Discriminating N-qudit states using geometric structure

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Quantum state discrimination is a fundamental task having many applications in quantum information processing. However up to now there has been no rigorous formulation for discriminating N-qudit states. In this article we provide a geometric method to obtain minimum error discrimination for N-qudit states. By using the geometric approach to minimum-error discrimination for N-qudit states, we supply the condition for the existence of optimal measurement that can be composed of null operators, which gives a key understanding for discriminating N-qudit states. Furthermore we present how the number of nonzero operators for optimal measurement can be reduced. Applying our method to symmetric N-qudit states we obtain optimal measurements, which are different from known ones.

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In quantum communication the sender prepares a quantum system used as a carrier of information and the receiver performs a quantum measurement to extract the encoded information [1–3]. However a mutually nonorthogonal quantum state cannot be discriminated perfectly. Therefore, obtaining accurate information is impossible [4]. In classical communication one uses perfectly discriminated states as carriers, which means that one can extract the exact information. Along these lines, it is an important task to find a quantum measurement that provides the information as accurately as possible. A strategy to get the optimal measurement can be classified by the existence of an inconclusive result. The strategies that allow an inconclusive result are unambiguous discrimination [5–8] and maximum confidence discrimination [9]. Unambiguous discrimination permits an inconclusive result with a certain probability; however, it does not allow the error to be a conclusive result. Here we take minimum error discrimination (MD) [4] as a strategy. Unlike the previous two strategies an inconclusive result does not occur. Therefore if the quantum states do not become orthogonal to each other, the guessing error must exist.

The purpose of MD is to find the quantum measurement that provides the minimum error. The optimal method to discriminate two quantum states is already known [4,10]. However a method for discriminating three or more general quantum states has not been supplied except for qubit states. Hunter [11] and Samsonov [12] considered the case of pure qubit states. In Refs. [13,14] a complete solution for three or four mixed qubit states with arbitrary *a priori* probability is derived. From the solution the method for discriminating N-qubit states can be provided because the minimum error measurement for qubit states can always be written as a positive-operator valued measure (POVM) with at most four nonzero elements [11,15,16]. Besides qubit states, there have been efforts to discriminate special quantum states [17–21].

Even though in Refs. [1,22–24], the necessary and sufficient condition for measurement to minimize the guessing error is obtained; however, by using the condition, a method to obtain the analytic solution for optimal measurement is not known yet. Deconinck *et al.* [16] and Bae *et al.* [25] introduced methods to MD for qubit states in terms of semidefinite programming [26]. In Refs. [13,14] the geometric representation was provided for the necessary and sufficient condition for the existence of optimal measurement, which can be composed of some null operators. Therefore one determines the maximum number of nonzero operators in measurement, which can discriminate qubit states with minimum error.

In this article we introduce the geometric approach to MD for d-level quantum states (qudit states) [27]. Through the method we provide the geometric condition for the existence of optimal measurement composed of some null operators. Using this condition we show that for MD to given N-qudit states, there exists the case that one does not need to consider every state. Furthermore we show how the number of nonzero operators for minimum error measurement can be reduced. And we prove that then it is sufficient to handle a specific d^2 number of quantum states (when one needs to consider d-dimensional complex Hilbert space $\mathcal{H}_d$) [15]. Applying this to cyclic states suggested by Chou and Hsu [21], we supply the method to obtain optimal measurement, which has at most d^2 nonzero operators.

One can consider MD as a process of sending a message from Alice to Bob. Before Alice send a message to Bob, she tells him that she encodes alphabet x with probability q_x into the quantum state corresponding to the density operator ρ_x on a d-dimensional complex Hilbert space $\mathcal{H}_d$, which we write as $\{q_x,\rho_x\}$. A density operator is a positive semidefinite Hermitian operator with unit trace. What Bob has to do is discriminate $\{\rho_x\}$ with minimum error. In order to receive Alice's message, he performs a measurement described by a POVM $\{M_x\}$ which consists of positive semidefinite Hermitian operators M_x on $\mathcal{H}_d$ and satisfies the completeness relation $\sum_x M_x = I_d$, where I_d is the identity operator on $\mathcal{H}_d$. When a click of M_x means the detection of quantum state ρ_x, the probability that Bob understand Alice's message x as y becomes $P(y|\rho_x) = \mathrm{tr}[\rho_x M_y]$ by Gleason's theorem [28]. Therefore in $\{q_x,\rho_x\}$ the probability for transmitting Alice's message to Bob correctly turns out to be $P_{\mathrm{corr}} = \sum_x q_x \mathrm{tr}[\rho_x M_x]$. The purpose of MD is to minimize the probability of detection error $P_{\mathrm{err}}(=1-P_{\mathrm{corr}})$ with measurement. It is equivalent to maximize P_{corr} using POVM. In MD the maximum value of P_{corr} is called the guessing probability, P_{guess}. Let m be an

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alphabet corresponding to the greatest *a priori* probability. That is, $q_m = \max_x q_x$. If $P_{\rm guess} = q_m$, Bob does not need to use measurement for the best effort [29].

The discrimination of $\{q_x,\rho_x\}_{x=1}^N$ can be understood as a discrimination of $\{q_x/Q_\chi,\rho_x\}_{x\in\chi}$ with probability $Q_\chi(=\sum_{x\in\chi} q_x)$ and $\{q_x/Q_{\chi^{\rm C}},\rho_x\}_{x\in\chi^{\rm C}}$ with the probability $Q_{\chi^{\rm C}}(=1-Q_\chi)$. Here χ and $\chi^{\rm C}$ are some subset of $\{x\}_{x=1}^N$ and its complementary set, respectively. If $\{M_x>0\}_{x\in\chi}$ and $P_{\rm guess}^{(\chi)}$ are an optimal POVM of $\{q_x,\rho_x\}_{x=1}^N$ and the guessing probability of $\{q_x/Q_\chi,\rho_x\}_{x\in\chi}$, respectively, then $P_{\rm guess} = Q_\chi \times P_{\rm guess}^{(\chi)}$ because the POVM does not guess the quantum states as $\rho_x(\forall x \in \chi^C)$. Therefore in $\{q_x,\rho_x\}_{x=1}^N$ if Bob can discriminate $|\chi|$ alphabets $\{x\}_{x\in\chi}$ optimally, the message transmission with minimum error is possible.

Now we will explain how to find such a subset χ of $\{x\}_{x=1}^N$ when Alice encodes messages using a d-level quantum state, that is, a qudit state. The Lagrange dual problem [16] for the original problem to maximize $P_{\rm corr}$ under POVM constraints is to mimimize $\mathrm{tr}[K]$ with $K=K^\dagger$ and $K \geqslant q_x\rho_x$. Using the convexity of the MD problem and separating the hyperplane theorem, one can see that $P_{\rm guess} \leqslant \mathrm{tr}[K]$ for any K satisfying constraints of the dual problem and there must exist a POVM $\{M_x\}_{x=1}^N$ and a feasible K satisfying $P_{\rm corr} = \mathrm{tr}[K]$ [24]. The approach to MD as the complementarity problem can be introduced using the set of complementary states $\{r_x,\tilde\rho_x\}_{x=1}^N$, which satisfies $K = q_x\rho_x + r_x\tilde\rho_x(\forall x)$ for optimal K [25]. Here r_x are nonnegative numbers and $\tilde\rho_x$ are density operators on $\mathcal{H}_d$. When $\{M_x\}_{x=1}^N$ is an optimal POVM and $\{r_x,\tilde\rho_x\}_{x=1}^N$ is a set of complementary states, the following relation should hold:

$$\begin{aligned} P_{\rm guess} &= \sum_{x=1}^N q_x \mathrm{tr}[\rho_x M_x] = \sum_{x=1}^N \mathrm{tr}[(K - r_x\tilde\rho_x)M_x] \\ &= \mathrm{tr}[K] - \sum_{x=1}^N r_x \mathrm{tr}[\tilde\rho_x M_x] \\ &= \mathrm{tr}[K] = q_x + r_x \quad \forall x. \end{aligned} \tag{1}$$

Because of this result, for optimality of both problems, one must need $r_x\mathrm{tr}[\tilde\rho_x M_x] = 0(\forall x)$, which is called the complementary slackness. If this condition is added to the constraints of the primal and the dual problem, one can have the necessary and sufficient conditions of $\{M_x\}_{x=1}^N$ and $\{r_x,\tilde\rho_x\}_{x=1}^N$ for optimality of both problems [30]. Let us introduce the geometric representation for the qudit state [27]:

$$\begin{aligned} \rho_x &= \frac{1}{d}\left(I_d + \sqrt{\frac{d(d-1)}{2}}\boldsymbol{v}_x\cdot\boldsymbol{\lambda}\right) \quad \forall x, \\ \tilde\rho_x &= \frac{1}{d}\left(I_d + \sqrt{\frac{d(d-1)}{2}}\boldsymbol{w}_x\cdot\boldsymbol{\lambda}\right) \quad \forall x, \end{aligned} \tag{2}$$

where $\boldsymbol{\lambda}$ is (d^2-1) generators of $SU(d)$ satisfying the Hermitian $\lambda_x = \lambda_x^\dagger$, traceless $\mathrm{tr}[\lambda_x]=0$, trace-orthogonal $\mathrm{tr}[\lambda_i\lambda_j] = 2\delta_{ij}$ properties, and $\boldsymbol{v}_x$ and $\boldsymbol{w}_x$ are Bloch vectors in the generalized Bloch ball $\Omega_d \subset \mathbb{R}^{d^2-1}$ for the qudit state. Using this representation one can see that

$$q_x\rho_x - q_y\rho_y = r_y\tilde\rho_y - r_x\tilde\rho_x, \tag{3}$$

and the constraints of the dual problem can be rewritten as $r_x - r_y = q_y - q_x(\forall x,y)$ and $r_x\boldsymbol{w}_x - r_y\boldsymbol{w}_y = q_y\boldsymbol{v}_y - q_x\boldsymbol{v}_x(\forall x,y)$. By the positive semidefinite condition of the POVM, the elements can be found as

$$M_x = p_x\left(I_d + \sqrt{\frac{d(d-1)}{2}}\boldsymbol{u}_x\cdot\boldsymbol{\lambda}\right) \quad \forall x, \tag{4}$$

where p_x are nonnegative numbers and $\boldsymbol{u}_x$ are Bloch vectors in Ω_d. From the completeness condition, one can obtain $\sum_x p_x = 1$ and $\sum_x p_x\boldsymbol{u}_x = 0$, which are the geometric representations for POVM constraints. In addition, by Eqs. (2) and (4), the complementary slackness condition becomes $p_x r_x[(d-1)\boldsymbol{u}_x\cdot\boldsymbol{w}_x + 1] = 0(\forall x)$. Then one has the following theorem.

Theorem 1. In $\{q_x,\rho_x\}_{x=1}^N$, there exists a minimum error measurement expressed by the POVM, which has N nonzero elements and its guessing probability is greater than the largest *a priori* probability if and only if there exist an optimal POVM and a set of complementary states described as $\{p_x>0,\boldsymbol{u}_x\in\Omega_d\}_{x=1}^N$ and $\{r_x>0,\boldsymbol{w}_x\in\Omega_d\}_{x=1}^N$, respectively, fulfilling the following conditions:

$$\begin{aligned} &\text{(i)}\ \ r_x\boldsymbol{w}_x - r_y\boldsymbol{w}_y = q_y\boldsymbol{v}_y - q_x\boldsymbol{v}_x \quad \forall x,y, \\ &\text{(ii)}\ \ \sum_{x=1}^N p_x = 1, \quad \sum_{x=1}^N p_x\boldsymbol{u}_x = 0, \\ &\text{(iii)}\ \ \boldsymbol{u}_x\cdot\boldsymbol{w}_x = -1/(d-1) \quad \forall x, \\ &\text{(iv)}\ \ r_x - r_y = q_y - q_x \quad \forall x,y. \end{aligned} \tag{5}$$

Suppose that $P_{\rm guess} > q_m$ in $\{q_x,\rho_x\}_{x=1}^N$ and $\{M_x\}_{x=1}^N$ is an optimal POVM whose every element is nonzero and $\{r_x,\tilde\rho_x\}_{x=1}^N$ is a set of complementary states. Let $\{p_x\geqslant 0,\boldsymbol{u}_x\in\Omega_d\}_{x=1}^N$ and $\{r_x\geqslant 0,\boldsymbol{w}_x\in\Omega_d\}_{x=1}^N$ be two sets of Bloch vectors corresponding to its optimal POVM and complementary states, respectively. What is needed is to show that they satisfy condition (iii) because constraints of the primal and the dual problem contain conditions (i), (ii), and (iv) of Eq. (5). Since $p_x r_x[(d-1)\boldsymbol{u}_x\cdot\boldsymbol{w}_x+1] = 0(\forall x)$ by complementary slackness, its proof can be shown through $p_x,r_x>0(\forall x)$. $p_x>0(\forall x)$ can be found from the fact every element of POVM is nonzero. $r_x>0(\forall x)$ can be shown through the following two facts: (i) because of $r_x - r_y = q_y - q_x(\forall x,y)$, r_m becomes the minimum of $\{r_x\}_{x=1}^N$, and (ii) $P_{\rm guess} > q_m$ and $P_{\rm guess} = q_m + r_m$ imply $r_m > 0$. Therefore, in $\{q_x,\rho_x\}_{x=1}^N$, if $P_{\rm guess} > q_m$ and there exists an optimal measurement whose POVM consists of N nonzero operators, there must exist $\{p_x>0,\boldsymbol{u}_x\in\Omega_d\}_{x=1}^N$ and $\{r_x>0,\boldsymbol{w}_x\in\Omega_d\}_{x=1}^N$ fulfilling the conditions in Eq. (5).

Now let us prove the inverse and assume that $\{p_x>0,\boldsymbol{u}_x\in\Omega_d\}_{x=1}^N$ and $\{r_x>0,\boldsymbol{w}_x\in\Omega_d\}_{x=1}^N$ satisfy the conditions in Eq. (5). It is clear that $\{M_x\}_{x=1}^N$ and $\{\tilde\rho_x\}_{x=1}^N$ obtained by substituting these parameters into Eqs. (2)–(4) satisfy the constraints of the primal and dual problems and the complementary slackness condition. Thus, $\{M_x\}_{x=1}^N$ is an optimal POVM whose every element is nonzero since every p_x is nonzero, and $\{r_x,\tilde\rho_x\}_{x=1}^N$ is a set of complementary states. Then we can see that the guessing probability is larger than q_m, from the fact that $P_{\rm guess} = q_m + r_m$ and $r_m > 0$. Therefore the inverse is proved.

This tells the following fact: In $\{q_x,\rho_x\}_{x=1}^N$, if $\{p_x>0,\boldsymbol{u}_x\in\Omega_d\}_{x=1}^N$ and $\{r_x>0,\boldsymbol{w}_x\in\Omega_d\}_{x=1}^N$ cannot fulfill the geometric optimality condition (GOC) of Eq. (5), it is not necessary to consider every alphabets $\{x\}_{x=1}^N$ for message transmission with minimum error. It is because the guessing probability $P_{\rm guess}$ is either q_m with a POVM $\{M_m = I_d\}$ or $Q_\chi \times P_{\rm guess}^{(\chi)}(>q_m)$ with some POVM $\{M_x > 0\}_{x\in\chi}$ ($\chi \subset \{x\}_{x=1}^N$). Note that χ is a proper subset of $\{x\}_{x=1}^N$. In the latter case, since $P_{\rm guess}^{(\chi)} > q_m/Q_\chi \geqslant (\max_{x\in\chi} q_x)/Q_\chi$ and $M_x > 0$ for any $x \in \chi$, there must exist $\{p_x > 0, \boldsymbol{u}_x \in \Omega_d\}_{x\in\chi}$ and $\{r_x > 0, \boldsymbol{w}_x \in \Omega_d\}_{x\in\chi}$ such that $\{p_x, \boldsymbol{u}_x\}_{x\in\chi}$ and $\{r_x/Q_\chi, \boldsymbol{w}_x\}_{x\in\chi}$ fulfill the GOC of $\{q_x/Q_\chi, \rho_x\}_{x\in\chi}$.

Let us consider MD for N-qudit states satisfying the following three conditions: (i) $q_x = 1/N(\forall x)$, (ii) the relative interior of $\{\boldsymbol{v}_x\}_{x=1}^N$ contains the origin of the Bloch ball, and (iii) ρ_x has $f(1 \leqslant f < d)$ eigenvalue $1/f$. The first condition implies that, if $\{r_x, \boldsymbol{u}_x\}_{x=1}^N$ are a set of Bloch vectors' corresponding complementary states, r_x are all equal. The second one means that there exists $\{p_x > 0\}_{x=1}^N$ satisfying $\sum_{x=1}^N p_x = 1$ and $\sum_{x=1}^N p_x \boldsymbol{v}_x = 0$. Since $\mathrm{tr}[\rho_x^2] = (1 + (d-1)\|\boldsymbol{v}_x\|_2^2)/d = 1/f$ and

$$I_d - f\rho_x = \frac{d-f}{d}\left(I_d - \sqrt{\frac{d(d-1)}{2}}\frac{f\boldsymbol{v}_x}{d-f}\cdot\boldsymbol{\lambda}\right) \geqslant 0$$

for any x, all of $\boldsymbol{v}_x$ fulfill $\|\boldsymbol{v}_x\|_2^2 = (d-f)/f(d-1)$ and $-f\boldsymbol{v}_x/(d-f) \in \Omega_d$. From these relations we can see that, if for any x, we can have $r_x = (d-f)/Nf$, $\boldsymbol{w}_x = -f\boldsymbol{v}_x/(d-f)\}_{x=1}^N$, and $\boldsymbol{u}_x = \boldsymbol{v}_x$, $\{p_x, \boldsymbol{u}_x\}_{x=1}^N$ and $\{r_x, \boldsymbol{w}_x\}_{x=1}^N$ can satisfy the GOC of $\{q_x, \rho_x\}_{x=1}^N$. Therefore $d/Nf[=1/N + (d-f)/Nf = q_x + r_x \quad \forall x]$ and $\{dp_x\rho_x(=dp_x[I_d + \sqrt{d(d-1)/2}\boldsymbol{v}_x\cdot\boldsymbol{\lambda}]/d = p_x[I_d + \sqrt{d(d-1)/2}\boldsymbol{u}_x\cdot\boldsymbol{\lambda}])\}_{x=1}^N$ are the guessing probability and a POVM of minimum-error measurement in $\{q_x, \rho_x\}_{x=1}^N$, respectively. Then if $\sum_{x=1}^N v_x = 0$, we have $\sum_{x=1}^N q_x\rho_x = I_d/d$. If we set $p_x = 1/N(\forall x)$, we can make an optimal POVM $\{dp_x\rho_x\}_{x=1}^N$ as a square root or a pretty good measurement [31].

Now we provide the method in which the number of nonzero operators for optimal measurement can be reduced. Suppose that $P_{\rm guess} = Q_\chi \times P_{\rm guess}^{(\chi)} > q_m$ ($\chi \subseteq \{x\}_{x=1}^N$). Note that this χ is not necessarily a proper subset of $\{x\}_{x=1}^N$. Then by theorem there exist $\{p_x > 0, \boldsymbol{u}_x \in \Omega_d\}_{x\in\chi}$ and $\{r_x > 0, \boldsymbol{w}_x \in \Omega_d\}_{x\in\chi}$ such that $\{p_x, \boldsymbol{u}_x\}_{x\in\chi}$ and $\{r_x/Q_\chi, \boldsymbol{w}_x\}_{x\in\chi}$ fulfill the GOC of $\{q_x/Q_\chi, \rho_x\}_{x\in\chi}$. Let M_x be POVM elements corresponding $\{p_x, \boldsymbol{u}_x\}_{x\in\chi}$. For $\{\bar{p}_x > 0\}_{x\in\Lambda}$ ($\Lambda \subseteq \chi$) fulfilling $\sum_{x\in\Lambda}(\bar{p}_x d/\mathrm{tr}[M_x])M_x = I_d$, $\{\bar{p}_x, \boldsymbol{u}_x\}_{x\in\Lambda}$ and $\{r_x/Q_\Lambda, \boldsymbol{w}_x\}_{x\in\Lambda}$ can satisfy the GOC of $\{q_x/Q_\Lambda, \rho_x\}_{x\in\Lambda}$.

Therefore the measurement associated with the POVM $\{(\bar{p}_x d/\mathrm{tr}[M_x])M_x\}_{x\in\Lambda}$ can discriminate $\{q_x/Q_\Lambda, \rho_x\}_{x\in\Lambda}$ with minimum error. Then we find that

$$Q_\Lambda \times P_{\rm guess}^{(\Lambda)} = q_x + r_x = Q_\chi \times P_{\rm guess}^{(\chi)} = P_{\rm guess}, \qquad (6)$$

for any $x \in \Lambda \subseteq \chi$. This implies that the suggested POVM is another one describing the minimum-error measurement. Then we obtain the following lemma.

Lemma 1. In the discrimination of a set of qudit states $\{q_x, \rho_x\}_{x=1}^N$, when $\{M_x > 0\}_{x\in\chi}(\chi \subseteq \{x\}_{x=1}^N)$ is a POVM describing the minimum-error measurement and the guessing probability is larger than the greatest *a priori* probability, if $\{\bar{p}_x > 0\}_{x\in\Lambda}(\Lambda \subseteq \chi)$ satisfies $\sum_{x\in\Lambda}(\bar{p}_x d/\mathrm{tr}[M_x])M_x = I_d$, $\{(\bar{p}_x d/\mathrm{tr}[M_x])M_x\}_{x\in\Lambda}$ is also a POVM describing the minimum error measurement in $\{q_x, \rho_x\}_{x=1}^N$.

By Carathéodory's theorem, if the relative interior of $\{\boldsymbol{u}_x\}_{x\in\chi}$ contains the origin of Ω_d, there must exist a subset Λ of χ satisfying the following conditions: (i) the convex hull of $\{\boldsymbol{u}_x\}_{x\in\Lambda}$ is simplex, and (ii)the relative interior of $\{\boldsymbol{u}_x\}_{x\in\Lambda}$ contains the origin of Ω_d. Furthermore the fact that $\Omega_d \subset \mathbb{R}^{d^2-1}$ tells us that the affine dimension of simplex of the convex hull of $\{\boldsymbol{u}_x\}_{x\in\Lambda}$ is less than d^2. This means that the number of elements of Λ should be equal to or less than d^2. Therefore we obtain the following corollary [15].

Corollary 1. For MD for any qudit state, there exists a minimum-error measurement expressed by nonzero operators equal to or less than d^2.

This tells us the following fact: If one can analyze MD for d^2-qudit states, one can obtain the guessing probability and an optimal POVM for MD to N-qudit states. When $N > d^2$, $\{p_x > 0, \boldsymbol{u}_x \in \Omega_d\}_{x=1}^N$ and $\{r_x > 0, \boldsymbol{w}_x \in \Omega_d\}_{x=1}^N$ are hard to fulfill the GOC of $\{q_x, \rho_x\}_{x=1}^N$. This is because condition (iv) of Eq. (5) builds equality conditions and one cannot express the GOC as inequality like the GOC [13,14] of qubit states. In fact when $N > d^2$ and $P_{\rm guess} > q_m$, the minimum-error measurement may not be a POVM with N nonzero elements. However for special symmetric states, though $N > d^2$, there exists a minimum-error measurement with N nonzero elements [17,21].

From now on, for MD to $N(>d^2)$-cyclic states, using Lemma 1, we show the method to provide a minimum-error measurement which has equal to or less than d^2 nonzero operators. The $N(>d^2)$-cyclic states $\{\rho_x\}_{x=1}^N$ (satisfying $\rho_{y+N} = \rho_y$ for $y \in \{x\}_{x=1}^N$) can be defined as $\{\rho_x = U^x\rho_0 U^{x\dagger}\}_{x=1}^N$, where a unitary operator U on $\mathcal{H}$ and a density operator ρ_0 on $\mathcal{H}$ are given by

$$U = \sum_{k=0}^{d-1} e^{2\pi i f_k/N}|\phi_k\rangle\langle\phi_k|, \quad \rho_0 = \sum_{i,j=0}^{d-1} c_{i,j}|\phi_i\rangle\langle\phi_j|. \qquad (7)$$

Here $|\phi_k\rangle$ are orthonormal eigenvectors of U and f_k are integers between 0 and $N-1$ satisfying $\rho_y \neq \rho_z$ for any $y,z \in \{x\}_{x=1}^N$ with $y \neq z$. If there exist $y,z \in \{x\}_{x=1}^N$ fulfilling $\rho_y = \rho_z$ and $y \neq z$, one can see that there exist $X \in \{x\}_{x=1}^{N-1}$ satisfying $\rho_X = \rho_0$ and $N \equiv 0 (\mathrm{mod}\, X)$. The discrimination of $\{1/N, \rho_x\}_{x=1}^N$ is equivalent to that of $\{1/X, \rho_x\}_{x=1}^X$ for X less than N. Therefore it is sufficient to consider the unitary operator U mentioned above. Since $\rho_N = \rho_0$, preparing $\{1/N, \rho_x\}_{x=0}^{N-1}$ is equivalent to preparing $\{1/N, \rho_x\}_{x=1}^N$. Here we choose the index from $x = 0$ to $N-1$. We assume that $f_0 = 0$ and as k increases f_k also increases. This means that $0 = f_0 < f_1 < \cdots < f_{d-1} < N$. The assumption is that this operator U can preserve its unitarity when $d = 2$; however it does not when $d > 2$. The results of Ref. [21] show that if $c_{i,j} \geqslant 0$ for any i,j, the measurement described as $\{M_x = U^x|\Phi\rangle\langle\Phi|U^{x\dagger}\}_{x=0}^{N-1}$ is a minimum error measurement in $\{1/N, \rho_x\}_{x=0}^{N-1}$ and the guessing probability becomes $(1/N)(\sum_{i,j=0}^{d-1} c_{i,j})$, where

$$|\Phi\rangle = \frac{1}{\sqrt{N}}\sum_{k=0}^{d-1}|\phi_k\rangle. \qquad (8)$$

If for $i,j \in \{0,\ldots,d-1\}$ satisfying $i \neq j$, $c_{i,j} = 0$, one can see that $\rho_0 = \rho_1 = \cdots = \rho_{N-1}$, which cannot fulfill an assumption such as $\rho_y \neq \rho_z$ for $y,z \in \{x\}_{x=1}^{N}$ with $y \neq z$. Therefore there should exist $i,j \in \{0,\ldots,d-1\}$ with $i \neq j$, satisfying $c_{i,j} > 0$. Furthermore $\sum_{i=0}^{d-1} c_{i,i} = \mathrm{tr}[\rho_0] = 1$ gives $P_{\rm guess} > 1/N = q_m$.

Here we obtain the necessary and sufficient condition for $\sum_{x\in\Lambda}(\bar{p}_x d/\mathrm{tr}[M_x])M_x = I_d$, when $\{\bar{p}_x > 0\}_{x\in\Lambda}(\Lambda \subset \{x\}_{x=0}^{N-1})$ satisfies the following special constraints: (i) $0 \in \Lambda$ and $N/2 \notin \Lambda$, (ii) if $x \in \Lambda \setminus \{0\}$, $N - x \in \Lambda$ and $\bar{p}_x = \bar{p}_{N-x}$, (iii) $|\Lambda| \leqslant d^2$, where $|\Lambda|$ is the number of elements of Λ. We now substitute $M_x = U^x|\Phi\rangle\langle\Phi|U^{x\dagger}(x \in \Lambda)$ and U of Eq. (7) into $\sum_{x\in\Lambda}(\bar{p}_x d/\mathrm{tr}[M_x])M_x$:

$$\sum_{x\in\Lambda} \frac{\bar{p}_x d\ M_x}{\mathrm{tr}[M_x]} = \sum_{k,l=0}^{d-1}\sum_{x\in\Lambda} \bar{p}_x U^x|\phi_k\rangle\langle\phi_l|U^{x\dagger} = \sum_{k,l=0}^{d-1} |\phi_k\rangle\langle\phi_l| \sum_{x\in\Lambda} \bar{p}_x e^{2\pi i(f_k-f_l)x/N}. \quad (9)$$

From $\sum_{x\in\Lambda} \bar{p}_x = 1$ one can see that $\sum_{x\in\Lambda}(\bar{p}_x d/\mathrm{tr}[M_x])M_x = I_d$ becomes

$$\sum_{x\in\Lambda} \bar{p}_x \cos\left[\frac{2\pi(f_k - f_l)x}{N}\right] = 0 \quad \forall k,l, \quad (10)$$

$$\sum_{x\in\Lambda} \bar{p}_x \sin\left[\frac{2\pi(f_k - f_l)x}{N}\right] = 0 \quad \forall k,l. \quad (11)$$

For $\{\bar{p}_x > 0\}_{x\in\Lambda}$ these condition can be rewritten as

$$\bar{p}_0 + 2\sum_{i=1}^{\frac{|\Lambda|-1}{2}} \bar{p}_{x_i^{(\Lambda)}} = 1, \qquad \bar{p}_0 + 2\sum_{i=1}^{\frac{|\Lambda|-1}{2}} \bar{p}_{x_i^{(\Lambda)}} \cos\left[\frac{2\pi(f_k - f_l)x_i^{(\Lambda)}}{N}\right] = 0 \quad \forall k,l, \quad (12)$$

where $x_i^{(\Lambda)}$ are elements of $\Lambda \setminus \{0\}$ satisfying $x_j^{(\Lambda)} < x_{j+1}^{(\Lambda)}$. Since the number of elements of Λ is odd and the sine function is an odd function, the conditions of Eq. (11) vanish for any k,l. If $\{\bar{p}_x > 0\}_{x\in\Lambda}$ satisfies the previous three conditions and the conditions of Eq. (12), by Lemma 1, the measurement expressed as $\{N\bar{p}_x U^x|\Phi\rangle\langle\Phi|U^{x\dagger}\}_{x\in\Lambda}$ can discriminate $\{1/N, U^x\rho_0 U^{x\dagger}\}_{x=0}^{N-1}$ with minimum error. In Eq. (12), the second equation when $k = l$ coincides with the first equation (the second equation when $k > l$ coincides with the equation when $k < l$). One should note that the conditions of Eq. (12) are independent of orthonormal eigenvectors of U.

For the case of $d = 2$, if $\{|\phi_0'\rangle = |\phi_0\rangle, |\phi_1'\rangle = (c_{1,0}/|c_{1,0}|)|\phi_1\rangle\}$ are used instead of $\{|\phi_0\rangle, |\phi_1\rangle\}$ as orthonormal eigenvectors of U, every element of ρ_0 is positive. U and ρ_0 in Eq. (7) can be written in the following way:

$$\begin{aligned} U &= \|\phi_0'\rangle\langle\phi_0'| + e^{2\pi i f_1/N}|\phi_1'\rangle\langle\phi_1'|, \\ \rho_0 &= c_{0,0}|\phi_0'\rangle\langle\phi_0'| + |c_{0,1}||\phi_0'\rangle\langle\phi_1'| \\ &\quad + |c_{1,0}||\phi_1'\rangle\langle\phi_0'| + c_{1,1}|\phi_1'\rangle\langle\phi_1'|. \end{aligned} \quad (13)$$

The conditions of Eq. (12) can be expressed by

$$\Gamma\begin{pmatrix}\bar{p}_0 \\ \bar{p}_{x_1}\end{pmatrix} = \begin{pmatrix}1 \\ 0\end{pmatrix}, \quad (14)$$

where

$$\Gamma = \begin{pmatrix} 1 & 2 \\ 1 & 2\cos\left[\frac{2\pi f_1 x_1^{(\Lambda)}}{N}\right] \end{pmatrix}. \quad (15)$$

Since $\det[\Gamma] \neq 0$ is equivalent to $\cos[2\pi f_1 x_1^{(\Lambda)}/N] < 1$, Γ can be invertible in this case. By multiplying Γ^{-1} to Eq. (14) $\bar{p}_0$ and $\bar{p}_{x_1^{(\Lambda)}}$ can be found as

$$\bar{p}_0 = \frac{-\cos\left[2\pi f_1 x_1^{(\Lambda)}/N\right]}{1 - \cos\left[2\pi f_1 x_1^{(\Lambda)}/N\right]}, \qquad \bar{p}_{x_1^{(\Lambda)}} = \frac{1}{2\left(1 - \cos\left[2\pi f_1 x_1^{(\Lambda)}/N\right]\right)}. \quad (16)$$

From this we can see that the condition for $\bar{p}_0, \bar{p}_{x_1^{(\Lambda)}} > 0$ becomes $\cos[2\pi f_1 x_1^{(\Lambda)}/N] < 0$. This condition contains $\det[\Gamma] \neq 0$. Lemma 2 summarizes this result.

Lemma 2. Let U and ρ_0 be a unitary operator on $\mathcal{H}_2$ and a density operator on $\mathcal{H}_2$, respectively, described as

$$U = |\phi_0\rangle\langle\phi_0| + e^{2\pi i f/N}|\phi_1\rangle\langle\phi_1|, \qquad \rho_0 = \sum_{i,j=0}^{1} c_{i,j}|\phi_i\rangle\langle\phi_j|, \quad (17)$$

where $|\phi_k\rangle$ are orthonormal eigenvectors of U and f is an integer between 1 and $N-1$ satisfying $\rho_y \neq \rho_z$ for any $y,z \in \{x\}_{x=0}^{N-1}$ with $y \neq z$ and N being an integer more than 4. Then if x' satisfies $\cos[2\pi f x'/N] < 0$, the following operators $\{M_0, M_{x'}, M_{N-x'}\}$ become a POVM associated with minimum-error measurement in $\{1/N, U^x\rho_0 U^{x\dagger}\}_{x=0}^{N-1}$:

$$\begin{aligned} M_0 &= \left(\frac{-\cos[2\pi f x'/N]}{1 - \cos[2\pi f x'/N]}\right)\tilde{M}_0, \\ M_{x'} &= \left(\frac{1}{2(1 - \cos[2\pi f x'/N])}\right)\tilde{M}_{x'}, \\ M_{N-x'} &= \left(\frac{1}{2(1 - \cos[2\pi f x'/N])}\right)\tilde{M}_{N-x'}, \end{aligned} \quad (18)$$

where

$$\tilde{M}_x = U^x|\bar{\Phi}\rangle\langle\bar{\Phi}|U^{x\dagger}, \qquad |\bar{\Phi}\rangle = \frac{1}{\sqrt{N}}\left(|\phi_0\rangle + \frac{\langle\phi_1|\rho_0|\phi_0\rangle}{|\langle\phi_1|\rho_0|\phi_0\rangle|}|\phi_1\rangle\right). \quad (19)$$

Finally let us consider an example when $d = 3$. If $f_1 = 1$ and $f_2 = 2$ for $d = 3$ and $N = 9$, one can find $f_1 - f_0 = f_2 - f_1 = 1$ and $f_2 - f_0 = 2$. This shows that the conditions of Eq. (12) become

$$\bar{p}_0 + 2\sum_{i=1}^{\frac{|\Lambda|-1}{2}} \bar{p}_{x_i^{(\Lambda)}} = 1, \qquad \bar{p}_0 + 2\sum_{i=1}^{\frac{|\Lambda|-1}{2}} \bar{p}_{x_i^{(\Lambda)}} \cos\left[\frac{2\pi x_i^{(\Lambda)}}{9}\right] = 0,$$

$$\bar{p}_0 + 2\sum_{i=1}^{\frac{|\Lambda|-1}{2}} \bar{p}_{x_i^{(\Lambda)}} \cos\left[\frac{4\pi x_i^{(\Lambda)}}{9}\right] = 0. \qquad (20)$$

When $\Lambda = \{0,3,6\}$ the first condition of Eq. (20) becomes $\bar{p}_0 + 2\bar{p}_3 = 1$. The second and third ones become $\bar{p}_0 - \bar{p}_3 = 0$. Therefore the conditions for $\bar{p}_0$ and $\bar{p}_3$ are expressed as

$$\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \bar{p}_0 \\ \bar{p}_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \qquad (21)$$

Without using an invertible matrix, we can see that $\bar{p}_0 = \bar{p}_3 = \bar{p}_6 = 1/3$ fulfills this equation. This means that the measurement $\{3U^x|\Phi\rangle\langle\Phi|U^{x\dagger}\}_{x\in\Lambda}$ can discriminate $\{q_x,\rho_x\}_{x=0}^{8}$ with minimum error.

In this article we provided a geometric method to obtain MD for qudit states. By supplying the geometric conditions for minimum error measurement including null operators, we could understand the structure to MD for N-qudit states. Furthermore we investigated the way to reduce the number of nonzero operators for minimum-error measurement and determined how many nonzero operators can exist. Finally we applied our method to symmetric states and obtained optimal measurements which are different from known ones.

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