "dissipative Church-Turing thesis" [41]. Specifically, it was shown that the time evolution of an open quantum system can be efficiently simulated by a unitary quantum circuit of a size

[*]rsweke@gmail.com
[†]sinayskiy@ukzn.ac.za
[‡]petruccione@ukzn.ac.za

scaling polynomially in the simulation time and size of the system.

Given these previous results we address in this paper the problem of constructing explicitly these efficient quantum circuits for the simulation of arbitrary Markovian processes within the traditional circuit model of quantum computation. In particular, we generalize into the superoperator regime recombination results, based on higher order Suzuki-Lie-Trotter formulas [42,43], from recent Hamiltonian simulation approaches [9–11]. These results allow us to efficiently implement the recombination methods of Bacon *et al.* [20], such that in order to construct efficient quantum circuits for the simulation of arbitrary Markovian dynamics of a qubit, it is only necessary to construct efficient circuits for the simulation of semigroups corresponding to the continuous one-parameter family of generators defined by Bacon *et al.* [20]. Furthermore, recently Wang *et al.* [44] have shown how to utilize convex properties of the set of single-qubit quantum channels [45] to simulate any such channel via unitary circuits requiring only a single-ancilla qubit, as opposed to the two-ancilla qubits required by straightforward implementations of the Stinespring dilation. We utilize these results for the construction of circuits for the simulation of the semigroups required by Bacon *et al.* [20], such that after recombination we obtain an explicit unitary circuit, with size scaling slightly superlinearly with respect to time, consisting only of controlled-NOT (CNOT) gates and single-qubit gates and requiring only a single-ancilla qubit, for the simulation up to any desired accuracy of an arbitrary single-qubit quantum dynamical semigroup.

The structure of this paper is as follows: We begin in Sec. II by introducing the setting and rigorously defining the problem we wish to address. Following this we proceed in Sec. III by presenting the method, introduced in [20], for the decomposition of an arbitrary generator of a single-qubit Markov semigroup. In Sec. IV we generalize results from [10] into the setting applicable for the problem addressed here, effectively demonstrating a method for the efficient recombination of the generators decomposed in Sec. III. Finally, in Sec. V we exploit the methods introduced in [44] in order to provide explicit efficient unitary circuits for the semigroups corresponding to the generators resulting from the decomposition in Sec. III.

II. PROBLEM AND SETTING

Given a system with finite-dimensional Hilbert space $\mathcal{H}_S = \mathbb{C}^d$, a quantum state of this system is described by a density matrix $\rho \in \mathcal{M}_d(\mathbb{C}) \cong \mathcal{B}(\mathcal{H}_S)$, where $\rho \geqslant 0, \mathrm{tr}[\rho] = 1$, and $\mathcal{B}(\mathcal{H}_S)$ is the algebra of bounded operators on $\mathcal{H}_S$. Quantum channels [17] provide the most general framework for describing the evolution of quantum states and are given by completely positive, trace-preserving maps,

$$T : \mathcal{B}(\mathcal{H}_S) \to \mathcal{B}(\mathcal{H}_S). \tag{1}$$

Given any quantum channel T, there exist Kraus operators $\{K_j \in \mathcal{B}(\mathcal{H}_S)\}$, such that

$$T(\rho) = \sum_{j=1}^{r} K_j \rho K_j^\dagger. \tag{2}$$

In the above, $\sum_{j=1}^{r} K_j^\dagger K_j = \mathbb{1}$ and $r = \mathrm{rank}(\tau) \leqslant d^2$ is the minimal number of Kraus operators, with $\tau \in \mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_S)$ the Jamiolkowski state,

$$\tau = (T \otimes \mathbb{1}_S)|\Omega\rangle\langle\Omega|, \tag{3}$$

where $\mathbb{1}_S$ is the identity on $\mathcal{H}_S$ and $|\Omega\rangle \in \mathcal{H}_S \otimes \mathcal{H}_S$ is any maximally entangled state [17]. Furthermore, it is always possible to dilate the total Hilbert space in order to include an environment such that the action of the channel on the system can be viewed as arising from the Hamiltonian-generated unitary evolution of the total system and environment. Technically, it is always possible to introduce a dilation space $\mathcal{H}_E$ with $\dim(\mathcal{H}_E) = [\dim(\mathcal{H}_S)]^2$ such that there exists a unitary matrix $U \in \mathcal{M}_{d^3}(\mathbb{C})$, where

$$T(\rho) = \mathrm{tr}_E\left[U(|e_0\rangle\langle e_0| \otimes \rho)U^\dagger\right] \tag{4}$$

and $|e_0\rangle\langle e_0| \in \mathcal{H}_E$ is some initial state of the environment. However, in the case where d is a factor of $\mathrm{rank}(\tau)$, then it is possible to construct a dilation with $\dim(\mathcal{H}_E) = r$ and $U \in \mathcal{M}_{dr}(\mathbb{C})$; such a dilation space is called a *minimal* dilation. Quantum channels as described above provide a complete picture of discrete-time evolution. However, in this paper we are concerned with the simulation of Markovian continuous-time evolutions, described by a continuous one-parameter semigroup of quantum channels $\{T_t\}$ satisfying

$$T_t T_s = T_{t+s}, \quad T_0 = \mathbb{1}, \tag{5}$$

for $t \in \mathbb{R}_+$, where $\rho(t) = T_t(\rho(0))$. Every continuous one-parameter semigroup of quantum channels $\{T_t\}$ has a unique generator

$$\mathcal{L} : \mathcal{B}(\mathcal{H}_S) \to \mathcal{B}(\mathcal{H}_S) \tag{6}$$

such that

$$T_t = e^{t\mathcal{L}} = \sum_{k=0}^{\infty} \frac{t^k \mathcal{L}^k}{k!} \tag{7}$$

and $\mathcal{L}$ satisfies the differential equation

$$\frac{d}{dt}\rho(t) = \mathcal{L}(\rho(t)), \tag{8}$$

known as a master equation. Furthermore, a linear superoperator $\mathcal{L} : \mathcal{B}(\mathcal{H}_S) \to \mathcal{B}(\mathcal{H}_S)$ is the generator of a continuous dynamical semigroup of quantum channels, if and only if it can be written in the form

$$\mathcal{L}(\rho) = i[\rho, H] + \sum_{k,l=1}^{d^2-1} A_{l,k}([F_k, \rho F_l^\dagger] + [F_k \rho, F_l^\dagger]), \tag{9}$$

where $H = H^\dagger \in \mathcal{M}_d(\mathbb{C})$ is Hermitian, $A \in \mathcal{M}_{d^2-1}(\mathbb{C})$ is positive semidefinite, and $\{F_i\}$ is a basis for the space of traceless matrices in $\mathcal{M}_d(\mathbb{C})$. Equation (9) is known as the Gorini, Kossakowsi, Sudarshan, and Lindblad form of the quantum Markov master equation and we refer to A as the GKS matrix [17]. For the remainder of this paper we choose the basis $\{F_i\}$, without loss of generality, to be the normalized Pauli operators $\frac{1}{\sqrt{2}}\{\sigma_x, \sigma_y, \sigma_z\}$.

In order to quantify the error in approximations of quantum channels we utilize the $(1 \to 1)$-norm for superoperators,

where, in general, the $(p \to q)$-norm of a superoperator is defined as [46]

$$\|T\|_{p\to q} := \sup_{\|A\|_p=1} \|T(A)\|_q. \tag{10}$$

The $(p \to q)$-norm defined above is induced from the Schatten p norm of an operator, defined as $\|A\|_p := (\mathrm{tr}(|A|^p))^{\frac{1}{p}}$. We use the $(1 \to 1)$-norm, as this is induced by the Schatten 1-norm, which corresponds up to a factor of $1/2$ with the trace distance, $\mathrm{dist}(\rho,\sigma) := \sup_{0\leqslant A\leqslant 1} \mathrm{tr}(A(\rho-\sigma))$, arising from a physical motivation of the operational distinguishability of quantum states [39]. At this stage it is possible to succinctly state the problem which is addressed in this paper.

Problem. Given a continuous one-parameter semigroup of single-qubit quantum channels $\{T_t\}$, generated by a generator $\mathcal{L}$, specified by a GKS matrix $A \geqslant 0 \in M_3(\mathbb{C})$ and a Hamiltonian $H = H^\dagger \in M_2(\mathbb{C})$, find a quantum circuit, acting on only the system qubit and a single-ancilla qubit and using at most $\mathrm{poly}(\|\mathcal{L}\|_{(1\to 1)}, t, 1/\epsilon)$ single-qubit and CNOT gates, that approximates the superoperator $T_t = e^{t\mathcal{L}}$ such that the maximum error in the final state, as quantified by the 1-norm, is at most ϵ.

It is important to note that each member T_t of an arbitrary semigroup of single-qubit channels $\{T_t\}$ is itself a single-qubit channel and therefore, in principle, when the methods of Wang *et al.* are used [44], can be simulated within 1-norm distance ϵ using $O(\log^{3.97}(1/\epsilon))$ gates from any specified single-qubit set S and one CNOT, acting on only the system qubit and a single ancilla. However, in order to utilize this method, which may even be improved [47,48] to require only $O(\log(1/\epsilon))$ such gates, it is necessary first to obtain a decomposition of the channel T_t into a convex sum of quasiextreme channels; this explicitly requires the specification of the generator. Therefore in order to exploit these methods for the simulation of a semigroup generated by an *arbitrary* generator, we utilize the decomposition-recombination strategy outlined in Sec. I. This strategy is inspired by approaches in Hamiltonian simulation [9–11], and as such, we simultaneously adopt the notion of efficiency developed within that context. Due to our restriction to the single-qubit case our notion of efficiency has no dependence on the system size, which remains a constant. However, as in [44], we restrict ourselves to quantum circuits requiring only a single-ancilla qubit, the smallest possible minimal dilation for a nonunitary single-qubit channel.

As we are restricting ourselves to single-qubit channels we begin by recalling some geometric properties of single-qubit states [45]. As $\{I,\sigma_x,\sigma_y,\sigma_z\}$ forms a basis for $\mathcal{M}_2(\mathbb{C})$, every density matrix ρ can be written in this basis as $\rho = 1/2(\mathbb{1} + \mathbf{r}\cdot\boldsymbol{\sigma})$, where $\boldsymbol{\sigma} = (\sigma_x,\sigma_y,\sigma_z)$ and $\mathbf{r} \in \mathbb{R}^3$ with $|\mathbf{r}| \leqslant 1$. Any single-qubit quantum channel can then be represented in this basis by a unique 4×4 matrix M, with the structure

$$M = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{m} & \tilde{M} \end{pmatrix}, \tag{11}$$

where $\tilde{M}$ is a 3×3 matrix, $\mathbf{0}$ and $\mathbf{m}$ are row and column vectors, respectively, and if we define

$$T(\rho) = \rho' = 1/2(\mathbb{1} + \mathbf{r}'\cdot\boldsymbol{\sigma}), \tag{12}$$

then M defines an affine map via

$$\mathbf{r}' = \tilde{M}\cdot\mathbf{r} + \mathbf{m}. \tag{13}$$

At this stage we can proceed to develop the solution to the problem defined above, as per the strategy outlined in Sec. I.

III. DECOMPOSITION OF THE ARBITRARY GENERATOR

As outlined in the description of our strategy, the first step is to provide a decomposition of an arbitrary generator $\mathcal{L}$, specified as per (9) by a GKS matrix $A \geqslant 0 \in \mathcal{M}_3(\mathbb{C})$ and a Hamiltonian $H = H^\dagger \in \mathcal{M}_2(\mathbb{C})$, into the combination of generators of simpler semigroups. This problem was initially addressed by Bacon *et al.* [20] and we follow their strategy here. As $A \geqslant 0$, one can use the spectral decomposition to write

$$A = \sum_{k=1}^{3} \lambda_k A_k, \tag{14}$$

and therefore, via linearity of $\mathcal{L}$,

$$\mathcal{L} = \mathcal{L}_H + \sum_{k=1}^{3} \lambda_k \mathcal{L}_k, \tag{15}$$

where

$$\mathcal{L}_H(\rho) = i[\rho, H] \tag{16}$$

and

$$\mathcal{L}_k(\rho) = \sum_{i,j=1}^{3} A_{k,(i,j)}([F_j,\rho F_i^\dagger] + [F_j\rho, F_i^\dagger]). \tag{17}$$

Relabeling $\mathcal{L}_0 := \mathcal{L}_H$ and defining $\lambda_0 = 1$ we can then write

$$\mathcal{L} = \sum_{k=0}^{3} \lambda_k \mathcal{L}_k, \tag{18}$$

giving us that

$$T_t = e^{t\mathcal{L}} = \exp\left(t\sum_{k=0}^{3}\lambda_k\mathcal{L}_k\right). \tag{19}$$

Furthermore, defining $T_{t'}^{(k)} := e^{t'\mathcal{L}_k}$ we see via a straightforward implementation of the Lie-Trotter formula [42] that

$$T_t = \lim_{n\to\infty}\left[\prod_{k=0}^{3} e^{[t\lambda_k(\mathcal{L}_k/2)]/n} \prod_{k'=3}^{0} e^{[t\lambda_{k'}(\mathcal{L}_{k'}/2)]/n}\right]^n \tag{20}$$

$$= \lim_{n\to\infty}\left[\prod_{k=0}^{3} T^{(k)}_{\left(\frac{t\lambda_k}{2n}\right)} \prod_{k'=3}^{0} T^{(k')}_{\left(\frac{t\lambda_{k'}}{2n}\right)}\right]^n. \tag{21}$$

Using the language of [20] we say that T_t can be constructed via *linear combination* of the semigroups $\{T_{t'}^{(k)}\}$. In Sec. IV we present a method for the efficient recombination of linear combinations; i.e., we provide a method for the approximation of T_t, up to arbitrary accuracy, using only a finite (polynomial in t) number of implementations of channels from the constituent semigroups $\{T_t^{(k)}\}$. Given such a method for the efficient simulation of linear combinations, it is then clear that

one can obtain an efficient algorithm for the simulation of T_t, provided that one can efficiently simulate the constituent channels $T_t^{(k)}$.

However, as per [20] we can utilize basis transformations to further decompose the constituent semigroups $\{T_t^{(k)}\}$ and, hence, simplify the task of implementing channels from these semigroups, which is tackled in Sec. V. First, note that for $k = 1$, $\mathcal{L}_k$ simply generates Hamiltonian evolution, which can be simulated using a single unitary operation on a single qubit. We therefore focus on the generators of dissipative evolution, for which $k \in [2,4]$. We begin by defining *unitary conjugation* of a channel T_t as the procedure transforming T_t according to $\mathcal{U}^\dagger T_t \mathcal{U}$, where $\mathcal{U}(\rho) = U\rho U^\dagger$ for some unitary operator U. Unitary conjugation preserves all Markovian semigroup properties and it is clear that the effect of unitary conjugation is to apply T_t in an alternative basis. In order to use unitary conjugation to further decompose the semigroups $\{T_t^{(k)}\}$ we utilize the following theorem, due to [20], establishing the manner in which unitary conjugation of a semigroup $\{T_t\}$ affects the GKS matrix defining the corresponding generator.

Theorem 1. For an N-dimensional system, unitary conjugation of the semigroup $\{T_t\}$ by $U \in \mathrm{SU}(N)$ results in conjugation of the GKS matrix by a corresponding element in the adjoint representation of SU(N).

One can then show [20] that, given A_k, as per (14), there exists $G_k \in \mathrm{SO}(3)$, the adjoint representation of SU(2), such that

$$A_k = G_k A_{(\theta_k)} G_k^T, \tag{22}$$

where

$$A_{(\theta_k)} = \begin{pmatrix} \cos^2(\theta_k) & -i\cos(\theta_k)\sin(\theta_k) & 0 \\ i\cos(\theta_k)\sin(\theta_k) & \sin^2(\theta_k) & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{23}$$

for $\theta_k \in [0,\pi/4]$. Therefore, as a result of Theorem 1 there exist unitary matrices $U_k \in \mathrm{SU}(2)$ such that

$$T_t^{(k)}(\rho) = U_k^\dagger\big[T_t^{(\theta_k)}\big(U_k \rho U_k^\dagger\big)\big]U_k, \tag{24}$$

where $T_t^{(\theta_k)} := e^{t\mathcal{L}_{(\theta_k)}}$ and

$$\mathcal{L}_{(\theta_k)}(\rho) = \sum_{i,j=1}^{3} A_{(\theta_k),(i,j)}([F_j,\rho F_i^\dagger] + [F_j\rho, F_i^\dagger]). \tag{25}$$

In light of the above, we can see that simulation of any channel from the semigroup $\{T_t^{(k)}\}$ requires only simulation of channels from the semigroup $\{T_t^{(\theta_k)}\}$, along with implementations of the single-qubit unitary U_k.

IV. RECOMBINATION

In this section we utilize methods developed within the context of Hamiltonian simulation [9–11] to show that higher order Suzuki integrators [42,43] can be used to simulate T_t up to arbitrary accuracy ϵ, using a finite sequence of implementations of $T_{t'}^{(j)} := e^{t'\mathcal{L}_j}$. In particular, we wish to place an upper bound on the number of implementations of $T_{t'}^{(j)}$ required within this sequence.

Given the generator $\mathcal{L} = \sum_{j=1}^m \mathcal{L}_j$ of a dynamical semigroup of quantum channels, as per (19) where $m = 4$, we begin by assuming that

$$\|\mathcal{L}_1\|_{1\to1} \geqslant \|\mathcal{L}_2\|_{1\to1} \geqslant \cdots \geqslant \|\mathcal{L}_m\|_{1\to1} \tag{26}$$

and defining the normalized component generators $\hat{\mathcal{L}}_j = \mathcal{L}_j/L_1$, where we have defined $L_1 := \|\mathcal{L}_1\|_{1\to1}$. We then follow [10] and define the basic Lie-Trotter product formula [42,43,49] as

$$S_2(\hat{\mathcal{L}}_1,\ldots,\hat{\mathcal{L}}_m,\lambda) = \prod_{j=1}^{m} e^{(\frac{\lambda}{2})\hat{\mathcal{L}}_j} \prod_{j'=m}^{1} e^{(\frac{\lambda}{2})\hat{\mathcal{L}}_{j'}} \tag{27}$$

$$= \prod_{j=1}^{m} T_{t_\lambda}^{(j)} \prod_{j'=m}^{1} T_{t_\lambda}^{(j')}, \tag{28}$$

where $t_\lambda = \lambda/(2L_1)$. Suzuki's higher order integrators are then defined using the recursion relation

$$S_{2k}(\lambda) = [S_{2k-2}(p_k\lambda)]^2[S_{2k-2}((1-4p_k)\lambda)][S_{2k-2}(p_k\lambda)]^2, \tag{29}$$

where $p_k = (4 - 4^{1/(2k-1)})^{-1}$ for $k > 1$ and for notational convenience we have used $S_{2k}(\lambda)$ and $S_{2k-2}(\lambda)$ to denote $S_{2k}(\hat{\mathcal{L}}_1,\ldots,\hat{\mathcal{L}}_m,\lambda)$ and $S_{2k-2}(\hat{\mathcal{L}}_1,\ldots,\hat{\mathcal{L}}_m,\lambda)$, respectively. Note that $S_{2k}(\lambda)$ consists of a product of

$$2(m-1)5^{k-1} + 1 \tag{30}$$

exponentials, so that we can define

$$N_{\exp} = [2(m-1)5^{k-1} + 1]x \tag{31}$$

as the number of exponentials, and hence channels $T_{t'}^{(j)}$, appearing in the expression $[S_{2k}(t/r)]^x$. In order to obtain the desired result, we then prove the following theorem, a direct generalization of the work in [10] to the superoperator setting.

Theorem 2. Let $1 \geqslant \epsilon > 0$ be such that $4met\|\mathcal{L}_2\|_{1\to1} \geqslant \epsilon$; then for any $k \in \mathbb{N}$ there exists r such that

$$\left\| \exp\left(t\sum_{j=1}^{m}\mathcal{L}_j\right) - \big[S_{2k}(\hat{\mathcal{L}}_1,\ldots,\hat{\mathcal{L}}_m,t/r)\big]^{rL_1} \right\|_{1\to1} \leqslant \epsilon \tag{32}$$

and the number of exponentials required is bounded by

$$\mathrm{N}_{\exp} \leqslant (2m-1)5^{k-1}\left[L_1 t\left(\frac{4emtL_2}{\epsilon}\right)^{1/2k}\frac{4me}{3}\left(\frac{5}{3}\right)^{k-1}\right], \tag{33}$$

where $L_2 := \|\mathcal{L}_2\|_{1\to1}$.

In order to prove Theorem 2 we first note that the following lemma can be proven using the exact same proof as described in [10], provided one replaces the 1-norm with the $(1\to1)$-norm and notes that $\|T\|_{1\to1} = 1$ for any quantum channel T [41,46], as the proof relies only on properties of the Taylor expansion of exponentials and generic properties of the norm, which hold for both Schatten norms and the induced superoperator norms [46].

Lemma 1. For $k \in \mathbb{N}$, $d_k\lambda < k+1$, $d_k = m(4/3)k(5/3)^{k-1}$, and

$$\|\hat{\mathcal{L}}_m\|_{1\to1} \leqslant \cdots \leqslant \|\hat{\mathcal{L}}_2\|_{1\to1} \leqslant \|\hat{\mathcal{L}}_1\|_{1\to1} = 1, \tag{34}$$

we have that

$$\left\| \exp\left(\lambda \sum_{j=1}^{m} \hat{\mathcal{L}}_j\right) - S_{2k}(\lambda) \right\|_{1\to 1} \leqslant \frac{4L_2}{(2k+1)!}(d_k\lambda)^{2k+1}, \tag{35}$$

where $S_{2k}(\lambda) = S_{2k}(\hat{\mathcal{L}}_1, \ldots, \hat{\mathcal{L}}_m, \lambda)$.

In addition to Lemma 1, the following lemma is required:

Lemma 2. Given quantum channels T and V we have that

$$||T^n - V^n||_{1\to 1} \leqslant n||T - V||_{1\to 1}. \tag{36}$$

Lemma 2 is a direct generalization to the superoperator setting of an important result describing the accumulation of errors due to gate approximations in unitary circuits. However, in the conventional operator setting the proof relies crucially on properties of Hermitian operators, and as a result, an alternative proof is required within this more general setting.

Proof of Lemma 2. It is clear that in the case where $n = 1$ the lemma is true. Assume that the lemma holds for $n = m$. We now show that it holds for $n = m + 1$ and, as a result, prove the result by induction:

$$\|T^{m+1} - V^{m+1}\|_{1\to 1} = \|TT^m - TV^m + TV^m - VV^m\|_{1\to 1} \tag{37}$$

$$\leqslant \|T\left(T^m - V^m\right)\|_{1\to 1} + \|\left(T - V\right)V^m\|_{1\to 1} \tag{38}$$

$$\leqslant \|T\|_{1\to 1}\|T^m - V^m\|_{1\to 1} + \|T - V\|_{1\to 1}\|V^m\|_{1\to 1} \tag{39}$$

$$\leqslant \|T^m - V^m\|_{1\to 1} + \|T - V\|_{1\to 1} \tag{40}$$

$$\leqslant (m+1)\|T - V\|_{1\to 1}. \tag{41}$$

In the above, (39) follows from (38) via submultiplicativity of the norm, and (40) follows from (39) due to the fact [41,46] that for any quantum channel T we have that $\|T\|_{1\to 1} = 1$. ■

Given these two lemmas it is now possible to follow [10] in order to prove Theorem 2.

Proof of Theorem 2. First, note that

$$\exp\left(t\sum_{j=1}^{m}\mathcal{L}_j\right) = \left[\exp\left(\frac{t}{r}\sum_{j=1}^{m}\hat{\mathcal{L}}_j\right)\right]^{rL_1}, \tag{42}$$

and as a result, we can utilize Lemma 1 and Lemma 2 to obtain

$$\left\| \exp\left(t\sum_{j=1}^{m}\mathcal{L}_j\right) - \left[S_{2k}\left(\frac{t}{r}\right)\right]^{rL_1} \right\|_{1\to 1} \leqslant 4tL_2\frac{d_k^{2k+1}}{(2k+1)!}\frac{t^{2k}}{r^{2k}}. \tag{43}$$

Therefore, taking

$$r \geqslant t\left(\frac{4tL_2d_k^{2k+1}}{\epsilon(2k+1)!}\right)^{1/(2k)} \tag{44}$$

ensures that (32) is satisfied. Furthermore, via the argument in [10] it suffices to take

$$r \geqslant t\left(\frac{4emtL_2}{\epsilon}\right)^{1/(2k)}\frac{2ed_k}{2k+1}, \tag{45}$$

such that we can define r as the lower bound,

$$r := t\left(\frac{4emtL_2}{\epsilon}\right)^{1/(2k)}\frac{2ed_k}{2k+1}, \tag{46}$$

which is easily seen to satisfy the assumptions of Lemma 1. From (31) one can then see that the total number of exponentials required is

$$\mathrm{N}_{\exp} \leqslant (2m-1)5^{k-1}rL_1, \tag{47}$$

so that substituting in the values of r and d_k one obtains (33). ■

As calculated in [10], if $\epsilon \leqslant mtL_2$, then the minimum value of the right-hand side of (33) is achieved for

$$k = \mathrm{round}\left(\sqrt{\frac{1}{2}\log_{25/3}\frac{4emtL_2}{\epsilon}}\right), \tag{48}$$

such that the number of exponentials required satisfies

$$N_{\exp} \leqslant \tfrac{8}{3}(2m-1)metL_1e^{2\sqrt{\frac{1}{2}\ln(25/3)\ln(4emtL_2/\epsilon)}}. \tag{49}$$

Furthermore, by definition of the $(1\to 1)$-norm we have that for any density matrix ρ and any superoperators P and Q,

$$\|P(\rho) - Q(\rho)\|_1 \leqslant \|P - Q\|_{1\to 1}, \tag{50}$$

and as such, the results of Theorem 2 bound the error in the output state obtained when approximating T_t with $[S_{2k}(t/r)]^{rL_1}$. At this point we have established that any channel T_t, a member of the semigroup $\{T_t\}$ generated by $\mathcal{L} = \sum_{j=1}^{m}\mathcal{L}_j$, can be simulated up to arbitrary accuracy using only a slightly superlinear, with respect to t, number of implementations of $T_{t'}^{(j)} = e^{t'\mathcal{L}_j}$.

V. SIMULATION OF CONSTITUENT SEMIGROUPS

Given the results in Secs. III and IV, all that remains is to illustrate a method for the construction of unitary circuits, consisting only of single-qubit and CNOT gates and requiring only a single-ancilla qubit, for the exact implementation of quantum channels from the semigroups $\{T_t^{(\theta_k)}\}$. We proceed by following the strategy, introduced in [44], of decomposing the channels $T_t^{(\theta_k)}$ into the convex sum of quasiextreme channels. These quasiextreme channels require only two Kraus operators for implementation and, hence, can be simulated using a unitary circuit acting on only a single-ancilla qubit. Furthermore, given a decomposition of $T_t^{(\theta_k)}$ into the convex sum of quasiextreme channels, $T_t^{(\theta_k)}$ can be simulated using classical random sampling of these channels.

In order to obtain this convex decomposition we proceed via the following steps: First, we utilize the damping basis [50,51] in order to find the affine map representation of $T_t^{(\theta_k)}$. From this affine map representation it is then easy to construct the Jamiolkowski state, from which it is possible to obtain the desired convex decomposition [45].

Using damping basis methods [50,51] (details are given in the Appendix) we find, as per (11)–(13), that the affine map

representation M of $T_t^{(\theta_k)}$ is given by

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \Lambda_1 & 0 & 0 \\ 0 & 0 & \Lambda_2 & 0 \\ m_3 & 0 & 0 & \Lambda_3 \end{pmatrix}, \tag{51}$$

where

$$\Lambda_1 = e^{(-2\sin^2(\theta_k)t)}, \tag{52}$$

$$\Lambda_2 = e^{(-2\cos^2(\theta_k)t)}, \tag{53}$$

$$\Lambda_3 = e^{(-2t)}, \tag{54}$$

$$m_3 = \sin(2\theta_k)(1-\Lambda_3). \tag{55}$$

Given this affine representation of $T_t^{(\theta_k)}$, the Jamiolkowski state

$$\tau_{(\theta_k)} = \left(T_t^{(\theta_k)} \otimes \mathbb{1}_S\right)|\psi_0\rangle\langle\psi_0|, \tag{56}$$

with $|\psi_0\rangle = (1/\sqrt{2})(|00\rangle + |11\rangle)$, is then given by [45]

$$\tau_{(\theta_k)} = \frac{1}{4}\begin{pmatrix} a^2 & 0 & 0 & \Lambda_1+\Lambda_2 \\ 0 & b^2 & \Lambda_1-\Lambda_2 & 0 \\ 0 & \Lambda_1-\Lambda_2 & c^2 & 0 \\ \Lambda_1+\Lambda_2 & 0 & 0 & d^2 \end{pmatrix}, \tag{57}$$

with

$$a = (1+m_3+\Lambda_3)^{1/2}, \tag{58}$$

$$b = (1-m_3-\Lambda_3)^{1/2}, \tag{59}$$

$$c = (1+m_3-\Lambda_3)^{1/2}, \tag{60}$$

$$d = (1-m_3+\Lambda_3)^{1/2}. \tag{61}$$

In order to utilize $\tau_{(\theta_k)}$ to obtain the desired convex decomposition of $T^{(\theta_k)}$, we follow the procedure established in [45]. First, for any quantum channel T we define $\beta(T) = 2\tau$ and note that $\beta(T)$ can always be written in the block form

$$\beta(T) = \begin{pmatrix} A & C \\ C^\dagger & B \end{pmatrix}. \tag{62}$$

Furthermore, if $\hat{T}$ is the adjoint [17] of T, then

$$\beta(\hat{T}) = \overline{U_{23}^\dagger \beta(T) U_{23}} \tag{63}$$

$$= \begin{pmatrix} A & C \\ C^\dagger & I-A \end{pmatrix}, \tag{64}$$

where

$$U_{23} = U_{23}^\dagger = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{65}$$

Given these facts we then utilize the following three results (Theorem 3, Lemma 3, and Lemma 4), all due to [45], in order to obtain the desired convex decomposition.

Theorem 3. A quantum channel T is a generalized extreme point of the set of all quantum channels of the same dimension if and only if $\beta(\hat{T})$ is of the form

$$\beta(\hat{T}) = \begin{pmatrix} A & \sqrt{A}U\sqrt{I-A} \\ \sqrt{I-A}U^\dagger\sqrt{A} & I-A \end{pmatrix} \tag{66}$$

for some unitary matrix U.

Lemma 3. A matrix

$$\begin{pmatrix} A & C \\ C^\dagger & B \end{pmatrix} \tag{67}$$

is positive semidefinite if and only if $A \geqslant 0$, $B \geqslant 0$, and $C = \sqrt{A}R\sqrt{B}$ for some contraction R. Moreover, the set of positive semidefinite matrices with fixed A and B is a convex set whose extreme points satisfy $C = \sqrt{A}U\sqrt{B}$ for some unitary matrix U.

Lemma 4. Any contraction in $M_2(\mathbb{C})$ can be written as the convex combination of two unitary matrices.

In light of the above three results, our strategy for obtaining a convex decomposition of an arbitrary channel T is as follows: Given $\beta(T)$ we find $\beta(\hat{T})$ using (63). As T is completely positive, this ensures that $\beta(\hat{T}) \geqslant 0$, and as such we write $\beta(\hat{T})$ in the form described in Lemma 3. As R is a contraction, we know, via Lemma 4, that R can be decomposed into the convex combination of two unitary matrices, and as a result we obtain that

$$\beta(\hat{T}) = \tfrac{1}{2}\beta(\hat{T}_1) + \tfrac{1}{2}\beta(\hat{T}_2), \tag{68}$$

where, due to Theorem 3, we see that T_1 and T_2 are quasiextreme channels (generalized extreme points of the set of quantum channels) providing the desired convex decomposition of T. Following these steps for $T_t^{(\theta_k)}$ we find that

$$\beta\left(\hat{T}_t^{(\theta_k)}\right) = \tfrac{1}{2}\beta\left(\hat{T}_{(t,1)}^{\theta_k}\right) + \tfrac{1}{2}\beta\left(\hat{T}_{(t,2)}^{\theta_k}\right), \tag{69}$$

where

$$\beta\left(\hat{T}_{(t,i)}^{\theta_k}\right) = \begin{pmatrix} A & \sqrt{A}U_i\sqrt{I-A} \\ \sqrt{I-A}U_i^\dagger\sqrt{A} & I-A \end{pmatrix}, \tag{70}$$

with

$$U_1 = \begin{pmatrix} 0 & e^{i\phi_1} \\ e^{i\phi_2} & 0 \end{pmatrix}, \qquad U_2 = \begin{pmatrix} 0 & e^{-i\phi_1} \\ e^{-i\phi_2} & 0 \end{pmatrix}, \tag{71}$$

$$\phi_1 = \arccos\left(\frac{\Lambda_1+\Lambda_2}{ab}\right), \tag{72}$$

$$\phi_2 = \arccos\left(\frac{\Lambda_1-\Lambda_2}{cd}\right), \tag{73}$$

and

$$A = \frac{1}{2}\begin{pmatrix} a^2 & 0 \\ 0 & c^2 \end{pmatrix}. \tag{74}$$

As in [44], in order to construct the unitary circuits implementing $T_{(t,i)}^{\theta_k}$ it is necessary to first find the Kraus operators K_1^i and K_2^i, where

$$T_{(t,i)}^{\theta_k}(\rho) = \sum_{j=1}^{2} \left(K_j^i\right)\rho\left(K_j^i\right)^\dagger. \tag{75}$$

To find these Kraus operators one then uses (63) to find the relevant Jamiolkowski state, before exploiting the standard Choi-Jamiolkowski correspondence [17]. Following these steps one obtains

$$K_1^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & c \\ de^{i\phi_2} & 0 \end{pmatrix}, \quad K_2^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} ae^{-i\phi_1} & 0 \\ 0 & b \end{pmatrix} \tag{76}$$

and

$$K_1^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & c \\ de^{-i\phi_2} & 0, \end{pmatrix} \quad K_2^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} ae^{i\phi_1} & 0 \\ 0 & b \end{pmatrix}. \tag{77}$$

Given these Kraus operators it is then possible to find a constant-size unitary circuit implementing $T_{(t,i)}^{\theta_k}$, consisting only of CNOT and single-qubit gates, in a variety of ways. The first method is to apply directly the results of [44] (requiring an additional two basis transformations), or alternatively, one can construct from the Kraus operators unitary matrices $U_i^{(\theta_k)}$, such that

$$T_{(t,i)}^{\theta_k}(\rho) = \mathrm{tr}_E\left[\left(U_i^{(\theta_k)}\right)(|0\rangle\langle 0| \otimes \rho)\left(U_i^{(\theta_k)}\right)^\dagger\right], \tag{78}$$

and proceed by obtaining a circuit decomposition of these unitary matrices. We provide an explicit demonstration of the latter strategy here. It is important to note that these unitary matrices are *not* unique [17]; however, for the purposes of this paper we choose to work with the following form for the unitary $U_1^{(\theta_k)}$,

$$U_1^{(\theta_k)} = \begin{pmatrix} e^{-i\phi_1}\cos(\beta) & 0 & 0 & -e^{-i\phi_2}\sin(\beta) \\ 0 & \cos(\alpha) & -\sin(\alpha) & 0 \\ 0 & \sin(\alpha) & \cos(\alpha) & 0 \\ e^{i\phi_2}\sin(\beta) & 0 & 0 & e^{i\phi_1}\cos(\beta) \end{pmatrix}, \tag{79}$$

where we have written

$$\cos(\beta) = \frac{1}{\sqrt{2}}a, \quad \sin(\beta) = \frac{1}{\sqrt{2}}d, \tag{80}$$

$$\cos(\alpha) = \frac{1}{\sqrt{2}}b, \quad \sin(\alpha) = \frac{1}{\sqrt{2}}c, \tag{81}$$

as a result of the observation that $a^2 + d^2 = 2$ and $b^2 + c^2 = 2$. Furthermore, note that $U_2^{(\theta_k)}$ can be simply obtained by swapping the signs occurring within each exponential function in $U_1^{(\theta_k)}$ and, as such, is not presented explicitly. In order to obtain an explicit circuit decomposition for $U_1^{(\theta_k)}$ we note that we can write $U_1^{(\theta_k)} = U_{1,A}^{(\theta_k)} U_{1,B}^{(\theta_k)}$, where $U_{1,A}^{(\theta_k)}$ and $U_{1,B}^{(\theta_k)}$ are the two-level unitary matrices

$$U_{1,A}^{(\theta_k)} = \begin{pmatrix} e^{-i\phi_1}\cos(\beta) & 0 & 0 & -e^{-i\phi_2}\sin(\beta) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ e^{i\phi_2}\sin(\beta) & 0 & 0 & e^{i\phi_1}\cos(\beta) \end{pmatrix} \tag{82}$$

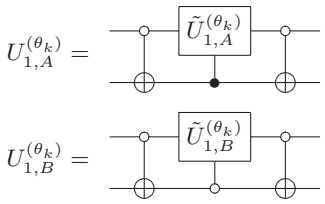

FIG. 1. Circuit decompositions for the unitary operators $U_{1,A}^{(\theta_k)}$ and $U_{1,B}^{(\theta_k)}$, where the unitary operator $U_1^{(\theta_k)}$, implementing the quasiextreme channel $T_{(t,i)}^{\theta_k}$ via (78), is given by $U_1^{(\theta_k)} = U_{1,A}^{(\theta_k)} U_{1,B}^{(\theta_k)}$. The single-qubit unitary operations $\tilde{U}_{1,A}^{(\theta_k)}$ and $\tilde{U}_{1,B}^{(\theta_k)}$ are defined in Eqs. (84) and (85), respectively.

and

$$U_{1,B}^{(\theta_k)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) & 0 \\ 0 & \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{83}$$

Furthermore, if we define the unitary matrices,

$$\tilde{U}_{1,A}^{(\theta_k)} = \begin{pmatrix} e^{-i\phi_1}\cos(\beta) & -e^{-i\phi_2}\sin(\beta) \\ e^{i\phi_2}\sin(\beta) & e^{i\phi_1}\cos(\beta) \end{pmatrix} \tag{84}$$

and

$$\tilde{U}_{1,B}^{(\theta_k)} = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}, \tag{85}$$

then we can implement $U_{1,A}^{(\theta_k)}$ and $U_{1,B}^{(\theta_k)}$ using the circuits shown in Fig. 1.

At this stage all that remains is to obtain circuit decompositions of the controlled-$\tilde{U}_{1,i}^{(\theta_k)}$ gates. In order to implement the controlled-$\tilde{U}_{1,B}^{(\theta_k)}$ gate we note the equivalence depicted in Fig. 2, where $A_B = R_y(\alpha)$ and $B_B = R_y(-\alpha)$, with R_y the standard exponentiation of the Pauli y matrix, given by

$$R_y(\theta) = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}. \tag{86}$$

Similarly, in order to implement the controlled-$\tilde{U}_{1,A}^{(\theta_k)}$ gate we note the equivalence depicted in Fig. 2, where $A_A = R_z(\phi_1 + \phi_2)R_y(\beta)$, $B_A = R_y(-\beta)R_z(-\phi_1)$, and $C_A = R_z(-\phi_2)$, with

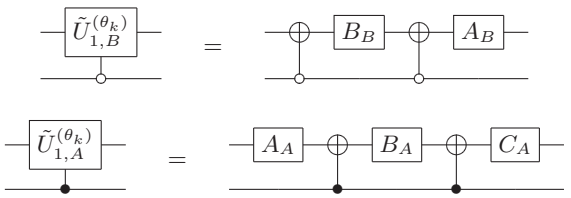

FIG. 2. Circuit decomposition for the controlled-$\tilde{U}_{1,i}^{(\theta_k)}$ operations, required for implementation of the unitary operators $U_{1,i}^{(\theta_k)}$, into only single-qubit and controlled-NOT gates. The single-qubit unitary gates are defined as $A_B = R_y(\alpha)$, $B_B = R_y(-\alpha)$, $A_A = R_z(\phi_1 + \phi_2)R_y(\beta)$, $B_A = R_y(-\beta)R_z(-\phi_1)$, and $C_A = R_z(-\phi_2)$, where $R_y(\theta)$ and $R_z(\theta)$ are defined in Eqs. (86) and (87), respectively.

R_z the standard exponentiation of the Pauli z matrix, given by

$$R_z(\theta) = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}. \tag{87}$$

VI. CONCLUSIONS AND OUTLOOK

Combining the results in the previous three sections we obtain the following algorithm, requiring only $O((\|\mathcal{L}\|_{1\to 1}t)^{1+1/2j}/\epsilon^{1/2j})$ single-qubit and CNOT gates, as a solution to the problem defined in Sec. II:

(1) Given $\mathcal{L}$, obtain, as per Sec. III, the spectral decomposition

$$\mathcal{L} = \sum_{k=0}^{4} \lambda_k \mathcal{L}_k, \tag{88}$$

and G_k and θ_k specifying the decomposition

$$A_k = G_k A_{(\theta_k)} G_k^T, \tag{89}$$

for all $k \in [1,3]$.

(2) Choose the desired approximation accuracy ϵ as well as the simulation time t. Using Eqs. (28) and (29), construct $S_{2j}(t/r)$ with

$$j = \text{round}\left(\sqrt{\log_{25/3}\frac{8etL_2}{\epsilon}}\right) \tag{90}$$

and

$$r = t\left(\frac{16etL_2}{\epsilon}\right)^{1/(2j)} \frac{2ed_j}{2j+1}. \tag{91}$$

(3) Implement $S_{2j}(t/r)$ $L_1 r$ times using

$$T_{t'}^{(k)}(\rho) = U_k^\dagger\left[T_{t'}^{(\theta_k)}(U_k \rho U_k^\dagger)\right]U_k, \tag{92}$$

where λ_k, L_1, and r have been incorporated into t', U_k is obtained from G_k as per Sec. III, and $T_{t'}^{(\theta_k)}$ is implemented via classical random sampling of the circuits derived in Sec. V.

In light of this result two natural avenues arise for extension of this work. The first is the investigation of improvements to the method presented here for the simulation of arbitrary single-qubit Markovian open quantum systems. In particular, it will be of interest to determine whether an optimality result, analogous to the "no fast-forward theorem" of Hamiltonian simulation [9], exists in this generalized context, in which case the results of this paper would be close to optimal for the single-qubit case. The second natural extension of this work is the development of methods allowing for the construction of explicit algorithms for the simulation of multiqubit and multiqudit Markovian open systems. However, the work presented in this paper relies heavily on the geometric properties of single-qubit channels, and as such, generalization of this work would require investigation into the geometric and convex structure of multiparticle quantum channels, at present an open question [52].

ACKNOWLEDGMENTS

The authors would like to thank the referee for the insightful comments and suggestions. This work is based upon research supported by the South African Research Chair Initiative of the Department of Science and Technology and National Research Foundation. Ryan Sweke acknowledges the financial support of the National Research Foundation SARChI program.

APPENDIX: DAMPING BASIS DERIVATION OF AFFINE MAP REPRESENTATION

Given the generator $\mathcal{L}$ of a semigroup of quantum channels (with $H = 0$) one can find the left and right eigenoperators L_i and R_i satisfying [51]

$$L_i\mathcal{L} = \lambda_{(L,i)}L_i, \tag{A1}$$

$$\mathcal{L}R_i = \lambda_{(R,i)}R_i, \tag{A2}$$

where the left action of a superoperator is defined so that

$$\text{tr}[(X\mathcal{L})\rho] = \text{tr}[(\mathcal{L}(\rho))X] \tag{A3}$$

for any Hermitian operator X and for all density matrices ρ. Using this left action one finds that $\text{tr}[L_i R_j] = \delta_{ij}$ and $\lambda_{(L,i)} = \lambda_{(R,i)}$. Furthermore, any density matrix $\rho(0)$ can be expressed in this basis (known as the damping basis), such that [50]

$$\rho(0) = \sum_i \text{tr}[L_i\rho(0)]R_i \tag{A4}$$

and

$$\rho(t) = e^{\mathcal{L}t}[\rho(0)] \tag{A5}$$

$$= \sum_i \text{tr}\big[L_i\rho(0)\big]\Lambda_i R_i, \tag{A6}$$

with $\Lambda_i = e^{\lambda_i t}$. Furthermore, the submatrix $\tilde{M}$ in the affine map representation of $T_t = e^{t\mathcal{L}}$ is then given by

$$\tilde{M} = \begin{pmatrix} \Lambda_2 & 0 & 0 \\ 0 & \Lambda_3 & 0 \\ 0 & 0 & \Lambda_4 \end{pmatrix}. \tag{A7}$$

Utilizing these methods for the semigroup $T_t^{(\theta_k)}$ generated by $\mathcal{L}_{(\theta_k)}$, as per (25), we find that

$$\lambda_2 = -2\sin^2(\theta_k), \tag{A8}$$

$$\lambda_3 = -2\cos^2(\theta_k), \tag{A9}$$

$$\lambda_4 = -2. \tag{A10}$$

The full affine representation, (51)–(55), is then found using (A7) and constructing $\mathbf{m}$ in (11) such that (12) and (13) hold.

[1] B. C. Sanders, in *Proceedings of the 5th International Conference on Reversible Computation (RC 2013)*, Lect. Notes Comput. Sci. **7948**, 1 (2013).

[2] K. L. Brown, W. J. Munro, and V. M. Kendon, Entropy **12**, 2268 (2010).
[3] I. M. Georgescu, S. Ashhab, and F. Nori, Rev. Mod. Phys. **86**, 153 (2014).
[4] R. P. Feynmann, Int. J. Theoret. Phys. **21**, 467 (1982).
[5] S. Lloyd, Science **273**, 1073 (1996).
[6] D. Aharonov and A. Ta-Shma, in *Proceedings of the 35th Annual ACM Symposium on Theory of Computing, 2003* (Association for Computing Machinery, New York, 2003), p. 20.
[7] A. M. Childs, Ph.D. thesis, Massachusetts Institute of Technology, 2004.
[8] A. M. Childs, R. Cleve, E. Deotto, E. Farhi, S. Guttman, and D. A. Spielman, in *Proceedings of the 35th Annual ACM Symposium on Theory of Computing, 2003* (Association for Computing Machinery, New York, 2003), p. 59.
[9] D. W. Berry, G. Ahokas, R. Cleve, and B. C. Sanders, Commun. Math. Phys. **270**, 359 (2007).
[10] A. Papageorgiou and C. Zhang, Quantum Inform. Process. **11**, 541 (2012).
[11] N. Wiebe, D. Berry, P. Høyer, and B. C. Sanders, J. Phys. A: Math. Theor. **43**, 065203 (2010).
[12] A. M. Childs and R. Kothari, in *Theory of Quantum Computation, Communication and Cryptography (TQC 2010)*, Lect. Notes Comput. Sci. **6519**, 94 (2011).
[13] A. M. Childs and D. W. Berry, Quantum Inform. Comput. **12**, 29 (2012).
[14] D. W. Berry, A. M. Childs, R. Cleve, R. Kothari, and R. D. Somma, arXiv:1312.1414 [quant-ph].
[15] H. P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, New York, 2002).
[16] H. Wang, S. Ashhab, and F. Nori, Phys. Rev. A **83**, 062317 (2011).
[17] M. M. Wolf, *Quantum Channels and Operations Guided Tour*, http://www-m5.ma.tum.de/foswiki/pub/M5/Allgemeines/MichaelWolf/.
[18] W. F. Stinespring, Proc. Am. Math. Soc. **6**, 211 (1955).
[19] B. M. Terhal, I. L. Chuang, D. P. DiVincenzo, M. Grassl, and J. A. Smolin, Phys. Rev. A **60**, 881 (1999).
[20] D. Bacon, A. M. Childs, I. L. Chuang, J. Kempe, D. W. Leung, and X. Zhou, Phys. Rev. A **64**, 062302 (2001).
[21] M. Ziman, P. Štelmachovič, and V. Bužek, Open Syst. Inform. Dynam. **12**, 81 (2005).
[22] M. Ziman and V. Bužek, in *Proceedings of 46th Karpacz Winter School of Theoretical Physics* (World Scientific, Singapore, 2011).
[23] M. Koniorczyk, V. Bužek, P. Adam, and A. Lázló, arXiv:quant-ph/0205008.
[24] M. Koniorczyk, V. Bužek, and P. Adam, Eur. Phys. J. D **37**, 275 (2006).
[25] T. Rybár *et al.*, J. Phys. B **45**, 154006 (2012).
[26] F. Verstraete, M. M. Wolf, and J. I. Cirac, Nature Phys. **5**, 633 (2009).
[27] J. T. Barreiro, M. Muller, P. Schindler, D. Nigg, T. Monz, M. Chwalla, M. Hennrich, C. F. Roos, P. Zoller, and R. Blatt, Nature **470**, 486 (2011).
[28] P. Schindler, M. Müller, D. Nigg, J. T. Barreiro, E. A. Martinez, M. Hennrich, T. Monz, S. Diehl, P. Zoller, and R. Blatt, Nature Phys. **9**, 361 (2013).
[29] B. Kraus, H. P. Buchler, S. Diehl, A. Kantian, A. Micheli, and P. Zoller, Phys. Rev. A **78**, 042307 (2008).
[30] M. J. Kastoryano, F. Reiter, and A. S. Sorensen, Phys. Rev. Lett. **106**, 090502 (2011).
[31] F. Reiter, M. J. Kastoryano, A. S. Sørensen, New J. Phys. **14**, 053022 (2012).
[32] L. T. Shen, X. Y. Chen, Z. B. Yang, H. Z. Wu, and S. B. Zheng, Phys. Rev. A **84**, 064302 (2011).
[33] S. G. Schirmer and X. Wang, Phys. Rev. A **81**, 062306 (2010).
[34] X. Chen, L. Shen, Z. Yang, H. Wu, and M. Chen, J. Opt. Soc. Am. B **29**, 1535 (2012).
[35] R. Sweke, I. Sinayskiy, and F. Petruccione, J. Phys. B **46**, 104004 (2013).
[36] G. Vacanti and A. Beige, New J. Phys. **11**, 083008 (2009).
[37] J. Busch, S. De, S. S. Ivanov, B. T. Torosov, T. P. Spiller, and A. Beige, Phys. Rev. A **84**, 022316 (2011).
[38] R. Sweke, I. Sinayskiy, and F. Petruccione, Phys. Rev. A **87**, 042323 (2013).
[39] M. J. Kastoryano, Ph.D. Thesis, University of Copenhagen, 2012.
[40] I. Sinayskiy and F. Petruccione, Quant. Inf. Proc. **11**, 1301 (2012).
[41] M. Kliesch, T. Barthel, C. Gogolin, M. J. Kastoryano, and J. Eisert, Phys. Rev. Lett. **107**, 120501 (2011).
[42] M. Suzuki, Phys. Lett. A **146**, 319 (1990).
[43] M. Suzuki, J. Math. Phys. **32**, 400 (1991).
[44] D. S. Wang, D. W. Berry, M. C. de Oliveira, and B. C. Sanders, Phys. Rev. Lett. **111**, 130504 (2013).
[45] M. B. Ruskai, S. Szarek, and E. Werner, Lin. Alg. Appl. **347**, 159 (2002).
[46] J. Watrous, Quantum Inform. Comput. **5**, 58 (2005).
[47] V. Kliuchnikov, D. Maslov, and M. Mosca, Phys. Rev. Lett. **110**, 190502 (2013).
[48] P. Selinger, arXiv:1212.6253 [quant-ph].
[49] H. F. Trotter, Proc. Am. Math. Soc. **10**, 545 (1959).
[50] S. Daffer, K. Wodkiewicz, and J. K. McIver, Phys. Rev. A **67**, 062312 (2003).
[51] H. J. Briegel and B. G. Englert, Phys. Rev. A **47**, 3311 (1993).
[52] M. B. Ruskai, arXiv:0708.1902 [quant-ph].