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Quantum metrology for a general Hamiltonian parameter

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Quantum metrology enhances the sensitivity of parameter estimation using the distinctive resources of quantum mechanics such as entanglement. It has been shown that the precision of estimating an overall multiplicative factor of a Hamiltonian can be increased to exceed the classical limit, yet little is known about estimating a general Hamiltonian parameter. In this paper, we study this problem in detail. We find that the scaling of the estimation precision with the number of systems can always be optimized to the Heisenberg limit, while the time scaling can be quite different from that of estimating an overall multiplicative factor. We derive the generator of local parameter translation on the unitary evolution operator of the Hamiltonian, and use it to evaluate the estimation precision of the parameter and establish a general upper bound on the quantum Fisher information. The results indicate that the quantum Fisher information generally can be divided into two parts: one is quadratic in time, while the other oscillates with time. When the eigenvalues of the Hamiltonian do not depend on the parameter, the quadratic term vanishes, and the quantum Fisher information will be bounded in this case. To illustrate the results, we give an example of estimating a parameter of a magnetic field by measuring a spin-$\frac{1}{2}$ particle and compare the results for estimating the amplitude and the direction of the magnetic field.

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I. INTRODUCTION

Quantum metrology [1,2] is a scheme that uses entanglement to increase the precision of parameter estimation by quantum measurements beyond the limit of its classical counterpart. In classical parameter estimation, the estimation precision scales as $\nu^{-\frac{1}{2}}$, where ν is the number of rounds of measurement. The scaling can be rewritten as $(N\nu)^{-\frac{1}{2}}$, where N is the number of qubits used in each round, for parameter estimation by quantum measurements if the N qubits are not entangled. This scaling is often termed as the *standard quantum limit* (SQL) [3], which characterizes the precision limit of quantum measurements in the presence of the shot noise. A more fundamental imprecision of quantum measurement originates from the Heisenberg uncertainty principle, which is one of the most fundamental properties of quantum mechanics, due to the probabilistic nature of quantum measurements. Research in quantum metrology has shown that with the assistance of n-qubit entanglement, the optimal scaling of the estimation precision can be raised to $N^{-1}\nu^{-\frac{1}{2}}$, i.e., the *Heisenberg limit*, implying an improvement of $N^{\frac{1}{2}}$ over the SQL.

Quantum metrology is rooted in the theory of quantum estimation, which was pioneered by Helstrom [4] and Holevo [5] who proposed the parameter-based uncertainty relation. Braunstein *et al.* [6,7] developed that theory from the view of the Cramér-Rao bound [8], which characterizes how well a parameter can be estimated from a probability distribution, and obtained the optimal Fisher information over different quantum measurement schemes for a given parameter-dependent quantum state. This is often called *quantum Fisher information.*

Given the importance of precision measurement in different fields of physics and engineering, the quantum Fisher information has attracted great interest from researchers. Giovannetti *et al.* [1] found that the scaling of the quantum Fisher information has an $N^{\frac{1}{2}}$ improvement compared to its classical counterpart if an N-qubit maximally entangled state is used. This stimulated the emergence of quantum metrology, which has been applied to different quantum systems to raise the precision of measurements.

The optimality of quantum metrology in terms of the scaling of the measurement precision was proved in [9] for different initial states and measurement schemes, and also by [10] from the viewpoint of the query complexity of a quantum network. Moreover, when there is interaction among the N entangled qubits or the Hamiltonian is nonlinear, the measurement precision can be further increased to beyond the Heisenberg limit [11–16].

Many applications of quantum metrology have been found, including quantum frequency standards [17,18], optical phase estimation [19–25], atomic clocks [26–30], atomic interferometers [31], quantum imaging [32,33], and quantum-enhanced positioning and clock synchronization [34]. The quantum Fisher information has also been studied in open systems [35–41], along with growing research on protocols assisted by error correction [42–44]. Moreover, quantum metrology with nonlinear Hamiltonians has received considerable attention [12,15,45–52]. For reviews of the field of quantum metrology, refer to [1,2].

Studies of quantum metrology have mainly focused on the precision of measuring an overall multiplicative factor of a Hamiltonian, e.g., the parameter g in a Hamiltonian gH, a setting particularly suitable for enhancing phase or frequency estimation in devices such as optical interferometers or atomic spectroscopes. However, generally speaking, a parameter can appear in a more general form in a Hamiltonian, not necessarily as an overall multiplicative factor. For example, the parameter can appear with different orders in the eigenvalues of the Hamiltonian or even in the eigenstates of the Hamiltonian. An understanding of the quantum limits in estimating this kind of general parameter is emerging (e.g., [53] from the view

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of information geometry), but is still rather limited so far, which restricts the potential range of applications of quantum mechanics to metrology.

This paper extends quantum metrology to estimating a general parameter of a Hamiltonian. We will show that the optimal scaling of the measurement precision with the number N of systems is still N^{-1}, but the time scaling will be different. In detail, it will be shown that the quantum Fisher information can generally be divided into two parts: one is linear in the time t, corresponding to the variation of the eigenvalues, and the other is oscillatory, corresponding to the variation of the eigenvectors of the Hamiltonian. The oscillating part is bounded no matter how long the time t is. We will obtain an upper bound on the Fisher information for the general case.

The study of this problem will extend the current knowledge of quantum metrology to a more general case, and more kinds of precision measurements will benefit from this extension, especially those that go beyond phase or frequency measurement. For instance, as we show as an example in this paper, it can enhance the precision of measuring the direction of a magnetic field by a spin-$\frac{1}{2}$ system, which is useful for calibrating the field. Therefore, the results of this paper will be useful to both theory and experiments in quantum metrology.

II. BACKGROUND

Let us first review some concepts of the estimation theory and their quantum counterparts. The task of parameter estimation is to determine a parameter from a set of data which depends on the parameter. A general procedure for estimating a parameter is as follows: first acquire a set of data $x_1,\ldots,x_\nu$ which obey a probability distribution dependent on the parameter $f_g(x)$, where g is the parameter to estimate; then estimate g from $x_1,\ldots,x_\nu$ by a certain estimator and obtain the estimated value $g_{\text{est}}(x_1,\ldots,x_\nu)$. While there are many different estimation strategies, such as the method of moments and maximum-likelihood estimation, the performances of those strategies differ. One of the most important benchmarks of a strategy is the estimation precision, which is usually characterized by the estimation error [6]:

$$\delta g \equiv \frac{g_{\text{est}}}{|d\langle g_{\text{est}}\rangle_g/dg|} - g, \tag{1}$$

where the factor $|d\langle g_{\text{est}}\rangle_g/dg|$ is to eliminate the local difference in the units between the estimator and the real parameter for different g. If the estimation procedure is repeated many times, the estimated value g_{est} may have fluctuations. So an appropriate measure to quantify the performance of an estimator is the root-mean-square error of the estimation results $\langle(\delta g)^2\rangle^{\frac{1}{2}}$. A cornerstone of the classical theory of parameter estimation is the Cramér-Rao bound [8], which bounds the precision limit of an estimator by the following relation:

$$\langle(\delta g)^2\rangle \geqslant \frac{1}{\nu F(g)} + \langle\delta g\rangle^2, \tag{2}$$

where F_g is the Fisher information defined as

$$F_g = \int [\partial_g \ln f_g(x)]^2 f_g(x)dx. \tag{3}$$

The second term on the right side of (2), $\langle\delta g\rangle^2$, characterizes the bias of the estimator. If the estimator is unbiased, i.e., $\langle g_{\text{est}}\rangle_g = g$, then $\langle\delta g\rangle = 0$.

The achievability (or the tightness) of the Cramér-Rao bound (2) is addressed by the Fisher theorem. Fisher proved that for asymptotically large ν, the Cramér-Rao bound can always be achieved by maximum-likelihood estimation (MLE) and the estimation result is unbiased. Because of this property, MLE has been widely adopted in parameter estimation protocols.

In quantum metrology, one measures a parameter-dependent state, say ρ_g, to estimate g. The process of a quantum metrology protocol splits into two stages. First, measure the state in some basis [or, more generally, perform a positive operator-valued measure (POVM) on it] and record the measurement result. When such a measurement is repeated ν times for the same ρ_g, we will acquire ν measurement results. These results depend on g, so they can be used as sample data to estimate g. The second stage is estimating the parameter g based on the measurement results by some appropriate estimation strategy. The precision of the estimation is bounded by (2) as usual. The complexity of quantum metrology comes from the many different choices of the measurements (or POVMs). Different choices lead to different precisions of the estimation results. The aim of quantum metrology is to increase the estimation precision by optimizing the measurement basis (or POVM).

Braunstein and Caves obtained the optimal Fisher information over all POVMs for a given ρ_g [6], which is called the quantum Fisher information, through the logarithmic derivative L_g:

$$F_g^{(Q)} = \text{Tr}(L_g^\dagger \rho_g L_g). \tag{4}$$

The logarithmic derivative L_g has several different but equivalent definitions. The most common is the symmetric logarithmic derivative (SLD), defined as $\partial_g\rho_g = (L_g\rho_g + \rho_g L_g)/2$. L_g in this definition is Hermitian, and the quantum Fisher information $F_g^{(Q)}$ (4) can be simplified to $\text{Tr}(\rho_g L_g^2)$. In the eigenbasis of ρ_g, an explicit form of L_g can be found:

$$L_g = 2\sum_{i,j} \frac{\langle\eta_i|\partial_g\rho_g|\eta_j\rangle}{\eta_i+\eta_j}|\eta_i\rangle\langle\eta_j|, \tag{5}$$

where the η_i's are the eigenvalues of ρ_g and the $|\eta_i\rangle$'s are the corresponding eigenstates.

In the current literature of quantum metrology, most research interest has been focused on estimating an overall multiplicative factor of a Hamiltonian, for example, estimating g in a Hamiltonian gH. Usually an initial pure state $|\Psi\rangle$ is used to undergo evolution by the Hamiltonian, so that

$$\rho_g = \exp(-igtH)|\Psi\rangle\langle\Psi|\exp(igtH). \tag{6}$$

In such a case, the quantum Fisher information $F_g^{(Q)}$ can be simplified to

$$F_g^{(Q)} = 4t^2\langle\Psi|\Delta H^2|\Psi\rangle. \tag{7}$$

It can be proved [9] that $\langle\Psi|\Delta H^2|\Psi\rangle_{\max} = \frac{1}{4}(E_{\max} - E_{\min})^2$, where $E_{\max}$ and $E_{\min}$ are the maximal and minimal eigenvalues of H, respectively. Since $E_{\max}$ and $E_{\min}$ grow linearly with the number of systems N, $F_g^{(Q)} \propto N^2$, which is the origin of the $\sqrt{N}$ improvement of the precision scaling in quantum metrology compared with the SQL.

III. *N* SCALING OF QUANTUM FISHER INFORMATION

Now we turn to the major problem of this paper. We are interested in quantum metrology for a general parameter in a Hamiltonian. Both the eigenvalues and the eigenstates of the Hamiltonian may depend on the parameter. We mainly consider the scaling of the quantum Fisher information with the number of the systems N in this section, and leave the more general results for the next section.

We first introduce the general framework of how to derive the quantum Fisher information of estimating a Hamiltonian parameter. Suppose the Hamiltonian of a single system is H_g, the initial state of the system is ρ_0, and the parameter we want to estimate is g. After the evolution under the Hamiltonian, the state of the system becomes $\rho_g = U_g \rho_0 U_g^\dagger$, where $U_g = \exp(-itH_g)$. The sensitivity of ρ_g to the parameter g can be characterized by the generator of the local parameter translation from ρ_g to ρ_{g+dg}, where dg is an infinitesimal change of g.

In detail, when g is changed to $g + dg$, ρ_g is updated to $\rho_{g+dg} = U_{g+dg}\rho_0 U_{g+dg}^\dagger$. Since $U_{g+dg} \approx U_g + \partial_g U_g dg$, the translation from ρ_g to ρ_{g+dg} can be written as

$$\begin{aligned}\rho_{g+dg} &\approx (U_g + dg\partial_g U_g)\rho_0(U_g^\dagger + dg\partial_g U_g^\dagger)\\ &= [I + dg(\partial_g U_g)U_g^\dagger]U_g\rho_0 U_g^\dagger(I + dgU_g\partial_g U_g^\dagger)\\ &\approx \exp(-ih_g dg)\rho_g \exp(ih_g dg),\end{aligned} \tag{8}$$

where

$$h_g = i(\partial_g U_g)U_g^\dagger. \tag{9}$$

Here, h_g is the generator of parameter translation with respect to g, and the subscript g is to indicate that this generator is *local* in g. It can be shown [6,7] that when the initial state of the system is a pure state $|\Psi\rangle$, the quantum Fisher information of the evolved state $U_g|\Psi\rangle$ is

$$F_g^{(Q)} = 4\langle\Psi|\Delta h_g^2|\Psi\rangle. \tag{10}$$

And the variance of h_g is maximized when $|\Psi\rangle = \frac{1}{\sqrt{2}}[|\lambda_{\max}(h_g) + e^{i\varphi}|\lambda_{\min}(h_g)\rangle]$ ($e^{i\varphi}$ is an arbitrary phase) [9], so the maximal quantum information is

$$F_{g,\max}^{(Q)} = [\lambda_{\max}(h_g) - \lambda_{\min}(h_g)]^2, \tag{11}$$

where $\lambda_{\max}(h_g)$ and $\lambda_{\min}(h_g)$ are the maximal and minimal eigenvalues of h, respectively.

When there are N systems, the total Hamiltonian is $H_{g,\text{total}} = H_{g,1} + \cdots + H_{g,N}$, where $H_{g,i}$ is the Hamiltonian for the ith system alone, i.e., $H_{g,i} = I^{\otimes i-1} \otimes H_g \otimes I^{\otimes N-i}$. Since $[H_i, H_j] = 0, \forall i,j = 1,\ldots,N$, we have

$$\begin{aligned}h_{g,\text{total}} &= i\frac{\partial e^{-itH_{g,\text{total}}}}{\partial g}e^{itH_{g,\text{total}}}\\ &= h_{g,1} + \cdots + h_{g,N}.\end{aligned} \tag{12}$$

As $H_{g,1},\ldots,H_{g,N}$ are the same Hamiltonian on different systems, it is obvious that

$$\begin{aligned}\lambda_{\max}(h_{g,\text{total}}) &= N\lambda_{\max}(h_g),\\ \lambda_{\min}(h_{g,\text{total}}) &= N\lambda_{\min}(h_g).\end{aligned} \tag{13}$$

So according to (11),

$$\max F_{g,\text{total}}^{(Q)} = N^2 F_{g,\max}^{(Q)}, \tag{14}$$

where $F_{g,\text{total}}^{(Q)}$ is the total quantum Fisher information of the N systems.

Equation (14) is interesting since it implies that the optimal scaling of the total Fisher information using N systems can always reach N^2, which beats the classical scaling limit and is universal for estimating an arbitrary parameter in the Hamiltonian. Of course, if there are interactions among the N systems, the optimal scaling of the Fisher information may be even higher, which has been found for estimating an overall multiplicative factor of a Hamiltonian [11,13]. In that case, the total Hamiltonian becomes $H_{g,\text{total}} = \sum_{i_1,\ldots,i_k} H_{g,\langle i_1,\ldots,i_k\rangle}$ if there are k-body interactions among the N systems. Obviously, the total Hamiltonian can grow nonlinearly with N in general, so the quantum Fisher information may increase faster than N^2. Such a case is beyond the scope of this paper and we do not consider it in detail here.

IV. QUANTUM FISHER INFORMATION FOR GENERAL HAMILTONIAN PARAMETERS

In this section, we study quantum metrology for general Hamiltonian parameters in detail. We consider only single systems here and focus on the time scaling of the quantum Fisher information, since the scaling with the number of systems was treated in the previous section. It can be seen from Eq. (11) that the key to the quantum Fisher information $F_g^{(Q)}$ is the generator h_g (9) of the local parameter translation from U_g to U_{g+dg}, so our main effort is to derive h_g in the following.

A. Result for $t \ll 1$

First, we study the derivative of $\exp[-itH(g)]$ with respect to g which is needed in Eq. (9). This derivative is nontrivial, since H_g does not commute with $\partial_g H_g$ in general. To obtain this derivative, we start from an integral formula for the derivative of an operator exponential [54]:

$$\begin{aligned}&\frac{\partial \exp[-i\beta H(\lambda)]}{\partial\lambda}\\ &\quad= -i\int_0^\beta \exp[-i\mu H(\lambda)]\frac{\partial H(\lambda)}{\partial\lambda}\exp[(i\mu - i\beta)H(\lambda)]d\mu,\end{aligned} \tag{15}$$

where $\mu,\beta \in \mathbb{R}$. By this formula and according to the definition of h_g (9), we get

$$h_g = \int_0^t \exp(-i\mu H_g)\partial_g H_g \exp(i\mu H_g)d\mu. \tag{16}$$

When $t \ll 1$, the first-order approximation of (16) is

$$h_g \approx t\partial_g H_g.$$

As shown in Appendix B, we can get an upper bound for the quantum Fisher information in this case,

$$F_{g,\max}^{(Q)} \leqslant \frac{t^2}{2}\mathrm{Tr}(\partial_g H_g)^2. \tag{17}$$

B. Result for general *t*

For larger t, direct calculation of the integral in Eq. (16) is not easy. Of course, one can use the Baker-Campbell-Hausdorff formula to expand the integrand, but that will yield an infinite series that is difficult to treat. So we resort to a different approach to work out h_g, which was first proposed in Ref. [54].

Denote the integrand of (16) as $Y(\mu)$:

$$Y(\mu) = \exp(-i\mu H_g)\frac{\partial H_g}{\partial g}\exp(i\mu H_g). \tag{18}$$

The derivative of $Y(\mu)$ with respect to μ satisfies

$$\frac{\partial Y}{\partial \mu} = -i[H_g,Y], \tag{19}$$

and the initial condition is $Y(0) = \partial_g H_g$.

To solve the differential equation (19), consider the following eigenvalue equation:

$$[H_g,\Gamma] = \lambda\Gamma. \tag{20}$$

In this equation, H_g can be treated as a superoperator acting on Γ. To distinguish H_g as a superoperator from that as an operator, we denote the superoperator of H_g as $\mathcal{H}_g$, and (20) can be rewritten as

$$\mathcal{H}_g\Gamma = \lambda\Gamma. \tag{21}$$

It is easy to verify that $\mathcal{H}_g$ is an Hermitian superoperator (see Appendix A). Therefore, $\mathcal{H}_g$ has d^2 real eigenvalues, some of which may be degenerate. Suppose the eigenvalues of $\mathcal{H}_g$ are $\lambda_1,\ldots,\lambda_{d^2}$, and that $\lambda_k = 0$ for $k = 1,\ldots,r$ and $\lambda_k \neq 0$ for $k = r+1,\ldots,d^2$, and denote the corresponding orthonormal eigenvectors as $\Gamma_1,\ldots,\Gamma_{d^2}$, satisfying $\mathrm{Tr}(\Gamma_i^\dagger\Gamma_j) = \delta_{ij}$. Then, $\partial_g H_g$ can be decomposed as

$$\partial_g H_g = \sum_{k=1}^{d^2} c_k\Gamma_k, \tag{22}$$

where $c_k = \mathrm{Tr}(\Gamma_k^\dagger\partial_g H_g)$. Since $Y(\mu)$ can also be decomposed in terms of $\Gamma_1,\ldots,\Gamma_{d^2}$, and $Y(0) = \partial_g H_g$, the solution of Eq. (19) is

$$Y(\mu) = \sum_{k=1}^{d^2}\mathrm{Tr}(\Gamma_k^\dagger\partial_g H_g)e^{-i\lambda_k\mu}\Gamma_k. \tag{23}$$

Now, we can insert the above solution for $Y(\mu)$ into (16), and since the first r eigenvalues of $\mathcal{H}_g$ are zero,

$$\begin{aligned} h_g &= t\sum_{k=1}^{r}\mathrm{Tr}(\Gamma_k^\dagger\partial_g H_g)\Gamma_k \\ &\quad -i\sum_{k=r+1}^{d^2}\frac{1-e^{-i\lambda_k t}}{\lambda_k}\mathrm{Tr}(\Gamma_k^\dagger\partial_g H_g)\Gamma_k. \end{aligned} \tag{24}$$

Equation (24) is the general solution for h_g. When one obtains the eigenvalues and eigenvectors of $\mathcal{H}_g$ from (20) and plugs them into (24), h_g can then be derived.

If we know the eigenvalues and eigenstates of H_g (as an ordinary operator), the solution for h_g (24) can be greatly simplified. Suppose H_g has n_g different eigenvalues, $E_1,\ldots,E_{n_g}$, the degeneracy of E_k is d_k, and the eigenstates corresponding to E_k are $|E_k^{(1)}\rangle,\ldots,|E_k^{(d_k)}\rangle$. The eigenvectors and eigenvalues of $\mathcal{H}_g$ are

$$\Gamma_{kl}^{(ij)} = \big|E_k^{(i)}\big\rangle\big\langle E_l^{(j)}\big|, \quad \lambda_{kl}^{(ij)} = E_k - E_l. \tag{25}$$

It is obvious that the degeneracy of the zero eigenvalue is $d_1^2+\cdots+d_{n_g}^2$, and the corresponding eigenvectors are $\Gamma_{kk}^{(ij)}$, $i,j = 1,\ldots,d_k$, $k = 1,\ldots,n_g$. The coefficients of these eigenvectors in h_g are

$$\begin{aligned} \mathrm{Tr}\big(\Gamma_{kk}^{(ij)\dagger}\partial_g H_g\big) &= \big\langle E_k^{(j)}\big|\partial_g H_g\big|E_k^{(i)}\big\rangle \\ &= \partial_g E_k\delta_{ij}. \end{aligned} \tag{26}$$

The eigenvectors with nonzero eigenvalues of $\mathcal{H}_g$ are $\Gamma_{kl}^{(ij)}$, $k \neq l$, and their coefficients in h_g are

$$\begin{aligned} \mathrm{Tr}\big(\Gamma_{kl}^{(ij)\dagger}\partial_g H_g\big) &= \big\langle E_l^{(j)}\big|\partial_g H_g\big|E_k^{(i)}\big\rangle \\ &= E_k\big\langle E_l^{(j)}\big|\partial_g E_k^{(i)}\big\rangle + E_l\big\langle\partial_g E_l^{(j)}\big|E_k^{(i)}\big\rangle \\ &= (E_k - E_l)\big\langle E_l^{(j)}\big|\partial_g E_k^{(i)}\big\rangle, \end{aligned} \tag{27}$$

where we have used $\langle E_l^{(j)}|\partial_g E_k^{(i)}\rangle + \langle\partial_g E_l^{(j)}|E_k^{(i)}\rangle = \partial_g\langle E_l^{(j)}|E_k^{(i)}\rangle = 0$.

By plugging (25)–(27) into (24), we finally have

$$\begin{aligned} h_g &= t\sum_{k=1}^{n_g}\frac{\partial E_k}{\partial g}P_k + 2\sum_{k\neq l}\sum_{i=1}^{d_k}\sum_{j=1}^{d_l}e^{\frac{-i(E_k-E_l)t}{2}} \\ &\quad\times\sin\frac{(E_k-E_l)t}{2}\big\langle E_l^{(j)}\big|\partial_g E_k^{(i)}\big\rangle\big|E_k^{(i)}\big\rangle\big\langle E_l^{(j)}\big|, \end{aligned} \tag{28}$$

where P_k is the projection onto the eigensubspace corresponding to E_k: $P_k = \sum_{i=1}^{d_k}|E_k^{(i)}\rangle\langle E_k^{(i)}|$. We have used $1 - e^{-i(E_k-E_l)t} = 2i\exp\frac{-i(E_k-E_l)t}{2}\sin\frac{(E_k-E_l)t}{2}$.

The form of h_g in Eq. (28) implies that the quantum Fisher information $F_g^{(Q)}$ can be divided into two parts: one is due to the dependence of the eigenvalues E_k on g, and this part is linear in the time t; the other is due to the dependence of the eigenstates $|E_k^{(i)}\rangle$ on g, and that part oscillates with time.

When the dimension of the system is low, one may find the eigenvalues and the eigenstates of the Hamiltonian explicitly, so Eq. (28) is a more direct and compact result for h_g. However, if the dimension of the system is very high, e.g., a condensed-matter system, then the eigenvalues and the eigenstates will be extremely difficult to obtain, and the general result (24) will be more helpful. In this case, the eigenvalues and eigenstates of $\mathcal{H}_g$ are still unavailable, but one can get some knowledge of the quantum Fisher information $F_g^{(Q)}$ from the symmetry of the Hamiltonian.

For example, if H is invariant under a unitary operation $U = \exp(-i\Omega)$, then $[H_g,\Omega] = 0$, which implies that Ω is an eigenvector of $\mathcal{H}_g$ with eigenvalue zero. Thus one can calculate the coefficient $\mathrm{Tr}(\Omega\partial_g H_g)$ and check whether Ω

belongs to the support of $\partial_g H_g$. If it does, then the quantum Fisher information $F_g^{(Q)}$ will scale as t^2 when $t \gg 1$. So we can see that even lacking details about the eigenvalues and eigenvectors of H_g, (24) can give some information about the scaling of $F_g^{(Q)}$ through the symmetry of H_g.

C. Upper bound on the quantum Fisher information $F_g^{(Q)}$

From (24) or (28), we can obtain an upper bound on the quantum Fisher information $F_g^{(Q)}$.

First, we note that

$$\langle \Delta h_g^2 \rangle_{\max} \leqslant \tfrac{1}{2}\mathrm{Tr}(h_g^\dagger h_g) \tag{29}$$

(see Appendix B for a proof), so from (10) and (24), we can derive

$$\begin{aligned} F^{(Q)}_{g,\max} \leqslant{}& 2t^2 \sum_{k=1}^{r} |\mathrm{Tr}(\Gamma_k^\dagger \partial_g H_g)|^2 \\ &+ 8 \sum_{k=r+1}^{d^2} \frac{|\mathrm{Tr}(\Gamma_k^\dagger \partial_g H_g)|^2}{\lambda_k^2} \sin^2 \frac{\lambda_k t}{2}. \end{aligned} \tag{30}$$

And when we know the eigenvalues and eigenstates of the Hamiltonian H_g, the upper bound can be simplified to

$$\begin{aligned} F^{(Q)}_{g,\max} \leqslant{}& 2t^2 \sum_{k=1}^{n_g} d_k (\partial_g E_k)^2 \\ &+ 8 \sum_{k\neq l} \sum_{i=1}^{d_k} \sum_{j=1}^{d_l} \left| \sin \frac{1}{2}(E_k - E_l)t \right|^2 \left| \langle E_l^{(j)} | \partial_g E_k^{(i)} \rangle \right|^2. \end{aligned} \tag{31}$$

In particular, if the eigenvalues of H_g are independent of g, the upper bound of $F_g^{(Q)}$ will not grow as t^2 when t is large, and the bound becomes

$$\begin{aligned} F^{(Q)}_{g,\max} &\leqslant 8 \sum_{k\neq l} \sum_{i=1}^{d_k} \sum_{j=1}^{d_l} \left| \sin \frac{1}{2}(E_k - E_l)t \right|^2 \left| \langle E_l^{(j)} | \partial_g E_k^{(i)} \rangle \right|^2 \\ &\leqslant 8 \sum_{k\neq l} \sum_{i=1}^{d_k} \sum_{j=1}^{d_l} \left| \langle E_l^{(j)} | \partial_g E_k^{(i)} \rangle \right|^2. \end{aligned} \tag{32}$$

In this case, the quantum Fisher information $F_g^{(Q)}$ is always finite, no matter how long the time t is, in sharp contrast to the time scaling of the Fisher information for estimating an overall multiplicative factor of a Hamiltonian.

V. EXAMPLE: A SPIN-$\frac{1}{2}$ PARTICLE IN A MAGNETIC FIELD

In this section, we consider an example to illustrate the results in the previous sections. We study the quantum Fisher information in estimating a parameter of a magnetic field by measuring a spin-$\frac{1}{2}$ particle in the field.

Suppose the magnetic field is $B\overrightarrow{n_\theta}$, where B is the amplitude of the magnetic field and $\overrightarrow{n_\theta} = (\cos\theta, 0, \sin\theta)$, gives its direction. The parameter θ denotes the angle between the direction of the magnetic field and the z axis. Now we place a spin-$\frac{1}{2}$ particle, e.g., an electron, in this magnetic field and our task is to estimate the angle θ by measuring this particle.

The interaction Hamiltonian between the the particle and the magnetic field is

$$H_\theta = B(\cos\theta\sigma_x + \sin\theta\sigma_z), \tag{33}$$

where σ_x and σ_z are Pauli operators. We have assumed $e = m = c = 1$ in the above Hamiltonian for simplicity.

The eigenvalues of H_θ are $\pm B$, and the corresponding eigenstates are

$$|{+B}\rangle = \begin{pmatrix} \cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right) \\ \sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right) \end{pmatrix}, \quad |{-B}\rangle = \begin{pmatrix} \sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right) \\ -\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right) \end{pmatrix}. \tag{34}$$

According to (28), the generator h of the local translation with respect to the parameter θ for an evolution of time t is

$$h = B \begin{pmatrix} 0 & e^{-iBt}\sin Bt \\ e^{iBt}\sin Bt & 0 \end{pmatrix}. \tag{35}$$

The eigenvalues of h are $\pm B \sin Bt$, so the maximum quantum Fisher information is

$$F^{(Q)}_{\max} = 4B^2 \sin^2 Bt. \tag{36}$$

We can also extend this result to a more general case. Suppose the direction of the magnetic field $\overrightarrow{n_\theta}$ has an arbitrary form with $\|\overrightarrow{n_\theta}\| = 1$, then the Hamiltonian of the interaction between the particle and the magnetic field is

$$H_\theta = B\overrightarrow{n_\theta} \cdot \overrightarrow{\sigma}, \tag{37}$$

where $\overrightarrow{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of the Pauli operators.

We can obtain h from (28),

$$h = B \sin Bt\,(\cos Bt\,\overrightarrow{\partial_\theta n_\theta} + i \sin Bt\,\overrightarrow{\partial_\theta n_\theta} \times \overrightarrow{n_\theta}) \cdot \overrightarrow{\sigma}. \tag{38}$$

Since $\overrightarrow{\partial_\theta n_\theta} \times \overrightarrow{n_\theta}$ is orthogonal to $\overrightarrow{\partial_\theta n_\theta}$, and $\|\overrightarrow{\partial_\theta n_\theta} \times \overrightarrow{n_\theta}\| = \|\overrightarrow{\partial_\theta n_\theta}\|$, the eigenvalues of h are

$$\pm B \|\overrightarrow{\partial_\theta n_\theta}\| \sin Bt. \tag{39}$$

Therefore, the maximum quantum Fisher information of estimating θ is

$$F^{(Q)}_{\max} = 4B^2 \|\overrightarrow{\partial_\theta n_\theta}\|^2 \sin^2 Bt. \tag{40}$$

From (36) and (40), we can see that the maximum quantum Fisher information oscillates with the time t, and the period of the oscillation is $\frac{\pi}{B}$. This implies that the maximum quantum Fisher information is always bounded in this case, and the upper bound is $4B^2\|\overrightarrow{\partial_\theta n_\theta}\|^2$. This is in sharp contrast to the case where the parameter to estimate is an overall multiplicative factor of the Hamiltonian (compare to the amplitude case below). In that case, the maximum quantum Fisher information grows as t^2, and is unbounded as $t \to \infty$.

By way of comparison, if instead we want to estimate a parameter in the amplitude B_g of the magnetic field, where g is the parameter to estimate, and the direction of the magnetic field $\overrightarrow{n}$ is fixed, then

$$h = \partial_g B_g \overrightarrow{n} \cdot \overrightarrow{\sigma}. \tag{41}$$

In this case, the maximum quantum Fisher information is

$$F^{(Q)}_{\max} = 4(\partial_g B_g)^2 t^2, \tag{42}$$

which recovers the time scaling t^2, which is known in quantum metrology for phase estimation.

The maximum quantum Fisher information (40) for estimating θ has an intuitive physical picture. The derivative $\overrightarrow{\partial_\theta n_\theta}$ characterizes how fast the direction $\overrightarrow{n_\theta}$ changes with the parameter θ. If $\overrightarrow{n_\theta}$ changes quickly with the parameter θ, it will be more sensitive to distinguish different θ, so the precision of estimating θ will be higher.

VI. CONCLUSION

In summary, in this paper we studied quantum metrology for estimating a general parameter of a Hamiltonian. We obtained the generator h_g of the infinitesimal parameter translation with respect to g, of which the variance is the quantum Fisher information, and also a general upper bound on the quantum Fisher information. The results show that the optimal scaling of the quantum Fisher information with the number of systems can always reach the Heisenberg limit, but the time scaling can be different from that of estimating an overall multiplicative factor. We considered estimating a parameter of a magnetic field by measuring a spin-$\frac{1}{2}$ particle as an example to illustrate the results, and compared estimating a parameter of the magnetic field amplitude to estimating a parameter of the magnetic field direction. When estimating a parameter of the magnetic field amplitude, the time scaling of the quantum Fisher information is t^2, but when estimating the parameter of the magnetic field direction, the quantum Fisher information oscillates as a sine function of t. This example clearly shows the difference between estimating an overall multiplicative factor and estimating a general parameter, and gives a physical picture illustrating the general results.

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APPENDIX A: PROOF OF THE HERMICITY OF $\mathcal{H}_g$

Suppose $\{\sigma_1,\ldots,\sigma_{d^2}\}$ is an orthonormal basis in the operator space; then the (i,j)th element of the superoperator $\mathcal{H}_g$ is

$$(\mathcal{H}_g)_{ij} = \mathrm{Tr}(\sigma_i^\dagger[H_g,\sigma_j]), \tag{A1}$$

and

$$(\mathcal{H}_g)^\dagger_{ij} = \mathrm{Tr}([\sigma_i^\dagger,H_g]\sigma_j). \tag{A2}$$

If $\mathcal{H}_g$ is Hermitian, it must satisfy $(\mathcal{H}_g)_{ij} = (\mathcal{H}_g)^\dagger_{ij}$. We can check whether this is true directly from (A1) and (A2). Note that

$$\begin{aligned}(\mathcal{H}_g)_{ij} - (\mathcal{H}_g)^\dagger_{ij} &= \mathrm{Tr}(\sigma_i^\dagger[H_g,\sigma_j]) + \mathrm{Tr}([H_g,\sigma_i^\dagger]\sigma_j)\\ &= \mathrm{Tr}([H_g,\sigma_i^\dagger\sigma_j])\\ &= 0,\end{aligned} \tag{A3}$$

so this proves the Hermicity of $\mathcal{H}_g$.

APPENDIX B: PROOF OF EQUATION (29)

First, we note that [9]

$$\langle \Delta h_g^2\rangle_{\max} = \tfrac{1}{4}(\lambda_{\max} - \lambda_{\min})^2, \tag{B1}$$

where $\lambda_{\max}$ and $\lambda_{\min}$ are the maximum and minimum eigenvalues of h, respectively.

On one hand, $|\lambda_{\max} - \lambda_{\min}| \leqslant |\lambda_{\max}| + |\lambda_{\min}|$, so

$$\begin{aligned}\langle \Delta h_g^2\rangle_{\max} &\leqslant \left(\frac{|\lambda_{\max}| + |\lambda_{\min}|}{2}\right)^2\\ &\leqslant \frac{|\lambda_{\max}|^2 + |\lambda_{\min}|^2}{2},\end{aligned} \tag{B2}$$

where the second inequality follows from the well-known *power mean inequality*: for any real positive numbers $x_1,\ldots,x_n$ and nonzero p, q,

$$\left(\frac{x_1^q + \cdots + x_n^q}{n}\right)^{\frac{1}{q}} \leqslant \left(\frac{x_1^p + \cdots + x_n^p}{n}\right)^{\frac{1}{p}} \quad \text{if } p \geqslant q. \tag{B3}$$

If we take $q = 1$ and $p = 2$, it will produce (B2).

On the other hand,

$$\mathrm{Tr}(h_g^\dagger h_g) = \sum_k |\lambda_k|^2 \geqslant |\lambda_{\max}|^2 + |\lambda_{\min}|^2,$$

where λ_k runs over all eigenvalues of h_g, so we have

$$\langle \Delta h_g^2\rangle_{\max} \leqslant \tfrac{1}{2}\mathrm{Tr}(h_g^\dagger h_g), \tag{B4}$$

which proves Eq. (29).

[1] V. Giovannetti, S. Lloyd, and L. Maccone, Science **306**, 1330 (2004).

[2] V. Giovannetti, S. Lloyd, and L. Maccone, Nat. Photon. **5**, 222 (2011).

[3] V. B. Braginsky and Y. I. Vorontsov, Sov. Phys. Usp. **17**, 644 (1975).

[4] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic, New York, 1976), Chap. VIII.4.

[5] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (North-Holland, Amsterdam, 1982).

[6] S. L. Braunstein and C. M. Caves, Phys. Rev. Lett. **72**, 3439 (1994).

[7] S. L. Braunstein, C. M. Caves, and G. J. Milburn, Ann. Phys. **247**, 135 (1996).

[8] H. Cramer, *Mathematical Methods of Statistics* (Princeton University, Princeton, NJ, 1946), p. 500.

[9] V. Giovannetti, S. Lloyd, and L. Maccone, Phys. Rev. Lett. **96**, 010401 (2006).

[10] M. Zwierz, C. A. Pérez-Delgado, and P. Kok, Phys. Rev. Lett. **105**, 180402 (2010).

[11] S. Boixo, S. T. Flammia, C. M. Caves, and J. M. Geremia, Phys. Rev. Lett. **98**, 090401 (2007).

[12] S. Choi and B. Sundaram, Phys. Rev. A **77**, 053613 (2008).

[13] S. M. Roy and S. L. Braunstein, Phys. Rev. Lett. **100**, 220501 (2008).

[14] S. Boixo, A. Datta, S. T. Flammia, A. Shaji, E. Bagan, and C. M. Caves, Phys. Rev. A **77**, 012317 (2008)

[15] M. Napolitano and M. W. Mitchell, New J. Phys. **12**, 093016 (2010).

[16] M. Napolitano, M. Koschorreck, B. Dubost, N. Behbood, R. J. Sewell, and M. W. Mitchell, Nature (London) **471**, 486 (2011).

[17] S. F. Huelga, C. Macchiavello, T. Pellizzari, A. K. Ekert, M. B. Plenio, and J. I. Cirac, Phys. Rev. Lett. **79**, 3865 (1997).

[18] J. J. Bollinger, Wayne M. Itano, D. J. Wineland, and D. J. Heinzen, Phys. Rev. A **54**, R4649 (1996).

[19] J. P. Dowling, Phys. Rev. A **57**, 4736 (1998).

[20] U. Dorner, R. Demkowicz-Dobrzanski, B. J. Smith, J. S. Lundeen, W. Wasilewski, K. Banaszek, and I. A. Walmsley, Phys. Rev. Lett. **102**, 040403 (2009).

[21] P. M. Anisimov, G. M. Raterman, A. Chiruvelli, W. N. Plick, S. D. Huver, H. Lee, and J. P. Dowling, Phys. Rev. Lett. **104**, 103602 (2010).

[22] J. Joo, W. J. Munro, and T. P. Spiller, Phys. Rev. Lett. **107**, 083601 (2011).

[23] M. G. Genoni, S. Olivares, and M. G. A. Paris, Phys. Rev. Lett. **106**, 153603 (2011).

[24] N. Thomas-Peter, B. J. Smith, A. Datta, L. Zhang, U. Dorner, and I. A. Walmsley, Phys. Rev. Lett. **107**, 113603 (2011).

[25] H. Yonezawa *et al.*, Science **337**, 1514 (2012).

[26] V. Bužek, R. Derka, and S. Massar, Phys. Rev. Lett. **82**, 2207 (1999).

[27] A. André, A. S. Sørensen, and M. D. Lukin, Phys. Rev. Lett. **92**, 230801 (2004).

[28] A. Louchet-Chauvet, J. Appel, J. J. Renema, D. Oblak, N. Kjaergaard, and E. S. Polzik, New J. Phys. **12**, 065032 (2010).

[29] J. Borregaard and A. S. Sørensen, Phys. Rev. Lett. **111**, 090801 (2013).

[30] E. M. Kessler, P. Kómár, M. Bishof, L. Jiang, A. S. Sørensen, J. Ye and M. D. Lukin, Phys. Rev. Lett. **112**, 190403 (2014).

[31] J. Jacobson, G. Bjork, and Y. Yamamoto, Appl. Phys. B **60**, 187 (1995).

[32] L. A. Lugiato, A. Gatti, and E. Brambilla, J. Opt. B: Quantum Semiclass. Opt. **4**, S176 (2002).

[33] C. A. Pérez-Delgado, M. E. Pearce, and P. Kok, Phys. Rev. Lett. **109**, 123601 (2012).

[34] V. Giovannetti, S. Lloyd, and L. Maccone, Nature (London) **412**, 417 (2001).

[35] R. Demkowicz-Dobrzański, J. Kołodyński and M. Guţă, Nat. Commun. **3**, 1063 (2012).

[36] A. W. Chin, S. F. Huelga, and M. B. Plenio, Phys. Rev. Lett. **109**, 233601 (2012)

[37] A. del Campo, I. L. Egusquiza, M. B. Plenio, and S. F. Huelga, Phys. Rev. Lett. **110**, 050403 (2013).

[38] M. Tsang, New J. Phys. **15**, 073005 (2013).

[39] J. Kołodyński and R. Demkowicz-Dobrzański, New J. Phys. **15**, 073043 (2013).

[40] S. Alipour, M. Mehboudi, and A. T. Rezakhani, Phys. Rev. Lett. **112**, 120405 (2014).

[41] S. Alipour, arXiv:1403.8033.

[42] W. Dür, M. Skotiniotis, F. Fröwis, and B. Kraus, Phys. Rev. Lett. **112**, 080801 (2014).

[43] E. M. Kessler, I. Lovchinsky, A. O. Sushkov, and M. D. Lukin, Phys. Rev. Lett. **112**, 150802 (2014).

[44] X.-M. Lu, S. Yu, and C. H. Oh, arXiv:1405.4052.

[45] A. Luis, Phys. Lett. A **329**, 8 (2004).

[46] A. Luis, Phys. Rev. A **76**, 035801 (2007).

[47] M. J. Woolley, G. J. Milburn, and C. M. Caves, New J. Phys. **10**, 125018 (2008).

[48] S. Boixo, A. Datta, M. J. Davis, S. T. Flammia, A. Shaji, and C. M. Caves, Phys. Rev. Lett. **101**, 040403 (2008).

[49] Á. Rivas and A. Luis, Phys. Rev. Lett. **105**, 010403 (2010).

[50] T. Tilma, S. Hamaji, W. J. Munro, and Kae Nemoto, Phys. Rev. A **81**, 022108 (2010).

[51] J. Joo, K. Park, H. Jeong, W. J. Munro, K. Nemoto, and T. P. Spiller, Phys. Rev. A **86**, 043828 (2012).

[52] M. J. W. Hall and H. M. Wiseman, Phys. Rev. X **2**, 041006 (2012).

[53] D. C. Brody and E.-M. Graefe, Entropy **15**, 3361 (2013).

[54] R. M. Wilcox, J. Math. Phys. **8**, 962 (1967).