

Systems of coupled \mathcal{PT} -symmetric oscillators

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The Hamiltonian for a \mathcal{PT} -symmetric chain of coupled oscillators is constructed. It is shown that if the loss-gain parameter γ is uniform for all oscillators, then as the number of oscillators increases, the region of unbroken \mathcal{PT} symmetry disappears entirely. However, if γ is localized in the sense that it decreases for more distant oscillators, then the unbroken \mathcal{PT} -symmetric region persists even as the number of oscillators approaches infinity. In the continuum limit the oscillator system is described by a \mathcal{PT} -symmetric pair of wave equations, and a localized loss-gain impurity leads to a pseudobound state. It is also shown that a planar configuration of coupled oscillators can have multiple disconnected regions of unbroken \mathcal{PT} symmetry.

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I. INTRODUCTION

A previous paper [1] considered a system consisting of a pair of coupled oscillators, one with loss and the other with gain. Such a system is \mathcal{PT} symmetric if the loss and gain parameters are equal. The energy of this \mathcal{PT} -symmetric system is exactly conserved because this system is described by a time-independent Hamiltonian. In the current paper we examine the systems that arise when the number of pairs of coupled oscillators is extended from 1 to N , where N can be arbitrarily large.

Let us review the case $N = 1$. A single pair of coupled oscillators, the first with loss and the second with gain, is described by the equations of motion

$$\ddot{x} + \omega^2 x + \mu \dot{x} = -\epsilon y, \quad \ddot{y} + \omega^2 y - \nu \dot{y} = -\epsilon x. \quad (1)$$

To treat this system at a classical level, we seek solutions to (1) of the form $e^{i\lambda t}$. The classical frequency λ then satisfies the quartic polynomial equation

$$\lambda^4 - i(\mu - \nu)\lambda^3 - (2\omega^2 - \mu\nu)\lambda^2 + i\omega^2(\mu - \nu)\lambda - \epsilon^2 + \omega^4 = 0. \quad (2)$$

This classical system becomes \mathcal{PT} symmetric if the loss and gain are balanced; that is, if we set $\mu = \nu = 2\gamma$. In this case the frequencies λ are given by

$$\lambda^2 = \omega^2 - 2\gamma^2 \pm \sqrt{\epsilon^2 - 4\gamma^2\omega^2 + 4\gamma^4}. \quad (3)$$

Note that there are four real frequencies when ϵ is in the range

$$\epsilon_1 = 2\gamma\sqrt{\omega^2 - \gamma^2} < \epsilon < \epsilon_2 = \omega^2. \quad (4)$$

This defines the *unbroken* \mathcal{PT} -symmetric region. In the broken \mathcal{PT} -symmetric region $\epsilon < \epsilon_1$ there are two pairs of complex-conjugate frequencies and in the broken \mathcal{PT} -symmetric region $\epsilon > \epsilon_2$ there are two real frequencies and one complex-conjugate pair of frequencies.

When $\mu \neq \nu$, system (1) cannot be derived from a quadratic *time-independent* Hamiltonian. To verify this, one can construct the most general quadratic Hamiltonian for the four variables p , q , x , and y with arbitrary constant coefficients. It is then a simple algebraic calculation to show that there is no choice of coefficients for which one obtains a Hamiltonian that leads to Eqs. (1). However, when the system is \mathcal{PT} symmetric ($\mu = \nu = 2\gamma$), (1) can be derived from the two-coupled-oscillator time-independent Hamiltonian

$$H_2 = pq + \gamma(yq - xp) + (\omega^2 - \gamma^2)xy + \frac{1}{2}\epsilon(x^2 + y^2). \quad (5)$$

This Hamiltonian is \mathcal{PT} symmetric because under parity reflection \mathcal{P} the loss and gain oscillators are interchanged [2],

$$\mathcal{P}: x \rightarrow -y, \quad y \rightarrow -x, \quad p \rightarrow -q, \quad q \rightarrow -p, \quad (6)$$

and under time reversal \mathcal{T} the signs of the momenta are reversed,

$$\mathcal{T}: x \rightarrow x, \quad y \rightarrow y, \quad p \rightarrow -p, \quad q \rightarrow -q. \quad (7)$$

The Hamiltonian H_2 is \mathcal{PT} symmetric but it is not invariant under \mathcal{P} or \mathcal{T} separately [3]. Because the balanced loss-gain system is described by a time-independent Hamiltonian, the energy (that is, the value of H_2) is conserved. However, the total energy (5) is not the usual sum of the kinetic and potential energies (such as $p^2 + q^2 + x^2 + y^2$).

If we set the coupling parameter ϵ to 0, H_2 describes the system studied by Bateman [4]. Bateman showed that an equation of motion having a friction term linear in velocity could be derived from a variational principle. To do this he introduced a time-reversed companion of the original damped harmonic oscillator. This auxiliary oscillator acts as an energy reservoir and can be viewed as a thermal bath. The classical Hamiltonian for the Bateman system was constructed by Morse and Feshbach [5] and the corresponding quantum theory was analyzed by many authors, including Bopp [6], Feshbach and Tikochinsky [7], Tikochinsky [8], Dekker [9], Celeghini, Rasetti, and Vitiello [10], Banerjee and Mukherjee [11], and Chruściński and Jurkowski [12].

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Only the *noninteracting* ($\epsilon = 0$) case was considered in these references.

The noteworthy feature of \mathcal{PT} -symmetric systems is that they exhibit transitions; the classical system described by H_2 exhibits two transitions. The first occurs at $\epsilon = \epsilon_1$. If $\epsilon < \epsilon_1$, the energy flowing into the y resonator cannot transfer fast enough to the x resonator, where energy is flowing out, so the system cannot be in equilibrium. However, when $\epsilon > \epsilon_1$, the energy flowing into the y resonator transfers to the x resonator and the entire system is in equilibrium. The frequencies of a classical system in equilibrium are real and the system exhibits *Rabi oscillations* (power oscillations between the two resonators) in which the two oscillators are 90° out of phase. Complex frequencies indicate exponential growth and decay and are a signal that the system is not in equilibrium. A second transition occurs at $\epsilon = \epsilon_2$; when $\epsilon > \epsilon_2$, the classical system is no longer in equilibrium. This transition is difficult to see in classical experiments because in the strong-coupling regime the loss and gain components would have to be so close that they would interfere with one another. For example, in the pendulum experiment in Ref. [13] the pendula would be so close that they could no longer swing freely, and in the optical resonator experiment in Ref. [14] the solid-state resonators would be damaged. This strong-coupling region is discussed for the case of coupled systems *without* loss and gain in Ref. [15], where it is called the *ultrastrong-coupling regime*.

In Ref. [1] it is shown that the classical and the quantum systems described by H_2 exhibit transitions at the same two values of the coupling parameter ϵ . When $\epsilon < \epsilon_1$ and when $\epsilon > \epsilon_2$ the quantum energies are complex, but in the unbroken \mathcal{PT} -symmetric region $\epsilon_1 < \epsilon < \epsilon_2$ the quantum energies are real.

This paper is organized as follows. In Sec. II we formulate the equations of motion for a linear chain of N identical pairs of \mathcal{PT} -symmetric loss-gain oscillators and we construct the Hamiltonians H_{2N} for such systems. We show that there are two ways to represent such Hamiltonians: one that we call a *sum* representation and another that we call a *product* representation. In the product representation it is easy to see that the Hamiltonian is not unique and that this nonuniqueness takes the form of a gauge invariance. Next, in Sec. III we construct the Hamiltonians for a general \mathcal{PT} -symmetric system of $2N$ coupled oscillators in which the coupling parameter ϵ and the loss-gain parameter γ are allowed to vary from oscillator to oscillator. In addition, we consider a system of $2N + 1$ coupled \mathcal{PT} -symmetric oscillators, where \mathcal{PT} symmetry requires that the central oscillator have neither loss nor gain. We also perform the $N \rightarrow \infty$ limit of H_{2N} . In this limit the equations of motion of the oscillators become coupled linear wave equations with balanced loss and gain.

In Sec. IV we ask whether a \mathcal{PT} -symmetric chain of $2N$ coupled oscillators can have an unbroken \mathcal{PT} -symmetric region. We show that as N increases, if γ and ϵ are the same for all oscillators, the region of unbroken \mathcal{PT} symmetry shrinks and then disappears entirely as $N \rightarrow \infty$. However, if the loss-gain parameter γ decreases to 0 for distant oscillators, then such systems always have an unbroken \mathcal{PT} -symmetric region for intermediate values of the coupling parameter ϵ surrounded by broken \mathcal{PT} -symmetric regions for small and

large values of ϵ . Specifically, for the cases in which γ_n decreases like $1/n$ or $1/n^2$, where $1 \leq n \leq N$ is the number of oscillators measured from the center of the system, we show that an unbroken \mathcal{PT} -symmetric region persists in the limit as $N \rightarrow \infty$. If one views loss-gain as the consequence of an impurity, then a configuration of oscillators for which γ decreases with increasing distance from the center can be seen as having a localized impurity. Thus, in Sec. V we investigate a special case for the continuum model in which there is a pointlike \mathcal{PT} -symmetric impurity localized at the origin. We find that this impurity gives rise to a pseudobound-state solution. In Sec. VI we consider the simplest case of a two-dimensional array of coupled oscillators, namely, three oscillators—one with loss, one with gain, and the third with neither loss nor gain. This system is interesting because it can exhibit *five* distinct regions as a function of the coupling constant, two having unbroken \mathcal{PT} symmetry and three having broken \mathcal{PT} symmetry. Finally, in Sec. VII we make some brief concluding remarks.

II. \mathcal{PT} -SYMMETRIC SYSTEM OF COUPLED CLASSICAL OSCILLATORS

In this section we describe the properties of a \mathcal{PT} -symmetric one-dimensional chain of $2N$ coupled oscillators with alternating loss and gain. We begin by making the simplifying assumptions that the natural frequency ω , the coupling to adjacent oscillators ϵ , and the loss-gain parameter γ are the same for all oscillators. The classical coordinates are $x_k(t)$ ($1 \leq k \leq 2N$) and the equations of motion are

$$\begin{aligned} \ddot{x}_1 + \omega^2 x_1 + 2\gamma \dot{x}_1 &= -\epsilon x_2, \\ \ddot{x}_2 + \omega^2 x_2 - 2\gamma \dot{x}_2 &= -\epsilon x_1 - \epsilon x_3, \\ \ddot{x}_3 + \omega^2 x_3 + 2\gamma \dot{x}_3 &= -\epsilon x_2 - \epsilon x_4, \\ \ddot{x}_4 + \omega^2 x_4 - 2\gamma \dot{x}_4 &= -\epsilon x_3 - \epsilon x_5, \\ &\dots \\ \ddot{x}_{2N} + \omega^2 x_{2N} - 2\gamma \dot{x}_{2N} &= -\epsilon x_{2N-1}. \end{aligned} \quad (8)$$

These equations of motion are \mathcal{PT} symmetric, where the definitions of \mathcal{P} and \mathcal{T} are generalized from (6) and (7) to

$$\begin{aligned} \mathcal{P}: x_k &\rightarrow -x_{2N-k+1}, & p_k &\rightarrow -p_{2N-k+1} \quad (1 \leq k \leq 2N), \\ \mathcal{T}: x_k &\rightarrow x_k, & p_k &\rightarrow -p_k \quad (1 \leq k \leq 2N). \end{aligned} \quad (9)$$

The equations of motion (8) imply that there is a conserved quantity. To construct this constant of the motion we multiply the first equation by \dot{x}_2 , the second equation by $\dot{x}_1 + \dot{x}_3$, the third equation by $\dot{x}_2 + \dot{x}_4$, the fourth equation by $\dot{x}_3 + \dot{x}_5$, and so on. If we add the resulting equations, γ drops out entirely and we obtain a time-independent quantity, which we can identify as the energy E_{2N} of the system:

$$\begin{aligned} E_{2N} &= \sum_{j=1}^{2N-1} (\dot{x}_j \dot{x}_{j+1} + \omega^2 x_j x_{j+1}) \\ &+ \frac{\epsilon}{2} (x_1^2 + x_{2N}^2) + \epsilon \sum_{j=2}^{2N-1} x_j^2 + \epsilon \sum_{j=1}^{2N-2} x_j x_{j+2}. \end{aligned} \quad (10)$$

The existence of a conserved quantity suggests that (8) is a Hamiltonian system, and indeed one can find a Hamiltonian from which these equations of motion can be derived. There are two ways to express the (nonunique) Hamiltonian that gives rise to (8); we can use what we call a *sum* or a *product* representation. We describe these two structures below.

A. Sum representation of the Hamiltonian

In the sum representation H_{2N} consists of four terms. First, there is a pure momentum term of the form

$$\begin{aligned}
 & [x_1x_2 + x_3x_4 + x_5x_6 + x_7x_8 + \dots + x_{2N-7}x_{2N-6} + x_{2N-5}x_{2N-4} + x_{2N-3}x_{2N-2} + x_{2N-1}x_{2N} \\
 & - x_1x_4 - x_3x_6 - x_5x_8 - \dots - x_{2N-7}x_{2N-4} - x_{2N-5}x_{2N-2} - x_{2N-3}x_{2N} \\
 & + x_1x_6 + x_3x_8 + \dots + x_{2N-7}x_{2N-2} + x_{2N-5}x_{2N} \\
 & - x_1x_8 - \dots - x_{2N-7}x_{2N} \\
 & \dots \\
 & (-1)^{N+1}x_1x_{2N}](\omega^2 - \gamma^2).
 \end{aligned} \tag{11}$$

Note the interesting structure of this term: The jumps in the products in (11) skip 0, 2, 4, 6, ... and change sign. A compact expression for H_{2N} is

$$\begin{aligned}
 H_{2N} = & \sum_{j=1}^{2N-1} p_j p_{j+1} + \frac{\epsilon}{2} \sum_{j=1}^{2N} x_j^2 + \gamma \sum_{j=1}^{2N} (-1)^j x_j p_j \\
 & + (\omega^2 - \gamma^2) \sum_{j=0}^{N-1} (-1)^j \sum_{k=1}^{N-j} x_{2k-1} x_{2j+2k}. \tag{12}
 \end{aligned}$$

To obtain the equations of motion (8) for this Hamiltonian from Hamilton's equations, we take one derivative of H_{2N} with respect to p_k to find \dot{x}_k :

$$\dot{x}_k = p_{k+1} + p_{k-1} + (-1)^k \gamma x_k. \tag{13}$$

We then take a time derivative,

$$\begin{aligned}
 \ddot{x}_k - (-1)^k \gamma \dot{x}_k = & -\frac{\partial H_{2N}}{\partial x_{k+1}} - \frac{\partial H_{2N}}{\partial x_{k-1}} \\
 = & -\epsilon x_{k+1} - \epsilon x_{k-1} + (-1)^k \gamma (p_{k+1} + p_{k-1}) \\
 & + (\omega^2 - \gamma^2)(\dots), \tag{14}
 \end{aligned}$$

and use the one-derivative equation (13) to recover the equations of motion (8).

B. Product representation of the Hamiltonian

In this representation it is easy to understand the nonuniqueness of the Hamiltonian that gives rise to the equations of motion (8). This nonuniqueness is a gauge invariance, where γ plays the role of an electric charge. Without changing the equations of motion we rewrite the sum representation H_2 in (5) so that the momentum terms appear in factored form:

$$H_2 = (p + \gamma y)(q - \gamma x) + \omega^2 xy + \epsilon(x^2 + y^2)/2. \tag{15}$$

$p_1 p_2 + p_2 p_3 + p_3 p_4 + \dots + p_{2N-1} p_{2N}$. Second, there is a momentum times a coordinate term proportional to γ : $\gamma(-p_1 x_1 + p_2 x_2 - p_3 x_3 + \dots + p_{2N} x_{2N})$. Third, there is a potential-energy-like term proportional to ϵ : $\frac{1}{2}\epsilon(x_1^2 + x_2^2 + x_3^2 + \dots + x_{2N}^2)$. (It is surprising that this term is proportional to ϵ because in the equations of motion ϵ appears to play the role of a coupling constant; ϵ does not appear to be a measure of the potential energy, which one associates with a frequency of oscillation.) Fourth, there is an oscillator coupling term proportional to $\omega^2 - \gamma^2$:

Similarly, the sum representation for H_4 ,

$$\begin{aligned}
 H_4 = & p_1 p_2 + p_2 p_3 + p_3 p_4 + \epsilon(x_1^2 + x_2^2 + x_3^2 + x_4^2)/2 \\
 & + \gamma(-x_1 p_1 + x_2 p_2 - x_3 p_3 + x_4 p_4) \\
 & + (\omega^2 - \gamma^2)(x_1 x_2 + x_3 x_4 - x_1 x_4), \tag{16}
 \end{aligned}$$

can be reconfigured in product form as

$$\begin{aligned}
 H_4 = & [p_1 + \gamma(x_2 - x_4)](p_2 - \gamma x_1) + (p_2 - \gamma x_1)(p_3 + \gamma x_4) \\
 & + (p_3 + \gamma x_4)[p_4 - \gamma(x_3 - x_1)] \\
 & + \omega^2(x_1 x_2 + x_3 x_4 - x_1 x_4) + \epsilon(x_1^2 + x_2^2 + x_3^2 + x_4^2)/2 \tag{17}
 \end{aligned}$$

without changing the equations of motion. The product representation of H_6 has the form

$$\begin{aligned}
 H_6 = & [p_1 + \gamma(x_2 - x_4 + x_6)](p_2 - \gamma x_1) + (p_2 - \gamma x_1) \\
 & \times [p_3 + \gamma(x_4 - x_6)] + [p_3 + \gamma(x_4 - x_6)] \\
 & \times [p_4 - \gamma(x_3 - x_1)] + [p_4 - \gamma(x_3 - x_1)] \\
 & \times (p_5 + \gamma x_6) + (p_5 + \gamma x_6)[p_6 - \gamma(x_5 - x_3 + x_1)] \\
 & + \omega^2(x_1 y_1 + x_3 x_4 + x_5 x_6 - x_1 x_4 - x_3 x_6 + x_1 x_6) \\
 & + \epsilon(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)/2. \tag{18}
 \end{aligned}$$

The general structure for the product representation of H_{2N} is now clear.

The advantage of the product representation is that if we consider the Hamiltonian to be quantum mechanical, we can identify a gauge invariance. Each momentum factor in the product representation has the form $[p + \gamma$ (sum of spatial coordinates)]. This term resembles the structure $p - eA$ in electrodynamics, which suggests that we can make a unitary (canonical) transformation analogous to a gauge transformation in electrodynamics. By virtue of the Heisenberg algebra $[x, p] = i$, it follows that $e^{-iax} p e^{iax} = p + a$, where a is a constant. Therefore, if we perform the unitary

transformation

$$e^{-i a_{mn} x_m x_n} H_{2N} e^{i a_{mn} x_m x_n} \quad (19)$$

on the Hamiltonian, the only terms that will be affected are the product terms because they contain the momentum operators. The only changes that will occur are that the momentum operators p_m and p_n will be shifted by terms that are linear in the coordinates x_n and x_m . There are $N(2N - 1)$ independent gauge transformations that can be performed on H_{2N} , and therefore we can introduce $N(2N - 1)$ arbitrary constants a_{mn} into H_{2N} . Furthermore, since the transformation is unitary, it leaves the equations of motion invariant [16].

C. Lagrangian

Having found a Hamiltonian for system (8), it is easy to construct a Lagrangian:

$$L_{2N} = \sum_{j=0}^{N-1} (-1)^j \sum_{k=1}^{N-j} [\gamma (\dot{x}_{2k-1} x_{2j+2k} - x_{2k-1} \dot{x}_{2j+2k}) - \omega^2 x_{2k-1} x_{2j+2k} + \dot{x}_{2k-1} \dot{x}_{2j+2k}] - \frac{\epsilon}{2} \sum_{j=1}^{2N} x_j^2. \quad (20)$$

III. GENERAL CASE OF NONCONSTANT ϵ , γ , AND ω

We can construct a Hamiltonian (in the sum representation) for a \mathcal{PT} -symmetric system of $2N$ oscillators even if the parameters ϵ , γ , and ω vary from oscillator to oscillator:

$$H_{2N} = \sum_{k=1}^N (-1)^k \gamma_k (x_k p_k - x_{2N+1-k} p_{2N+1-k}) + \sum_{k=1}^{N-1} \epsilon_k (x_k x_{2N-k} + x_{k+1} x_{2N+1-k}) + \epsilon_N (x_N^2 + x_{N+1}^2)/2 + \sum_{k=1}^N p_k p_{2N+1-k} + \sum_{k=1}^N (\omega_k^2 - \gamma_k^2) x_k x_{2N+1-k}. \quad (21)$$

We can also construct a Hamiltonian for $2N + 1$ oscillators:

$$H_{2N+1} = \sum_{k=1}^N (-1)^k \gamma_k (x_k p_k - x_{2N+2-k} p_{2N+2-k}) + \sum_{k=1}^N \epsilon_k (x_k x_{2N+1-k} + x_{k+1} x_{2N+2-k}) + (x_{N+1}^2 + p_{N+1}^2)/2 + \sum_{k=1}^N p_k p_{2N+2-k} + \sum_{k=1}^N (\omega_k^2 - \gamma_k^2) x_k x_{2N+2-k}. \quad (22)$$

The even Hamiltonian H_{2N} leads to the equations of motion

$$\begin{aligned} \ddot{x}_1 + \omega_1^2 x_1 + 2\gamma_1 \dot{x}_1 &= -\epsilon_1 x_2, \\ \ddot{x}_2 + \omega_2^2 x_2 - 2\gamma_2 \dot{x}_2 &= -\epsilon_1 x_1 - \epsilon_2 x_3, \\ &\dots \\ \ddot{x}_N + \omega_N^2 x_N - (-1)^N 2\gamma_N \dot{x}_N &= -\epsilon_{N-1} x_{N-1} - \epsilon_N x_{N+1}, \\ \ddot{x}_{N+1} + \omega_N^2 x_{N+1} + (-1)^N 2\gamma_N \dot{x}_{N+1} &= -\epsilon_N x_N - \epsilon_{N-2} x_{N+2}, \\ &\dots \\ \ddot{x}_{2N-1} + \omega_2^2 x_{2N-1} + 2\gamma_2 \dot{x}_{2N-1} &= -\epsilon_1 x_{2N} - \epsilon_2 x_{2N-2}, \\ \ddot{x}_{2N} + \omega_1^2 x_{2N} - 2\gamma_1 \dot{x}_{2N} &= -\epsilon_1 x_{2N-1}, \end{aligned} \quad (23)$$

and the odd Hamiltonian H_{2N+1} gives the equations of motion

$$\begin{aligned} \ddot{x}_1 + \omega_1^2 x_1 + 2\gamma_1 \dot{x}_1 &= -\epsilon_1 x_2, \\ \ddot{x}_2 + \omega_2^2 x_2 - 2\gamma_2 \dot{x}_2 &= -\epsilon_1 x_1 - \epsilon_2 x_3, \\ &\dots \\ \ddot{x}_{N+1} + \omega_{N+1}^2 x_{N+1} &= -\epsilon_N (x_N + x_{N+2}), \\ &\dots \\ \ddot{x}_{2N} + \omega_2^2 x_{2N} + 2\gamma_2 \dot{x}_{2N} &= -\epsilon_1 x_{2N+1} - \epsilon_2 x_{2N-1}, \\ \ddot{x}_{2N+1} + \omega_1^2 x_{2N+1} - 2\gamma_1 \dot{x}_{2N+1} &= -\epsilon_1 x_{2N}. \end{aligned} \quad (24)$$

A. Continuum limit $N \rightarrow \infty$

In this subsection we show how to take the limit as the number of oscillators approaches infinity. For simplicity, let us consider two rows of identical particles of mass m . These masses are coupled by springs, as illustrated in Fig. 1.

The top row of particles is subject to damping (friction) forces and the bottom row is subject to undamping forces. Each particle in the top row is coupled by horizontal springs (of force constant per unit length k/Δ) to the adjacent particles to the left and right. Thus, the particle at x_n is coupled to its neighbors at x_{n-1} and at x_{n+1} . The neighboring particles exert a net force on the n th mass of strength $\frac{k}{\Delta}(x_{n+1} - 2x_n + x_{n-1})$, where Δ is the equilibrium spacing. The constant k is the *tension* in the horizontal chain of masses. Also, there are fixed springs above the top row of masses that exert a restoring force per unit length of $-\mu_1 v_1^2 \Delta$ on each of the x masses. This force tends to pull the x masses back to their

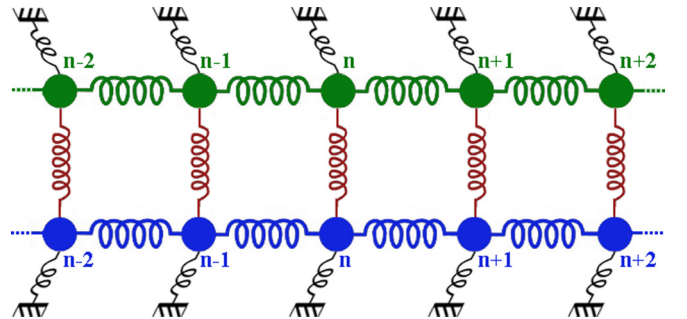


FIG. 1. (Color online) Infinite \mathcal{PT} -symmetric array of identical particles coupled by springs. The masses in the top row, whose position coordinates are $x_n(t)$, experience loss, and the masses in the bottom row, which are located at $y_n(t)$, experience gain.

equilibrium positions. The parameter μ_1 has dimensions of mass density (mass per unit length) and the parameter ν_1 is a frequency having dimensions of 1/time. The force on the n th mass due to these vertical springs is $-\mu_1\nu_1^2\Delta x_n$. Finally, the particle at x_n in the top row is coupled to the particle at position y_n in the bottom row by a vertical spring of force per unit length $\mu_2\nu_2^2\Delta$. (Here, μ_2 is a mass density and ν_2 is a frequency.) The force exerted on the mass at x_n due to the particle at y_n is $\mu_2\nu_2^2\Delta(y_n - x_n)$. The particles in the top row lose energy due to friction (drag), where the dissipation per unit length is given by Γ . Thus, the equation of motion of the n th particle is

$$m\ddot{x}_n + \Gamma\Delta\dot{x}_n = \frac{k}{\Delta}(x_{n+1} - 2x_n + x_{n-1}) - \mu_1\nu_1^2\Delta x_n + \mu_2\nu_2^2\Delta(y_n - x_n). \quad (25)$$

Let $m = \rho\Delta$, where ρ is the horizontal mass per unit length. We then divide (25) by Δ and take the limit as $\Delta \rightarrow 0$ to get the continuum wave equation

$$\rho u_{tt} + \Gamma u_t = k u_{xx} - \mu_1\nu_1^2 u + \mu_2\nu_2^2(v - u). \quad (26)$$

Finally, we divide by ρ and define the quantities $c^2 \equiv k/\rho$, $\gamma \equiv \Gamma/\rho$, $\omega^2 \equiv (\mu_1\nu_1^2 + \mu_2\nu_2^2)/\rho$, and $\epsilon \equiv -\mu_2\nu_2^2/\rho$. This leads to the wave equation

$$u_{tt} + 2\gamma u_t + \omega^2 u - c^2 u_{xx} = -\epsilon v. \quad (27)$$

Similarly, from the equation for the particle at y_n we obtain the wave equation

$$v_{tt} - 2\gamma v_t + \omega^2 v - c^2 v_{xx} = -\epsilon u. \quad (28)$$

These equations are the continuous analogs of (8).

In anticipation of the calculation in Sec. V, we rewrite these equations in a more convenient form by defining $S(x,t) \equiv u(x,t) + v(x,t)$ and $D(x,t) \equiv u(x,t) - v(x,t)$. The coupled wave equations satisfied by S and D are

$$\begin{aligned} S_{tt} + \omega^2 S - c^2 S_{xx} + \epsilon S &= -2\gamma(x)D_t, \\ D_{tt} + \omega^2 D - c^2 D_{xx} - \epsilon D &= -2\gamma(x)S_t, \end{aligned} \quad (29)$$

where we have now taken the loss-gain parameter γ to depend on x .

IV. EXISTENCE OF AN UNBROKEN \mathcal{PT} -SYMMETRIC REGION

The question addressed in this section is whether a region of unbroken \mathcal{PT} symmetry persists as the number of oscillators N increases. First, we consider the case in which the loss-gain parameter γ is the same for all oscillators and show that the unbroken region disappears as N increases. Next, we demonstrate numerically that if γ decreases for the more distant oscillators, a region of unbroken \mathcal{PT} symmetry persists as $N \rightarrow \infty$.

A. Case of constant γ

To find the frequencies of system (8), we seek solutions of the form $x_k = A_k e^{i\lambda t}$. The frequencies λ can then be found by imposing the condition that $\det[M_{2N}] = 0$ (Cramer's rule), where M_{2N} is the $2N \times 2N$ tridiagonal matrix

$$M_{2N} = \begin{pmatrix} a - ib & -\epsilon & 0 & 0 & 0 & 0 & \dots \\ -\epsilon & a + ib & -\epsilon & 0 & 0 & 0 & \dots \\ 0 & -\epsilon & a - ib & -\epsilon & 0 & 0 & \dots \\ 0 & 0 & -\epsilon & a + ib & -\epsilon & 0 & \dots \\ 0 & 0 & 0 & -\epsilon & a - ib & -\epsilon & \dots \\ 0 & 0 & 0 & 0 & -\epsilon & a + ib & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (30)$$

and a and b are given by $a = \lambda^2 - \omega^2$ and $b = 2\lambda\gamma$.

Let $P_N = \det[M_{2N}]$ ($N = 1, 2, \dots$) be the polynomial obtained by computing the determinant of the matrix M_{2N} . The first five of these polynomials are

$$\begin{aligned} P_1 &= -\epsilon^2 + x, \\ P_2 &= \epsilon^4 - 3x\epsilon^2 + x^2, \\ P_3 &= -\epsilon^6 + 6x\epsilon^4 - 5x^2\epsilon^2 + x^3, \\ P_4 &= \epsilon^8 - 10x\epsilon^6 + 15x^2\epsilon^4 - 7x^3\epsilon^2 + x^4, \\ P_5 &= -\epsilon^{10} + 15x\epsilon^8 - 35x^2\epsilon^6 + 28x^3\epsilon^4 - 9x^4\epsilon^2 + x^5, \end{aligned} \quad (31)$$

where $x = a^2 + b^2 = \lambda^4 + \lambda^2(4\gamma^2 - 2\omega^2) + \omega^4$. These polynomials satisfy the recursion relation

$$P_N = (x - 2\epsilon^2)P_{N-1} - \epsilon^4 P_{N-2} \quad (N \geq 2), \quad (32)$$

where we take $P_0 = 1$.

Given these polynomials, we can calculate the frequencies λ to see what happens to the unbroken \mathcal{PT} -symmetric region as N increases. In Fig. 2 we plot the imaginary part of λ for $N = 1, 2, 3$, and 4 for fixed $\omega = 1$ and $\gamma = 0.1$. It is clear that as N increases, the size of the unbroken region in the coupling parameter ϵ shrinks and at $N = 4$ it disappears entirely.

To study analytically the shrinking of the unbroken region with increasing N , we solve the constant-coefficient recursion relation (32). The exact solution is

$$P_N = \sqrt{\pi} \sum_{k=0}^N (-1)^k \frac{4^{k-N} (2N-k)!}{(N-k)! k! \Gamma(N-k+1/2)} x^{N-k} \epsilon^{2k}. \quad (33)$$

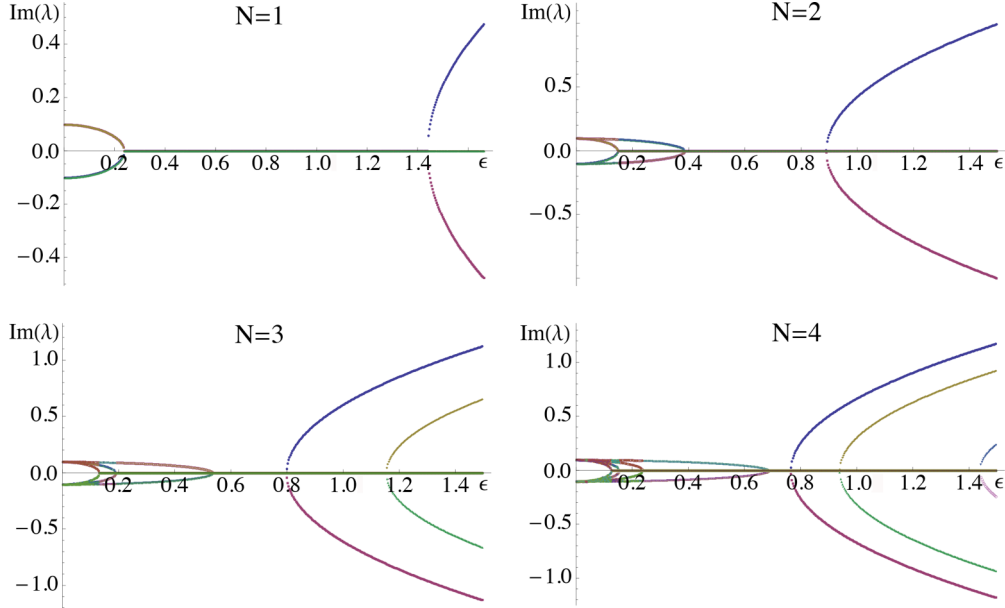


FIG. 2. (Color online) Imaginary parts of the frequencies λ for $N = 1, 2, 3$, and 4 as functions of the coupling constant ϵ for $\omega = 1$ and $\gamma = 0.1$. Frequencies are the zeros of the polynomials P_N in (31). Observe that the extent of the unbroken \mathcal{PT} -symmetric region (where the frequencies are all real) decreases as N increases and disappears entirely when $N = 4$.

Substituting $x = -4\epsilon^2 y$ and $\Delta = \sqrt{y(y+1)}$, we express these polynomials more simply:

$$P_N = \frac{\epsilon^{2N}}{2\Delta} (-1)^N [(1+2y-2\Delta)^N (\Delta-y) + (1+2y+2\Delta)^N (\Delta+y)]. \quad (34)$$

The zeros of P_N are the roots of the equation $\sqrt{y} + \sqrt{y+1} = (-1)^{1/(4N+2)}$. Since $y = -[(\lambda^2 - \omega^2)^2 + 4\lambda^2 \gamma^2]/(4\epsilon^2)$ is negative, we substitute $y = -z^2$. The equation for z then reads $iz + \sqrt{1-z^2} = (-1)^{1/(4N+2)}$, whose solutions are

$$z = \sin[\pi(2k+1)/(4N+2)] \quad (k = 0, 1, \dots, 4N+1). \quad (35)$$

Consequently, the equation for λ becomes $4\epsilon^2 z^2 = (\lambda^2 - \omega^2)^2 + 4\lambda^2 \gamma^2$, whose roots are

$$\lambda_{1,2,3,4} = \pm \sqrt{\omega^2 - 2\gamma^2 \pm 2\sqrt{\gamma^2(\gamma^2 - \omega^2) + \epsilon^2 z^2}}. \quad (36)$$

We consider two cases. For $N = 1$ there are four roots, $z = \sin \theta = \pm 1, \pm \frac{1}{2}$ with $e^{6i\theta} = -1$. These correspond to the six values $\theta = \{\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}\}$. The solutions $z = \pm 1$ are spurious because the denominator of (34) contains $\Delta = \sqrt{z^2(z^2-1)}$ and this expression vanishes at $z = \pm 1$. Thus, the only admissible solutions are $z = \pm 1/2$. Substituting $z^2 = 1/4$ into (36), we obtain the four roots of the polynomial P_1 in (31).

For the case $N = 2$ there are six roots,

$$z = \sin \theta = \{-1, -(1+\sqrt{5})/4, (1-\sqrt{5})/4, (\sqrt{5}-1)/4, (1+\sqrt{5})/4, 1\},$$

with $e^{10i\theta} = -1$. These correspond to the 10 values

$$\theta = \left\{ \frac{1}{10}\pi, \frac{3}{10}\pi, \frac{5}{10}\pi, \frac{7}{10}\pi, \frac{9}{10}\pi, \frac{11}{10}\pi, \frac{13}{10}\pi, \frac{3}{2}\pi, \frac{17}{10}\pi, \frac{19}{10}\pi \right\}.$$

The solutions $z = \pm 1$ are spurious and there are only four genuine roots, $z = \pm(1 \pm \sqrt{5})/4$, and two values, $z^2 = (1 \pm \sqrt{5})^2/16$, to substitute into (36) to get the eight roots of the polynomial P_2 in (31).

In general, in the region of unbroken \mathcal{PT} symmetry the roots λ in (36) are all real. Thus,

$$0 < \gamma < \sqrt{\omega^2/2 - \sqrt{\omega^4/4 - \epsilon^2 z_{\min}^2}}, \quad (37)$$

$$\gamma \sqrt{\omega^2 - \gamma^2} / \sqrt{z_{\min}} < \epsilon < \omega^2 / (2z_{\max}),$$

where $z_{\min} = \sin[\pi/(4N+1)]$ and $z_{\max} = \sin[\pi(2N-1)/(4N+1)]$. Condition (37) identifies the region in the parameter space (γ, ϵ) where the \mathcal{PT} symmetry is unbroken. Note that as $N \rightarrow \infty$, $z_{\min} \rightarrow 0$ and $z_{\max} \rightarrow 1$. Thus, as $N \rightarrow \infty$ the only allowed γ is 0 (so that there is no loss and gain), and the range of ϵ shrinks to $0 \leq \epsilon < \omega^2/2$.

Let us examine further how the allowed γ decreases as a function of increasing N . We can see from Fig. 2 that at the lower end of the unbroken region the curves open to the left and at the upper end of this region the curves open to the right. For fixed N and fixed $\epsilon = \omega^2/(2z_{\max})$, if we increase γ , the left-opening curves will eventually touch the right-opening curves and the unbroken region in ϵ will disappear. We designate as γ_{crit} the critical value of γ at which the unbroken region in ϵ disappears. When we compute γ_{crit} as a function of N and plot in Fig. 3 these values of γ_{crit} versus $1/N$, we see clearly that the critical value of γ decreases to 0. Thus, if there are too many oscillators, there cannot be a region of unbroken \mathcal{PT} symmetry in a system with uniform nonzero loss and gain.

The only way for an unbroken region of \mathcal{PT} symmetry to survive as $N \rightarrow \infty$ is for the loss-gain parameter to decrease with increasingly distant oscillators. Our numerical calculations show that if the loss-gain parameter is $\gamma/(N-n+1)$ (where n ranges from 1 to N), there will be an unbroken region

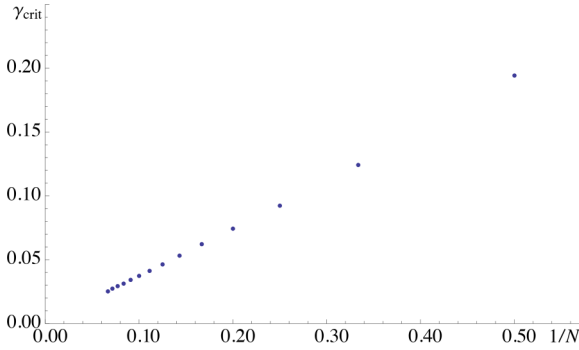


FIG. 3. (Color online) Plot of γ_{crit} as a function of $1/N$. This sequence evidently converges to 0 with increasing N . Thus, a system of coupled oscillators with a uniform loss-gain parameter $\gamma > 0$ has no unbroken \mathcal{PT} -symmetric region if N is sufficiently large.

if γ is less than about 0.1 (Fig. 4, left), and if the loss-gain parameter is $\gamma/(N - n + 1)^2$ (where n ranges from 1 to N), there will be an unbroken region if γ is less than about 0.2 (Fig. 4, right).

V. LOCALIZED IMPURITY IN THE CONTINUUM MODEL

In Sec. IV we have shown that if the effect of loss and gain is localized about the central oscillators and decays for more distant oscillators, then the unbroken \mathcal{PT} -symmetric region can survive as $N \rightarrow \infty$. This suggests that for the continuum model developed in Sec. III A it would be interesting to examine what happens when $\gamma(x)$ decreases with increasing $|x|$. The simplest case to study is that for which $\gamma(x) = \gamma\delta(x)$; that is, the case of a localized pointlike \mathcal{PT} -symmetric loss-gain impurity at the origin. Studies of this type have been performed for tight-binding models by Joglekar *et al.* [17,18] and Longhi [19].

Let us assume that the loss-gain parameter is a localized function of x at the origin, $\gamma(x) = \gamma\delta(x)$, and seek a solution to (29) with frequency Ω :

$$S(x,t) = e^{i\Omega t} s(x), \quad D(x,t) = e^{i\Omega t} d(x). \quad (38)$$

If we assume that $a^2 = \omega^2 - \Omega^2 + \epsilon > 0$ and that $-b^2 = \omega^2 - \Omega^2 - \epsilon < 0$, where a and b are positive, the coupled

wave equations become coupled ordinary differential equations:

$$\begin{aligned} c^2 s''(x) - a^2 s(x) &= 2i\Omega\gamma\delta(x)d(x) \quad \text{and} \\ c^2 d''(x) + b^2 d(x) &= 2i\Omega\gamma\delta(x)s(x). \end{aligned} \quad (39)$$

The functions $s(x)$ and $d(x)$ are continuous at $x = 0$ and the δ function gives rise to a discontinuity in the derivatives of s and d at $x = 0$:

$$\begin{aligned} 2i\gamma\Omega d(0) &= c^2[s'(0^+) - s'(0^-)] \quad \text{and} \\ 2i\gamma\Omega s(0) &= c^2[d'(0^+) - d'(0^-)]. \end{aligned} \quad (40)$$

A simple solution to (39) has the form

$$s(x) = e^{-a|x|/c} \quad \text{and} \quad d(x) = i \frac{ac}{\gamma\Omega} \cos \frac{bx}{c} + i \frac{\gamma\Omega}{bc} \sin \frac{b|x|}{c}. \quad (41)$$

This solution is \mathcal{PT} symmetric, where \mathcal{P} changes the sign of x and interchanges u and v , which in turn changes the sign of d while leaving the sign of s unchanged, and \mathcal{T} performs complex conjugation.

This solution can be viewed as a pseudobound-state solution in the sense that $s(x)$ decays exponentially as $|x| \rightarrow \infty$. However, while $d(x)$ also has a cusp at $x = 0$, it is not localized and oscillates as $|x| \rightarrow \infty$. This solution resembles that found by Hatano *et al.* [20,21] and Longhi [19]. It is interesting that no localized bound-state solution exists if $a^2 = \omega^2 - \Omega^2 + \epsilon > 0$ and $b^2 = \omega^2 - \Omega^2 - \epsilon > 0$, where a and b are positive.

VI. THREE PLANAR OSCILLATORS

It appears that for all one-dimensional chains of oscillators there is just one region of unbroken \mathcal{PT} symmetry. However, it is possible to have more than one region of unbroken \mathcal{PT} symmetry if the oscillators are coupled in a planar array. For example, let us consider three oscillators in a plane, where the first (the x oscillator) has loss, the second (the y oscillator) has gain, and the third (the z oscillator) has neither loss nor gain. The x and y oscillators are coupled directly and are also coupled indirectly through the z oscillator. The Hamiltonian for this system is

$$\begin{aligned} H &= \frac{\omega_2^2}{4} q^2 + \frac{\omega_2^2}{2} pr + y^2 + 2 \frac{\omega_1^2 - \gamma^2}{\omega_2^2} xz - 2 \frac{\epsilon_1}{\omega_2^2} (xy + yz) \\ &\quad - \frac{\epsilon_2}{\omega_2^2} (x^2 + z^2) + \gamma(zr - xp). \end{aligned} \quad (42)$$

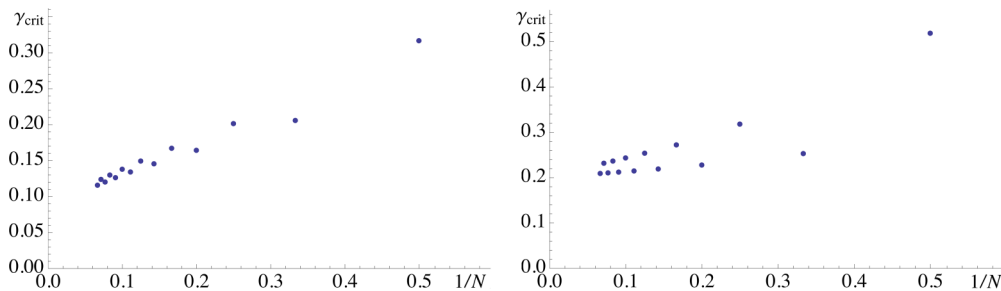


FIG. 4. (Color online) Analog of Fig. 3: Oscillatory convergence of γ_{crit} when the loss-gain parameter γ_n decreases like γ/n (left) and like γ/n^2 (right). Evidently, if the loss-gain parameter decays to 0 for more distant oscillators, a region of unbroken \mathcal{PT} symmetry can persist as $N \rightarrow \infty$.

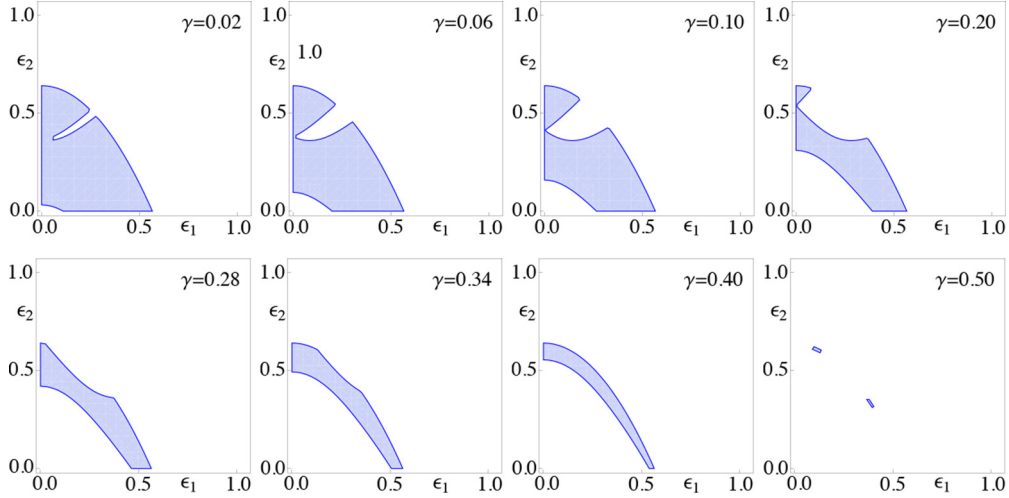


FIG. 5. (Color online) Regions in the space of parameters (ϵ_1 horizontal axis; ϵ_2 , vertical axis) for which the \mathcal{PT} symmetry is unbroken; that is, the roots of $P(\lambda)$ in (46) are all real and positive. The frequency $\omega = 0.8$ and the damping parameter has the values $\gamma = 0.02, 0.06, 0.10, 0.20, 0.28, 0.34, 0.40$, and 0.50 . As γ increases, the unbroken \mathcal{PT} -symmetric regions in (ϵ_1, ϵ_2) space decrease in size and eventually disappear. Unlike the case of linear chains of \mathcal{PT} -symmetric coupled oscillators, as ϵ_2 increases from 0 for fixed ϵ_1 , there is a range of γ and ω such that one can observe *five* regions of broken, unbroken, broken, unbroken, and broken \mathcal{PT} symmetry. For example, there are five regions when $\gamma = 0.10$, $\epsilon_1 = 0.10$, and $0 \leq \epsilon_2 \leq 0.70$.

This Hamiltonian gives the equations of motion

$$\begin{aligned} \ddot{x} + \omega_1^2 x + 2\gamma \dot{x} &= \epsilon_1 y + \epsilon_2 z, & \ddot{y} + \omega_2^2 y &= \epsilon_1(x + z), \\ \ddot{z} + \omega_1^2 z - 2\gamma \dot{z} &= \epsilon_1 y + \epsilon_2 x. \end{aligned} \quad (43)$$

This oscillator system can have *two* regions of unbroken \mathcal{PT} symmetry. Without loss of generality, we choose $\omega_2 = 1$ and $\omega_1 = \omega$, so that H in (42) becomes

$$\begin{aligned} H &= \frac{1}{4}q^2 + \frac{1}{2}pr + y^2 + 2(\omega^2 - \gamma^2)xz - 2\epsilon_1(xy + yz) \\ &\quad - \epsilon_2(x^2 + z^2) + \gamma(zr - xp) \end{aligned} \quad (44)$$

and the system of equations (43) becomes

$$\begin{aligned} \ddot{x} + \omega^2 x + 2\gamma \dot{x} &= \epsilon_1 y + \epsilon_2 z, & \ddot{y} + y &= \epsilon_1(x + z), \\ \ddot{z} + \omega^2 z - 2\gamma \dot{z} &= \epsilon_1 y + \epsilon_2 x. \end{aligned} \quad (45)$$

To find the frequencies of this classical system, we seek solutions to (45) of the form $x(t) = Ae^{i\lambda t}$, $y(t) = Be^{i\lambda t}$, and $z(t) = Ce^{i\lambda t}$. We use Cramer's rule to eliminate the coefficients A , B , and C and find that the resulting equation for the frequency λ is

$$\begin{aligned} P(\lambda) &= \lambda^6 + \lambda^4(4\gamma^2 - 2\omega^2 - 1) + \lambda^2(\omega^4 + 2\omega^2 - 2\epsilon_1^2 \\ &\quad - \epsilon_2^2 - 4\gamma^2) + 2\epsilon_1^2(\epsilon_2 + \omega^2) + \epsilon_2^2 - \omega^4. \end{aligned}$$

With the substitution $\mu = \lambda^2$, this polynomial becomes

$$p(\mu) = \mu^3 - \alpha\mu^2 + \beta\mu - \sigma, \quad (46)$$

with coefficients $\alpha = 1 + 2\omega^2 - 4\gamma^2$, $\beta = \omega^4 + 2\omega^2 - 2\epsilon_1^2 - \epsilon_2^2 - 4\gamma^2$, and $\sigma = \omega^4 - 2\epsilon_1^2(\epsilon_2 + \omega^2) - \epsilon_2^2$. Positive real roots of (46) are obtained by searching for the regions in the parameter space where the minimum $\mu_m = (\alpha - \sqrt{\alpha^2 - 3\beta})/3$ and maximum $\mu_p = (\alpha + \sqrt{\alpha^2 - 3\beta})/3$ are real and positive,

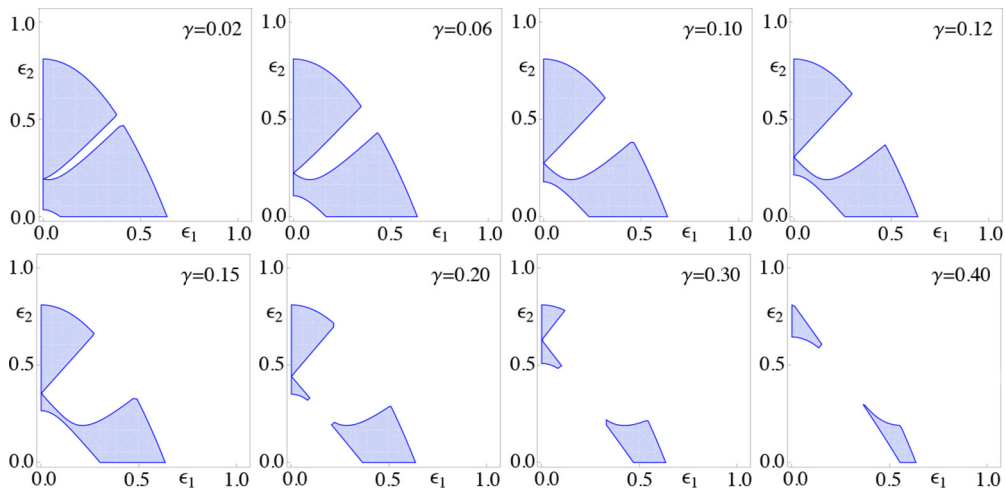


FIG. 6. (Color online) Same as Fig. 5, but with $\omega = 0.9$.

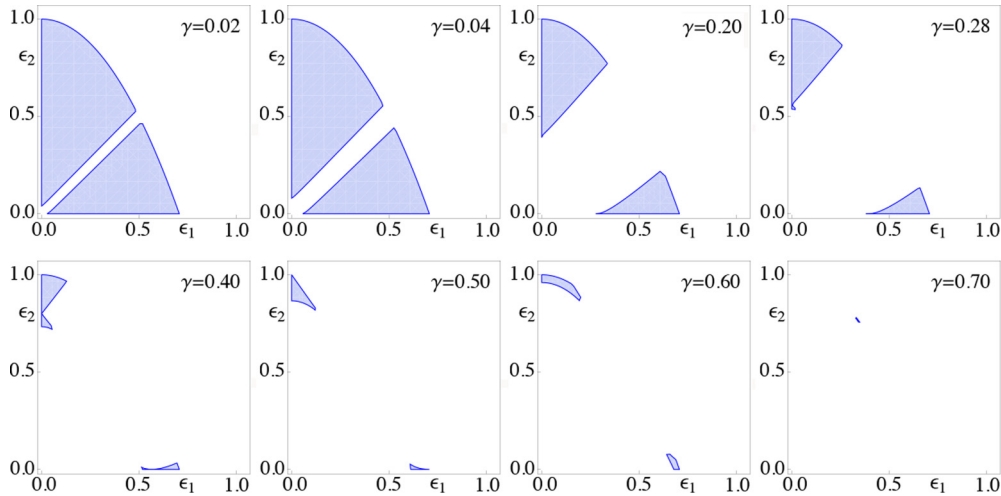


FIG. 7. (Color online) Same as Fig. 5, but with $\omega = 1.0$.

and $p(\mu_m) > 0$ and $p(\mu_p) < 0$. Figures 5, 6, 7, and 8 display the regions of unbroken \mathcal{PT} symmetry [where the roots of $P(\lambda)$ in (46) are all real] for various values of ω , γ , ϵ_1 , and ϵ_2 . For special ranges of the parameters ω , γ , and ϵ_1 one can get five distinct regions of broken and unbroken \mathcal{PT} symmetry as ϵ_2 increases continuously from 0. (For the case of linear chains of \mathcal{PT} -symmetric coupled oscillators one can have, at most, three regions.) The imaginary parts of the frequencies λ as functions of ϵ_2 are plotted in Fig. 9. The unbroken \mathcal{PT} -symmetric regions are characterized by the vanishing of $\text{Im} \lambda$.

VII. CONCLUDING REMARKS

The purpose of this paper has been to examine physically constructable \mathcal{PT} -symmetric systems consisting of many coupled oscillators. (Similar studies have been done for \mathcal{PT} -symmetric arrays of optical waveguides with loss and gain [22,23].) We have implemented \mathcal{PT} symmetry by arranging the oscillators so that loss and gain are balanced pairwise. We

have examined one-dimensional systems consisting of both even and odd numbers of oscillators and have also studied the limiting behavior as the number of oscillators approaches infinity. We have shown that the Hamiltonians associated with these systems can be formulated in two ways: first, as a sum representation and, second, as a product representation. The latter representation has a gaugelike coupling structure that can be used to demonstrate that the Hamiltonian is not unique.

We have shown that when the oscillators are arranged in a one-dimensional chain, for sufficiently many oscillators there cannot be a region of unbroken \mathcal{PT} symmetry (where the frequencies are all real) unless the loss-gain parameter γ decays with the distance from the center of the chain. Our numerical calculations show that if γ decays rapidly enough, then a region of unbroken \mathcal{PT} symmetry will always exist, even if the number of oscillators is infinite. We have also shown that in the continuum limit, a localized gain-loss impurity can give rise to a pseudobound state.

Our analysis shows that for a one-dimensional chain of oscillators, as the coupling constant ϵ increases from 0, one can find, at most, only three regions: two regions of broken \mathcal{PT}

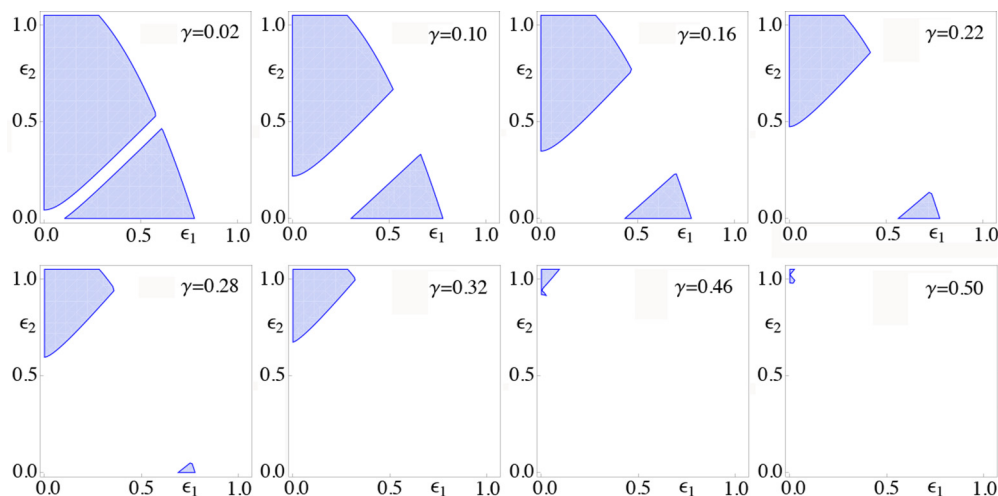


FIG. 8. (Color online) Same as Fig. 5, but with $\omega = 1.1$.

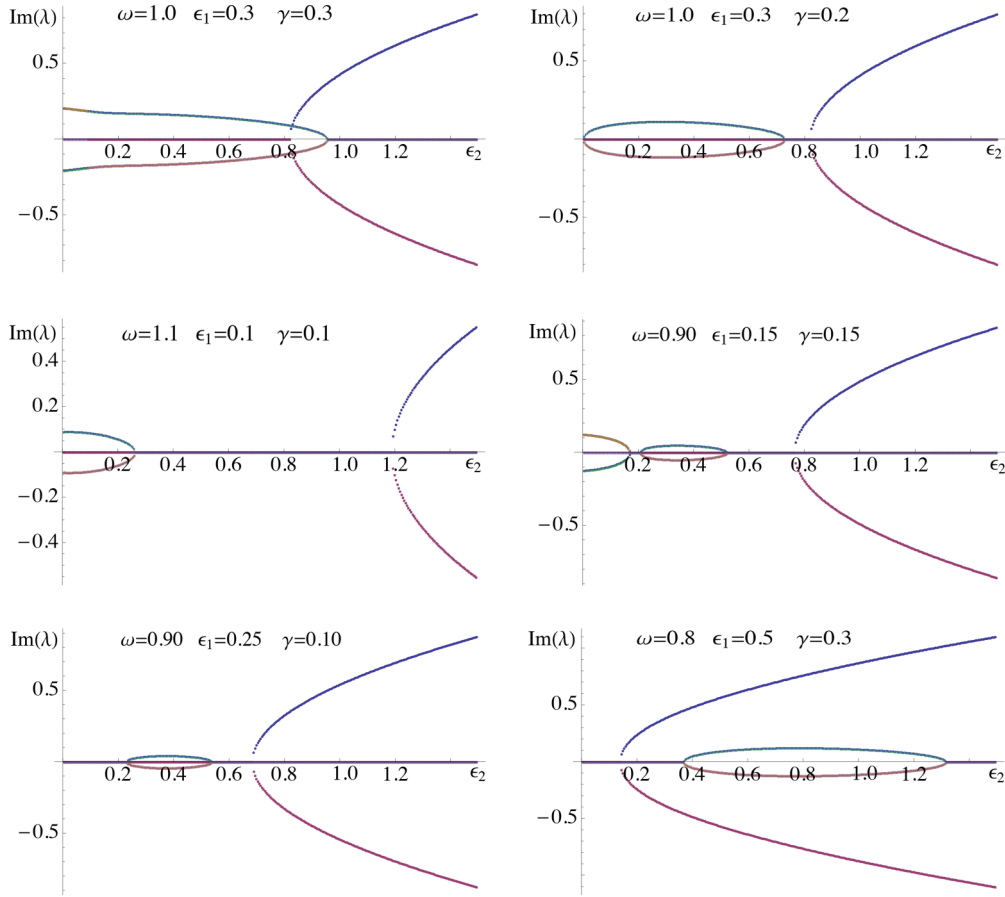


FIG. 9. (Color online) Imaginary parts of the frequencies λ plotted as a function of ϵ_2 for various values of the parameters ω , γ , and ϵ_1 . The regions of unbroken \mathcal{PT} symmetry occur when the imaginary parts vanish and all frequencies are real.

symmetry surrounding a region of unbroken \mathcal{PT} symmetry. However, a two-dimensional array of oscillators can exhibit more than three regions. For example, a triangle of coupled oscillators can exhibit five regions. Optics experiments are currently under way to study such a system [24].

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