Fine-structure constant for gravitational and scalar interactions

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(Received 5 March 2014; published 15 August 2014)

Starting from the coupling of a relativistic quantum particle to the curved Schwarzschild space time, we show that the Dirac-Schwarzschild problem has bound states and calculate their energies including relativistic corrections. Relativistic effects are shown to be suppressed by the gravitational fine-structure constant $\alpha_G = G\, m_1\, m_2/(\hbar c)$, where G is Newton's gravitational constant, c is the speed of light, and m_1 and $m_2 \gg m_1$ are the masses of the two particles. The kinetic corrections due to space-time curvature are shown to lift the familiar (n, j) degeneracy of the energy levels of the hydrogen atom. We supplement the discussion by a consideration of an attractive scalar potential, which, in the fully relativistic Dirac formalism, modifies the mass of the particle according to the replacement $m \to m(1 - \lambda/r)$, where r is the radial coordinate. We conclude with a few comments regarding the (n, j) degeneracy of the energy levels, where n is the principal quantum number, and j is the total angular momentum, and illustrate the calculations by way of a numerical example.

DOI: 10.1103/PhysRevA.90.022112 PACS number(s): 03.65.Pm, 11.10.−z, 03.70.+k, 95.85.Ry

I. INTRODUCTION

As one combines relativistic quantum mechanics [1,2] with general relativity [3–5], one has to formulate the Dirac equation on a curved space time [6–12]. One of the most paradigmatic calculations concerns the Dirac-Schwarzschild Hamiltonian [9,13,14], which is obtained for a Dirac particle in the static Schwarzschild metric. The Dirac-Schwarzschild problem constitutes the analog of the Dirac-Coulomb problem [15–17], which is otherwise relevant for the Dirac particle bound to a central Coulomb potential, as opposed to a central gravitational field. The main problem is that, unlike for the Dirac-Coulomb problem, the gravitational central-field Dirac-Schwarzschild problem cannot be treated based on the correspondence principle alone.

Namely, the gravitational potential $-G\, m_1\, m_2/r$ cannot simply be inserted into the Dirac-Schwarzschild Hamiltonian. One first has to couple [6–8] the Dirac particle to the curved space time, using a fully covariant formalism, and then, identify the translation operator for the time coordinate with the Dirac Hamiltonian. This identification becomes unique in the Dirac-Schwarzschild problem when we demand that the time coordinate have a smooth limit to the flat-space time in the regime of large separation [10–12,14].

We recall that for the Dirac-Coulomb problem, one simply adds the Coulomb potential $-Ze^2/(4\pi\epsilon_0\, r)$ to the free Dirac Hamiltonian, in the sense of a minimal coupling of the bound electron to the central electrostatic field of the nucleus [15–17]. Here, Z is the nuclear charge number, e is the elementary charge, ϵ_0 is the vacuum permittivity, and r is the distance from the center of the potential. Both the Dirac-Schwarzschild as well as the Dirac-Coulomb Hamiltonians take into account the gauge boson exchange (graviton exchange and Coulomb photon exchange, respectively) to all orders, but only in the classical approximation. This is sufficient to calculate the corrections of order α_G^4 and $\alpha_{\rm QED}^4$, where α_G and $\alpha_{\rm QED}$ denote the gravitational and electrodynamic fine-structure constants, respectively.

We anticipate that the familiar (n, j) degeneracy of the energy levels of the Dirac-Coulomb problem will be lifted for gravitational coupling, which implies that, for example, the gravitationally coupled $2S$ and $2P_{1/2}$ levels are not degenerate. For the electromagnetically coupled hydrogen atom, the corresponding degeneracy is lifted only by the Lamb shift; the theoretical explanation involves a manifestly quantized electromagnetic field [18]. The reason for the lifted degeneracy, in the case of gravitational coupling, is different: Namely, we observe that it is due to the space-time curvature corrections to the kinetic term in the Dirac-Schwarzschild Hamiltonian. This finding is illustrated by a comparison to the energy levels of an attractive scalar potential, which are also calculated here, including relativistic corrections.

This paper is organized as follows. In Sec. II, we consider the fine structure of the energy levels of the Dirac-Schwarzschild Hamiltonian and express the result in terms of the gravitational fine-structure constant α_G, and of the quantum numbers of the bound state. In passing, we clarify that the quantum mechanical gravitational central-field problem has bound states. For clarity, but without loss of generality, we consider a gravitationally coupled "atom" consisting of electron and proton. In Sec. III, we compare to the energy levels of an attractive scalar potential. Having clarified the physical origin of the correction terms which lift the (n, j) degeneracy, we continue in Sec. IV with the identification of a set of physical parameters for a gravitationally coupled system, where the calculations reported here might be phenomenologically relevant. These concern an electron gravitationally coupled, in a Rydberg state, to a black hole of mass $10^{-11}\, M_E$, where M_E is the mass of the Earth. In the derivations, we use the electron mass m_e and the proton mass m_p, Newton's gravitational constant G, Planck's reduced quantum unit of action $\hbar$, and the speed of light c. Units with $\hbar = c = \epsilon_0 = 1$ are used in this paper unless explicitly stated otherwise (in some manipulations, it will be of advantage to temporarily switch back to the SI mksA unit system).

II. DIRAC-SCHWARZSCHILD FINE STRUCTURE

We start from the Dirac-Schwarzschild Hamiltonian H for a particle of mass m_e in the central gravitational field of a particle (or planet) of mass $m_p \gg m_e$ (see Ref. [11]),

$$H = \frac{1}{2}\left\{\vec{\alpha}\cdot\vec{p}, \left(1 - \frac{G\,m_p}{2r}\right)\right\} + \beta\, m_e\left(1 - \frac{G\,m_p}{r}\right). \quad (1)$$

The mass parameters m_e and m_p are canonically associated with the electron and proton masses. However, the considerations reported in the following remain valid, without loss of generality, for any small mass $m_1 = m_e$ in the gravitational field of a larger, central mass $m_2 = m_p$. The nonrecoil approximation is employed. The vector of the Dirac $\vec{\alpha}$ matrices and the Dirac β matrix are used in the standard representation [10,11,15–17].

After a Foldy-Wouthuysen transformation [19], one obtains the Dirac-Schwarzschild Hamiltonian $H_{\rm DS}$. It is characterized by an overall prefactor matrix β, which expresses the particle-antiparticle symmetry inherent to the gravitationally coupled Dirac theory [see Eq. (28) of Ref. [9] and Eq. (21) of Ref. [11] for a manifestly Hermitian form]. In order to obtain the leading relativistic corrections, one may restrict the wave function to the "upper" two-component spinor, and the Dirac-Schwarzschild Hamiltonian $H_{\rm DS}$ to its upper (2×2) submatrix,

$$H_{\rm DS} = \frac{\vec{p}^{\,2}}{2m_e} - \frac{G m_e m_p}{r} - \frac{\vec{p}^{\,4}}{8m_e^3} - \frac{3Gm_p}{4m_e}\left\{\vec{p}^{\,2}, \frac{1}{r}\right\} + \frac{3\pi\, G m_p}{2m_e}\,\delta^{(3)}(\vec{r}) + \frac{3Gm_p\,\vec{\sigma}\cdot\vec{L}}{4m_e r^3}. \quad (2)$$

The vector of (2×2) Pauli matrices is denoted as $\vec{\sigma}$. The momentum operator in Eq. (2) is given as $\vec{p} = -i\hbar\vec{\nabla}_r$, where we temporarily restore SI mkSA units for absolute clarity. We employ the following scaling to dimensionless quantities ρ,

$$r = \frac{\hbar^2}{G\,m_e^2\,m_p}\,\rho, \quad \vec{\nabla}_r = \frac{G\,m_e^2\,m_p}{\hbar^2}\,\vec{\nabla}_\rho, \quad (3a)$$

$$\vec{p} = -i\,\frac{G\,m_e^2\,m_p}{\hbar}\,\vec{\nabla}_\rho. \quad (3b)$$

Here, $\vec{\nabla}_\rho$ is the dimensionless gradient operator, with respect to the dimensionless coordinate ρ. The scaled leading-order term has the Schrödinger-like structure,

$$H_S = \frac{\vec{p}^{\,2}}{2m_e} - \frac{Gm_em_p}{r} = \alpha_G^2 m_e c^2\left(-\frac{1}{2}\,\vec{\nabla}_\rho^2 - \frac{1}{\rho}\right). \quad (4)$$

For the electron-proton system, employing the CODATA [20] value of $G = 6.67384(80)\times 10^{-11}\ {\rm N}\,\frac{{\rm m}^2}{{\rm kg}^2}$, one obtains

$$\alpha_G = \frac{G\,m_e\,m_p}{\hbar\,c} = 3.21637(39)\times 10^{-42}. \quad (5)$$

Today, Newton's gravitational constant G remains [20] one of the least well-known physical constants to date, with a relative uncertainty of 1.2×10^{-4}. We should note that the numerically small value of the gravitational fine-structure constant α_G given in Eq. (5) is tied to the physical system under consideration, namely, the electron-proton system. The gravitational Bohr radius of the electron-proton system is

$$a_{0,G} = \frac{\hbar^2}{G\,m_e^2\,m_p} \approx 1.20\times 10^{29}\ {\rm m}, \quad (6)$$

which is very large but depends on the masses employed. For other systems composed of elementary particles or black holes of various masses, the value of the gravitational fine-structure constant is different. One may remark that Eddington [21] observed that the electromagnetic fine-structure constant $\alpha_{\rm QED} \approx 1/137.036$ and the gravitational fine-structure constant $\alpha_G^{(ee)}$ for two gravitationally interacting electrons fulfill the approximate numerical relationship,

$$\frac{\alpha_{\rm QED}}{\alpha_G^{(ee)}} = \frac{e^2}{4\pi\epsilon_0 G\,m_e^2} \approx 4.2\times 10^{42} \approx \sqrt{N_C}\,, \quad (7)$$

where N_C is the number of charged particles in the universe. We shall not comment on this numerical coincidence here except for reemphasizing that the gravitational interactions of elementary particles are much weaker than electromagnetic and "weak" interactions, as well as strong interactions. Still, to fix ideas, it is instructive to consider the bound electron-proton system. The Schrödinger eigenenergies of the eigenproblem $H_S|\phi\rangle = E_n|\phi\rangle$ are given as follows,

$$E_n = -\frac{\alpha_G^2 m_e c^2}{2n^2}. \quad (8)$$

For the relativistic correction term given in Eq. (2), it is instructive to consider the scaling of the various relativistic correction terms separately, with full reference to the SI mksA unit system,

$$-\frac{\vec{p}^{\,4}}{8\,m_e^3\,c^2} = -\frac{\hbar^4\,\vec{\nabla}_r^4}{8\,m_e^3\,c^2} = -\frac{1}{8}\,\alpha_G^4\,m_e\,c^2\,\vec{\nabla}_\rho^4, \quad (9a)$$

$$-\frac{\hbar^2}{c^2}\,\frac{3Gm_p}{4m_e}\left\{\vec{p}^{\,2}, \frac{1}{r}\right\} = \frac{3}{4}\,\alpha_G^4\,m_e\,c^2\left\{\vec{\nabla}_\rho^2, \frac{1}{\rho}\right\}, \quad (9b)$$

$$\frac{\hbar^2}{c^2}\,\frac{3\pi\,Gm_p}{2m_e}\,\delta^{(3)}(\vec{r}) = \alpha_G^4\,m_e\,c^2\,\frac{3\,\pi}{2}\,\delta^{(3)}(\vec{\rho}), \quad (9c)$$

$$\frac{\hbar^2}{c^2}\,\frac{3Gm_p\,\vec{\sigma}\cdot\vec{L}}{4m_e\,r^3} = \alpha_G^4\,m_e\,c^2\,\frac{3\,\vec{\sigma}\cdot\vec{L}}{4\,\rho^3}, \quad (9d)$$

where the (2×2) spin matrices $\vec{\sigma}$ measure the intrinsic angular momentum of the particle. These considerations manifestly identify the relativistic correction terms to be of order α_G^4. The scaled Dirac-Schwarzschild Hamiltonian with relativistic corrections thus is given as follows,

$$H_{\rm DS} = \alpha_G^2 m_e c^2\left(-\frac{1}{2}\,\vec{\nabla}_\rho^2 - \frac{1}{\rho}\right) + \alpha_G^4\,m_e\,c^2 \times\left(-\frac{1}{8}\,\vec{\nabla}_\rho^4 + \frac{3}{4}\left\{\vec{\nabla}_\rho^2, \frac{1}{\rho}\right\} + \frac{3\,\pi}{2}\,\delta^{(3)}(\vec{\rho}) + \frac{3\,\vec{\sigma}\cdot\vec{L}}{4\,\rho^3}\right). \quad (10)$$

Using formulas given on p. 17 of Ref. [22], we may evaluate the relativistic corrections as a function of the bound-state quantum numbers (n is the principal quantum number, ℓ is the orbital angular momentum quantum number, and j is the total angular momentum quantum number). The calculation proceeds via first-order perturbation theory, starting from the Schrödinger-Pauli wave function $\psi_{n\ell j}(\vec{\rho}) = R_{n\ell}(\rho)\,\chi_{\varkappa\mu}(\hat{\rho})$, where

$$\varkappa = 2(\ell - j)(j + 1/2) = (-1)^{j+\ell+1/2}\left(j + \tfrac{1}{2}\right) \quad (11)$$

is the Dirac angular quantum number [15,23]. Some exemplary radial parts $R_{n\ell}(\rho)$ of the Schrödinger-Pauli wave functions are given on p. 15 of Ref. [22]. Knowing j and ℓ, one may calculate $\varkappa$ using Eq. (11). Conversely, one may calculate ℓ with the help of the formula $\ell = |\varkappa + 1/2| - 1/2$. The relativistic corrections amount to

$$E_{n\ell j} = -\frac{\alpha_G^2 m_e c^2}{2n^2} + \alpha_G^4 m_e c^2 \left(\frac{15}{8n^4} - \frac{(7j+5)\,\delta_{\ell,j+1/2}}{(j+1)(2j+1)n^3} - \frac{(7j+2)\,\delta_{\ell,j-1/2}}{j\,(2j+1)\,n^3} \right) = -\frac{\alpha_G^2 m_e c^2}{2n^2} + \frac{\alpha_G^4 m_e c^2}{n^3}\left(\frac{15}{8n} - \frac{14\varkappa+3}{2\,|\varkappa|\,(2\varkappa+1)} \right). \tag{12}$$

The S state energy can be obtained from Eq. (12) with the help of the term $\ell = 0$ and $j = 1/2$; S states are the only ones for which the expectation value of the Dirac-δ term in Eq. (10) is nonvanishing; the result reads as

$$E_{nS_{1/2}} = -\frac{\alpha_G^2 m_e c^2}{2n^2} + \alpha_G^4 m_e c^2 \left(\frac{15}{8n^4} - \frac{11}{2n^3} \right). \tag{13}$$

The $2S_{1/2}$, $2P_{1/2}$, and $2P_{3/2}$ levels are given as follows,

$$E_{2S_{1/2}} = -\tfrac{1}{8}\,\alpha_G^2 m_e c^2 - \tfrac{73}{128}\,\alpha_G^4 m_e c^2, \tag{14a}$$

$$E_{2P_{1/2}} = -\tfrac{1}{8}\,\alpha_G^2 m_e c^2 - \tfrac{91}{384}\,\alpha_G^4 m_e c^2, \tag{14b}$$

$$E_{2P_{3/2}} = -\tfrac{1}{8}\,\alpha_G^2 m_e c^2 - \tfrac{55}{384}\,\alpha_G^4 m_e c^2. \tag{14c}$$

While there is no degeneracy, the hierarchy $E_{2S_{1/2}} < E_{2P_{1/2}} < E_{2P_{3/2}}$ follows a somewhat general paradigm of bound-state theory [24]; namely, that states with higher total angular momentum quantum numbers have higher energy.

III. FINE STRUCTURE FOR A SCALAR POTENTIAL

The Dirac Hamiltonian with a $(1/r)$-scalar potential [12] reads as follows (in natural units),

$$H = \vec{\alpha}\cdot\vec{p} + \beta\left(m_e - \frac{\lambda}{r} \right), \tag{15}$$

where β denotes the corresponding Dirac matrix. After the Foldy-Wouthuysen transformation, we have

$$H_{\rm SP} = \beta\left(m_e + \frac{\vec{p}^{\,2}}{2m_e} - \frac{\lambda}{r} - \frac{\vec{p}^{\,4}}{8m_e^3} + \frac{\lambda}{4m_e^2}\left\{ \vec{p}^{\,2}, \frac{1}{r} \right\} - \frac{\pi\,\lambda}{2m_e^2}\,\delta^{(3)}(\vec{r}) - \frac{\lambda\vec{\Sigma}\cdot\vec{L}}{4m_e^2\,r^3} \right). \tag{16}$$

The scaling to dimensionless variables is analogous to Eq. (3a),

$$r = \frac{1}{\lambda\,m_e}\,\rho, \quad \vec{\nabla}_r = m_e\,\lambda\,\vec{\nabla}_\rho, \tag{17}$$

$$\vec{p} = -i\,m_e\,\lambda\,\vec{\nabla}_\rho, \quad \alpha_S \equiv \lambda. \tag{18}$$

The role of the "scalar fine-structure constant" is taken by the variable $\alpha_S = \lambda$, and the scaled Hamiltonian reads as follows,

$$H_{\rm SP} = \alpha_S^2\,m_e\left(-\frac{1}{2}\,\vec{\nabla}_\rho^2 - \frac{1}{\rho} \right) + \alpha_S^4\,m_e\left(-\frac{1}{8}\,\vec{\nabla}_\rho^4 - \frac{1}{4}\left\{ \vec{\nabla}_\rho^2, \frac{1}{\rho} \right\} - \frac{\pi}{2}\,\delta^{(3)}(\vec{\rho}) - \frac{\vec{\sigma}\cdot\vec{L}}{4\,\rho^3} \right). \tag{19}$$

The energy levels are given as (in SI mksA units)

$$E_{n\ell j} = -\frac{\alpha_S^2\,m_e\,c^2}{2n^2} + \alpha_S^4\,m_e\,c^2\left(-\frac{1}{8n^4} + \frac{1}{n^3\,(j+1)} \right). \tag{20}$$

Here, an important observation can be made: In contrast to Eq. (12), the result for the relativistic corrections of order α_S^4 in the case of the scalar potential has a compact functional form, and the (n,j) degeneracy familiar from the Dirac-Coulomb problem (see Appendix) is restored. We also note that the Dirac-Schwarzschild Hamiltonian (2) and the scalar Dirac Hamiltonian (16) both entail "$(1/r)$ modifications of the mass term," namely, the terms,

$$\beta\,m_e\left(1 - \frac{G\,m_p}{r} \right) \Leftrightarrow \beta\,m_e\left(1 - \frac{\lambda}{r} \right). \tag{21}$$

However, in addition to this modification, the Dirac-Schwarzschild Hamiltonian (2) contains a modification of the kinetic term $\vec{\alpha}\cdot\vec{p}$ which is responsible for the lifting of the (n,j) degeneracy, as a comparison of Eqs. (12) and (20) shows.

IV. NUMERICAL EXAMPLE

Let us consider a "tiny black hole" of mass $m_{\rm BH}$ to be 10^{-11} times the mass M_E of the Earth,

$$M_E \approx 5.9742\times10^{24}\,{\rm kg}, \quad m_{\rm BH} = 5.9742\times10^{13}\,{\rm kg}, \tag{22}$$

and assume that the electric dipole polarizability of the very dense black hole is vanishing. The Schwarzschild radius $r_{S,{\rm BH}}$ is given as follows,

$$r_{S,{\rm BH}} = \frac{2\,G\,m_{\rm BH}}{c^2} = 8.8731\times10^{-14}\,{\rm m}. \tag{23}$$

The gravitational fine-structure constant for an electron gravitationally bound to the black hole is given as

$$\alpha_{G,{\rm BH}} = \frac{G\,m_e\,m_{\rm BH}}{\hbar c} = 0.1148. \tag{24}$$

The gravitational Bohr radius is

$$a_{0,{\rm BH}} = \frac{\hbar^2}{G\,m_e^2\,m_{\rm BH}} = 3.3612\times10^{-12}\,{\rm m}. \tag{25}$$

In accordance with Eq. (3a), we define the Cartesian components of the scaled dimensionless coordinate $\vec{\rho}$ as follows,

$$\rho_x = \frac{x}{a_{0,{\rm BH}}}, \quad \rho_y = \frac{x}{a_{0,{\rm BH}}}, \quad \rho_z = \frac{x}{a_{0,{\rm BH}}}. \tag{26}$$

In Fig. 1, we present a "scatter plot" of the bound state with quantum numbers $n = 10$, $\ell = 9$, and magnetic orbital angular momentum projection $m = |\ell| = 9$ ("circular Rydberg state"), where the points representing the wave function are distributed

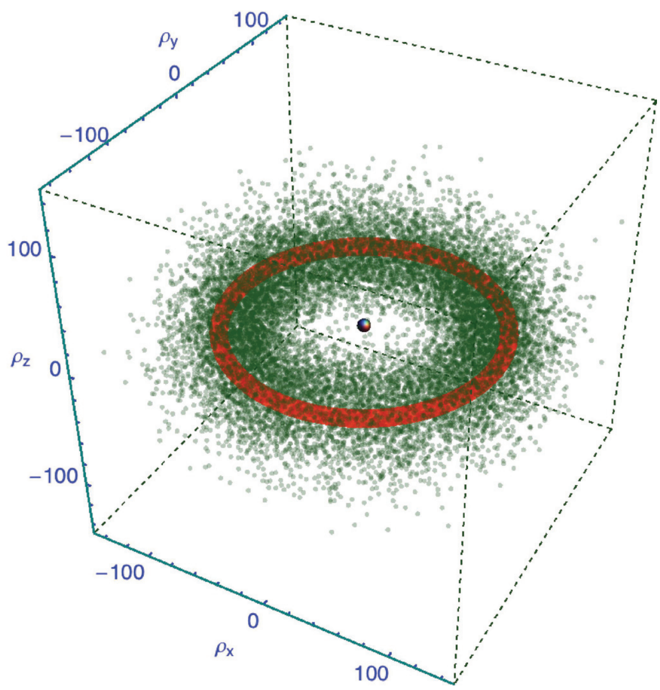

FIG. 1. (Color) Scatter plot of the probability density of an electron in a circular Rydberg state with quantum numbers $n = 10$, $\ell = m = 9$, gravitationally bound to a black hole of mass 10^{-11} times Earth's mass. The green points are distributed according to the probability density $|\psi|^2$ of finding the electron at a particular point in space. A corresponding circular classical trajectory with a radius of 105 gravitational Bohr radii is indicated in red for comparison, and the black hole at the center is indicated as a black dot.

according to the probability density given by the absolute square of the wave function $|\psi|^2$. The probability density of the Rydberg state inside the Schwarzschild radius is negligible and the expectation value of the zitterbewegung term in the Dirac-Schwarzschild Hamiltonian (2) vanishes. The nonrelativistic Schrödinger-type approximation is justified because the gravitational fine-structure constant $\alpha_{G,\rm BH}$ is small against unity. According to p. 17 of Ref. [22], the radial expectation value in the Schrödinger state is $\langle|\vec{\rho}|\rangle = 105$ gravitational Bohr radii. A classical circular trajectory circling the black hole is indicated in Fig. 1 for comparison.

According to Eq. (12), the bound-state energies for the two states with $j = 9 \pm 1/2$ are given as follows,

$$E_{n=10,\ell=9,j=19/2} = \left(-\frac{\alpha_{G,\rm BH}^2}{200} - \frac{263\,\alpha_{G,\rm BH}^4}{1520000}\right) m_e c^2 = -33.7397\,\text{eV}, \tag{27a}$$

$$E_{n=10,\ell=9,j=17/2} = \left(-\frac{\alpha_{G,\rm BH}^2}{200} - \frac{173\,\alpha_{G,\rm BH}^4}{912000}\right) m_e c^2 = -33.7412\,\text{eV}. \tag{27b}$$

The higher value of the total angular momentum j moves the state with $j = 19/2$ energetically upward. Both energies (27a) and (27b) are numerically close to the nonrelativistic approximation, which reads as $-\alpha_{G,\rm BH}^2 m_e c^2/200 = -33.7243$ eV. These bound-state energies are exclusively due to the gravitational interaction. In the case of a residual electromagnetic interaction, corrections to atomic energy levels due to curved space time have been discussed in Refs. [25,26].

V. EVENT HORIZON

In order to investigate the influence of the event horizon near the Schwarzschild radius onto the bound-state energy levels, let us first recall that the dominant gravitational binding potential given in Eq. (21) (before the Foldy-Wouthuysen transformation) is given as

$$\beta\, m_e\, w(r_1) = \beta\, m_e \frac{4r_1 - r_S}{4r_1 + r_S} \approx \beta\, m_e \left(1 - \frac{r_S}{2r_1}\right), \tag{28}$$

where we denote the Schwarzschild radius as $r_S = 2G\, m_p$ and we expand for large r_1 in the second step. The radial variable in the rescaled Schwarzschild-Eddington metric [see Eq. (33) of Ref. [10]] is denoted as r_1 (this radial coordinate leads to an isotropic metric and has been denoted as r in the discussion here up to this point). The radial variable r_1 is connected with the radial coordinate r_0 in the Schwarzschild metric [see Eq. (30) of Ref. [10]] as follows,

$$r_0 = r_1 \left(1 + \frac{r_S}{4\, r_1}\right)^2, \tag{29}$$

$$r_1 = \frac{1}{2}\left(r_0 - \frac{r_S}{2} \pm \sqrt{r_0(r_0 - r_S)}\right). \tag{30}$$

For large radial coordinates, r_0 and r_1 are almost identical,

$$r_1 = \frac{1}{2}\left(r_0 - \frac{r_S}{2} + \sqrt{r_0\,(r_0 - r_S)}\right) = r_0 - \frac{r_S}{2} - \frac{r_S^2}{16\, r_0} + O\left(r_0^{-2}\right), \quad r_0 \gg r_S. \tag{31}$$

By contrast, r_1 describes a circle of radius $r_S/4$ about the origin as r_0 sweeps the interval $0 < r_0 < r_S$,

$$r_1 = \frac{1}{2}\left(r_0 - \frac{r_S}{2} - i\,\sqrt{r_0\,(r_S - r_0)}\right) = -\frac{r_S}{4} - \frac{i}{2}\sqrt{r_0\, r_S} + \frac{r_0}{2} + O\left(r_0^{3/2}\right), \quad r_0 \ll r_S. \tag{32}$$

The event horizon is reached at $r_0 = r_S$, which corresponds to $r_1 = r_S/4$. In particular, we have

$$|r_1| = \frac{r_S}{4}, \quad 0 < r_0 < r_S. \tag{33}$$

Inside the event horizon, the binding potential w given in Eq. (28) takes the form,

$$w = \frac{4r_1 - r_S}{4r_1 + r_S} = \frac{r_0 - i\,\sqrt{r_0(r_S - r_0)} - r_S}{r_0 - i\,\sqrt{r_0(r_S - r_0)}}. \tag{34}$$

This expression is manifestly complex and allows us to calculate an approximation to the imaginary part of the bound-state energy by an explicit integration, without recourse to a barrier penetration amplitude (we observe that w does not develop a particularly pronounced singularity for small r_0 but goes "only" as $r_0^{-1/2}$). Taking into account that w takes the role of the binding potential in the region of the small radial coordinate, we evaluate the imaginary part of the energy shift as follows. We first employ the nonrelativistic approximation and the Schrödinger wave function as follows,

$$\psi_{n\ell m}(\vec{r}_1) = R_{n\ell}(r_1)\, Y_{\ell m}(\theta,\varphi),$$
$$R_{n\ell}(r_1) \approx \frac{2^{\ell+1}}{n^{\ell+2}} \frac{(\alpha_G\, m)^{\ell+3/2}}{\Gamma[2(\ell+1)]}\, r_1^{\ell} \sqrt{\frac{\Gamma(m+\ell+1)}{\Gamma(n-\ell)}}, \qquad (35)$$

where the radial wave function is expanded for small argument. Using Eqs. (28), (33), (35) and the integral,

$$\int_0^{r_S} dr_0\, r_0^2\, w = -\frac{i}{16}\,\pi\, r_S^3, \qquad (36)$$

we can finally write a good approximation to the imaginary part of the bound-state energy level as follows,

$$\begin{aligned} i\,\mathrm{Im}\,E &= -\frac{i}{2}\,\Gamma \approx \int_{|\vec{r}_0|<r_S} d^3r_0\,(m\,w)\left|\psi_{n\ell m}\left(|\vec{r}_1| = \frac{r_S}{4}\right)\right|^2 \\ &= -\frac{i}{16}\,\pi\, m\, r_S^3 \left|R_{n\ell}\left(|\vec{r}_1| = \frac{r_S}{4}\right)\right|^2 \\ &= -2\pi\, i\, \frac{\alpha_G^{6+4\ell}\, m}{n^{2\ell+4}} \frac{\Gamma(n+\ell+1)}{[\Gamma(2(\ell+1)]^2\, \Gamma(n-\ell)}. \end{aligned} \qquad (37)$$

The factor $\alpha_G^{6+4\ell}$ makes this imaginary part completely negligible for highly excited circular Rydberg states such as those discussed in Sec. III ($n = 10$, $\ell = 9$, order α_G^{42} in agreement with the intuitive wisdom that the event horizon cannot have a large effect on Rydberg energy levels where the particle has negligible probability density near the origin).

The effect of the region close to the event horizon onto the real as opposed to the imaginary part of the gravitational energy levels can be estimated as follows. Namely, according to Eq. (28), the gravitational potential, proportional to r_S/r, ceases to be a good approximation near the event horizon. Hence, the expectation value of the nonrelativistic gravitational Schrödinger Hamiltonian,

$$H_S(\vec{r}_1) = \frac{\vec{p}^{\,2}}{2m_e} - G\,\frac{m_e\, m_p}{r_1}, \qquad (38)$$

ceases to be a good approximation in the region near $r_0 \approx r_S$, which we map onto the region,

$$\frac{r_S}{4} < r_1 < r_S, \quad r_S < r_0 < \frac{25}{16}\, r_S. \qquad (39)$$

Consequently, we can estimate the theoretical uncertainty $\mathrm{Re}\,\delta E$ in the real part of the bound-state energy level, by the integral,

$$\begin{aligned} \mathrm{Re}\,\delta E &= \int_{|\vec{r}_1|<r_S} d^3r_1\, \psi^*_{n\ell m}(\vec{r}_1)\, H_S(\vec{r}_1)\, \psi_{n\ell m}(\vec{r}_1) \\ &= -\frac{16^{\ell+1}}{2\ell+3}\, \frac{\alpha_G^{8+4\ell}\, m}{n^{2\ell+6}}\, \frac{\Gamma(n+\ell+1)}{[\Gamma(2(\ell+1)]^2\, \Gamma(n-\ell)}. \end{aligned} \qquad (40)$$

Again, the presence of the factor $\alpha_G^{8+4\ell}$ suppresses this term for highly excited circular Rydberg states discussed in Sec. III, e.g., for the state with $n = 10$ and $\ell = 9$.

VI. RADIATIVE CORRECTIONS

Inspired by the work of Hartle and Hawking [27], we may investigate the question at which order the quantization of the gravitational interaction influences the energy levels. This calculation should otherwise lead to the gravitational analog of the Bethe logarithm [18]. We first observe the similarity of the Schrödinger-picture photon field operator [see Eq. (5) of Ref. [28]],

$$\begin{aligned} A^i(\vec{r}) = \sum_{h=\pm1} \int \frac{d^3k}{\sqrt{(2\pi)^3}} \frac{1}{\sqrt{2\,k}}\, \epsilon_h^i(\vec{k}) \\ \times [a_h(\vec{k})\, e^{i\,\vec{k}\cdot\vec{r}} + a_h^\dagger(\vec{k})\, e^{-i\,\vec{k}\cdot\vec{r}}], \end{aligned} \qquad (41)$$

where h denotes the helicity, with the spin-2 graviton operator [see Eq. (5) of Ref. [29] and Eq. (68) of Ref. [30]],

$$\begin{aligned} h^{ij}(x) = \kappa \sum_{h=\pm2} \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\,k}}\, \epsilon_h^{ij}(\vec{k}) \\ \times[a_h(\vec{k})\, \exp(i\vec{k}\cdot\vec{r}) + a_h^\dagger(\vec{k})\, \exp(i\vec{k}\cdot\vec{r})], \end{aligned} \qquad (42)$$

where $\kappa = \sqrt{8\pi G}$. Here, the polarization vectors for the massless gauge bosons (spin-1 and spin-2) are denoted by ϵ, the helicity is denoted as h (only two helicity projections are allowed for the massless fields), and $k = |\vec{k}|$. Finally, the $a_h(\vec{k})$ and $a_h^\dagger(\vec{k})$ are field annihilation and creation operators. The graviton field operator parametrizes the deviations from the flat-space metric $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$. The Schwarzschild metric [10] is equal to $\eta_{\mu\nu}$ up to corrections of order r_S.

After tracing of the photon degrees of freedom in second-order time-independent perturbation theory [31], the nonrelativistic electromagnetic self-energy shift [31–34], with an interaction Hamiltonian $-e\,\vec{A}\cdot\vec{p}$ ("velocity" gauge) is obtained from an integral of the form

$$E_{\mathrm{QED}} \propto \alpha_{\mathrm{QED}} \int dk k^2 \frac{1}{2k} \left\langle \epsilon^i\, p^i\, \frac{1}{H_S - E_S + k}\, \epsilon^j\, p^j \right\rangle, \qquad (43)$$

where $[1/(H_S - E_S + k)]$ is the nonrelativistic Schrödinger propagator, with H_S being the Schrödinger-Coulomb Hamiltonian and E_S the nonrelativistic bound-state energy (electromagnetic), while the polarization vector components ϵ^i of the photons needs to be summed over the helicities [not explicitly indicated in Eq. (43)]. The following order-of-magnitude estimates are valid,

$$k \sim \alpha_{\mathrm{QED}}^2, \quad \frac{1}{H_S - E_S + \omega} \sim \frac{1}{\alpha_{\mathrm{QED}}^2}, \qquad (44)$$

and thus

$$E_{\mathrm{QED}} \sim \alpha_{\mathrm{QED}}^5\, m_e. \qquad (45)$$

Indeed, the sum over the hydrogen spectrum, which is left after the introduction of an ultraviolet cutoff in the integral (43), is commonly referred to as the Bethe logarithm. It enters the hydrogen spectrum at order α_{QED}^5.

For the gravitational bound-state problem, the fully relativistic interaction is given by the covariant coupling $\partial_\mu \to \nabla_\mu = \partial_\mu - \Gamma_\mu$, which leads to a replacement,

$$\vec{\alpha}\cdot\vec{p} = -i\,\vec{\alpha}\cdot\partial_i \to -i\,\alpha^i\,\frac{\partial}{\partial r^i} + i\,\alpha^i\,\Gamma^i, \qquad (46)$$

to yield the gravitationally coupled Dirac Hamiltonian. Here, Γ_μ is the connection matrix [see Eq. (25) of Ref. [10]], and Cartesian spatial components are denoted by Latin superscripts, while the vector $\vec{\alpha}$ of Dirac matrices is used in the usual conventions [10] (see also Ref. [35]). Two metric tensors enter into Γ_μ, but we can assume that only one of them corresponds to a quantized excitation of the gravitational field, with a virtual one-graviton state excited from the vacuum via the action of the field operator (42). In full analogy to the quantum electromagnetic problem, we can employ the following order-of-magnitude estimates for the exchange of a virtual graviton with an energy commensurate with the binding energy scale,

$$k \sim \alpha_G^2, \quad (\vec{k}\cdot\vec{r}) \sim \frac{\alpha_G^2}{\alpha_G} = \alpha_G, \tag{47a}$$

$$\frac{1}{H_S - E_S + \omega} \sim \frac{1}{\alpha_G^2}, \tag{47b}$$

where now H_S and E_S refer to the gravitational Hamiltonian and energy, respectively. Equation (43) is replaced by the gravitational bound-state self-energy,

$$\begin{aligned} E_{\rm SE} &\propto G \int dk k^2 \, \frac{1}{k} \left\langle \epsilon^{ij}\,(\vec{k}\cdot\vec{r})\, \frac{1}{H_S - E_S + k}\, \epsilon^{ij}(\vec{k}\cdot\vec{r}) \right\rangle \\ &\sim \underbrace{G}_{\mathcal{O}(\alpha_G)} \underbrace{\int dk k}_{\mathcal{O}(\alpha_G^4)} \left\langle \underbrace{\epsilon^{ij}\,(\vec{k}\cdot\vec{r})}_{\mathcal{O}(\alpha_G)} \underbrace{\frac{1}{H_S - E_S + k}}_{\mathcal{O}(\alpha_G^{-2})} \underbrace{\epsilon^{ij}(\vec{k}\cdot\vec{r})}_{\mathcal{O}(\alpha_G)} \right\rangle, \end{aligned} \tag{48}$$

so that

$$E_{\rm SE} \sim \alpha_G^5\, m_e, \tag{49}$$

in full analogy to Eq. (45). In writing Eq. (48), we have taken into account the fact that the graviton polarization tensor ϵ^{ij} does not contain any coordinates of the bound particle; one has to expand to first order in $\vec{k}\cdot\vec{r}$ in order to obtain an energy shift beyond a mass renormalization [36]. The result (49) indicates that any generalization of the fine-structure terms given in Eq. (12) beyond the order α_G^4 requires a consideration of the quantized gravitational field at the next order in the α_G expansion.

VII. CONCLUSIONS

We have divided the current paper into five parts. The first of these (see Sec. II) deals with the leading-order relativistic corrections to the energies of bound states of the Dirac-Schwarzschild Hamiltonian (2), while the second part (Sec. III) investigates the bound states of a Dirac Hamiltonian with a scalar $(1/r)$ potential. The latter potential modifies the mass term of the Dirac particle; it is commonly referred to as a scalar potential because of its properties under Lorentz transformations [12]. Having clarified the origin of the terms that lift the (n,j) degeneracy otherwise observed for scalar Dirac bound states and for the Dirac-Coulomb problem (see Sec. III and Appendix, respectively), we then turn our attention back to the Dirac-Schwarzschild problem in Sec. IV, and consider a numerical example for bound states of a "small" black hole of mass 10^{-11} times Earth's mass (third part of our investigation). This parameter combination leads to gravitational electronic bound states [the coupling constant $\alpha_{G,\rm BH}$ given in Eq. (24) is small against unity]. It is thus possible to compare to a classical treatment for circular Rydberg states, in terms of the trajectory shown in Fig. 1. In the nonrelativistic approximation, the circular symmetry (Schrödinger approximation) is restored, while the relativistic corrections, including the Fokker precession term (spin-orbit coupling term) enter the relativistic energies given in Eqs. (27a) and (27b). Finally, limitations due to the event horizon (Sec. V) and due to radiative corrections ("gravitational Bethe logarithm") are discussed in Sec. VI. An approximate formula for the imaginary part of the gravitational binding energy (decay width) due to the region inside the event horizon is given in Eq. (49). Order-of-magnitude estimates for the self-energy radiative correction [quantized gravity; see Eq. (49)], and an estimate for the theoretical uncertainty of the real part of the energy due to the region near the horizon [see Eq. (40)] are also provided.

In our investigations, we clarify, in particular, that the quantum mechanical gravitational central-field problem has bound states. This result holds in the framework of curved space times (general relativity; see Ref. [5]) and takes into account the fact that it is impossible, in contrast to the Dirac-Coulomb problem, to simply insert the gravitational potential $(-G\, m_1\, m_2/r)$ into the Dirac Hamiltonian by the corresponding principle. We evaluate the fine-structure formula for the Dirac-Schwarzschild Hamiltonian [see Eq. (12)], and calculate the α_G^4 corrections to the energy. The bound-state energies are obtained as a function of "good" quantum numbers.

Let us briefly comment on the appropriate quantum numbers for the Dirac-Schwarzschild problem. Because of the symmetries of the problem [10,11], the principal quantum number n, the total angular momentum quantum number j, and the Dirac angular momentum quantum number $\varkappa$ constitute a set of "good" quantum numbers. The familiar spin-angular function $\chi_{\varkappa\,\mu}(\hat{r})$ is assembled from the fundamental spinors and the spherical harmonics via Clebsch-Gordan coefficients [15,23,37]. It has the property,

$$(\vec{\sigma}\cdot\vec{L} + 1)\,\chi_{\varkappa\,\mu}(\hat{r}) = -\varkappa\,\chi_{\varkappa\,\mu}(\hat{r}), \tag{50}$$

where $\varkappa$ is defined according to Eq. (11). Knowing j and $\varkappa$, one may calculate the orbital angular momentum quantum number $\ell = |\varkappa + 1/2| - 1/2$ even if the orbital angular momentum operator $\vec{L}$ itself does not commute with the Dirac-Schwarzschild Hamiltonian (2). Because $\varkappa$ can be mapped onto the orbital angular momentum quantum number ℓ (i.e., onto the "spin orientation with respect to the orbital angular momentum"), the main result (12) is consistent.

For both the scalar Dirac Hamiltonian (16) as well as the Dirac-Coulomb Hamiltonian (A2), the explicit ℓ dependence of the spin-orbit coupling accidentally cancels out against the "implicit" ℓ dependence of the matrix elements of the momentum, and the position operator [see Ref. [22] and Eq. (A5)].

ACKNOWLEDGMENTS

The author acknowledges helpful conversation with I. Nándori and M. M. Bush. The research has been supported by the National Science Foundation (Grants No. PHY-1068547 and No. PHY-14303973).

APPENDIX: DIRAC-COULOMB HAMILTONIAN

For comparison, we briefly recall the Dirac-Coulomb Hamiltonian [15,38],

$$H = \vec{\alpha} \cdot \vec{p} + \beta\, m_e - \frac{Z\alpha_{\rm QED}}{r}, \tag{A1}$$

where Z is the nuclear charge number, and $\alpha_{\rm QED} \approx 1/137.036$ is the QED fine-structure constant. The nonrecoil approximation is employed. After a Foldy-Wouthuysen transformation, the Hamiltonian takes the form,

$$H_{\rm DC} = \frac{\vec{p}^{\,2}}{2m_e} - \frac{Z\alpha_{\rm QED}}{r} - \frac{\vec{p}^{\,4}}{8m_e^3} + \frac{\pi\, Z\alpha_{\rm QED}}{2m_e^2}\,\delta^{(3)}(\vec{r}) + \frac{Z\alpha_{\rm QED}}{4m_e^2 r^3}\,\vec{\Sigma}\cdot\vec{L}. \tag{A2}$$

The scaling corresponding to Eqs. (3a) and (17) reads as follows,

$$r = \frac{\hbar}{m_e c}\,\rho, \quad \vec{\nabla}_r = \frac{m_e c}{\hbar}\,\vec{\nabla}_\rho, \quad \vec{p} = -i\,\frac{m_e c}{\hbar}\,\vec{\nabla}_\rho. \tag{A3}$$

The familiar [15–17] scaled Dirac-Coulomb Hamiltonian is obtained as

$$H_{\rm DS} = \alpha_{\rm QED}^2 m_e c^2 \left(-\frac{1}{2}\,\vec{\nabla}_\rho^2 - \frac{1}{\rho}\right) + \alpha_{\rm QED}^4\, m_e\, c^2 \times \left(-\frac{1}{8}\,\vec{\nabla}_\rho^4 + \frac{\pi}{2}\,\delta^{(3)}(\vec{\rho}) + \frac{\vec{\sigma}\cdot\vec{L}}{4\,\rho^3}\right). \tag{A4}$$

The energy levels are given as follows [15,38],

$$E_{n\ell j} = -\frac{\alpha_{\rm QED}^2\, m_e c^2}{2n^2} + \alpha_{\rm QED}^4\, m_e c^2 \left(\frac{3}{8n^4} - \frac{1}{n^3\,(2j+1)}\right). \tag{A5}$$

When evaluating the matrix elements according to formulas given on pp. 15–17 of Ref. [22], one first obtains a functionally different formula for $j = \ell + 1/2$ as opposed to $j = \ell - 1/2$ but they coincide for given j. This is analogous to Eq. (20).

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