PHYSICAL REVIEW A **90**, 020301(R) (2014)

Witnessing entanglement in hybrid systems

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(Received 24 April 2014; published 1 August 2014)

We extend the definition of entanglement witnesses based on spin structure factors to the case of scatterers with quantum mechanical motion. We show that this allows for hybrid entanglement detection and specialize the witness for a chain of trapped ions. Within this framework, we also show how the collective vibronic state of the chain can act as an undesired quantum environment affecting the spin–spin-entanglement detection. Furthermore, we investigate several specific cases where these witness operators allow us to detect hybrid entanglement.

DOI: 10.1103/PhysRevA.90.020301 PACS number(s): 03.67.Bg, 03.65.Yz, 03.67.Lx, 32.80.Qk

Introduction. Entanglement is a genuine quantum property of composite systems and is regarded as a resource for quantum information processing [1,2]. Hence, revealing entanglement experimentally is crucial to prove its usefulness for specific information protocols. To this end, entanglement witness operators have been proposed and experimentally implemented [3–5]. The major advantage is that fewer experimental resources than standard quantum state tomography are required when using witness operators. A method to detect multipartite entanglement via structure factors was recently proposed [6,7] and successfully tested in a quantum optical experiment [8]. Related work in condensed matter systems was reported in [9,10]. In this paper we further investigate the properties of the entanglement witnesses defined in [6] by removing the constraint of classically localized particles. This kind of scenario can be easily implemented in ion traps, where the ions can be cooled down so that the quantum nature of their harmonic motion emerges. Both the vibrational and the electronic degrees of freedom of these systems can be precisely manipulated with laser beams and entangled states of the internal (spin), external (phonons), and hybrid spin-phonon degrees of freedom have been experimentally realized. [11,12]. In this paper we investigate two related scenarios. In the first scenario we focus, as in Refs. [6–9], on spin-entanglement witnesses and we ask ourselves if and how the presence of other quantum degrees of freedom (the vibrational quanta in the trapped-ion case) affects the detection of entanglement between spins. In the second scenario we identify several cases for which the generalized witnesses introduced here allow us to detect hybrid entanglement. Furthermore, entanglement witnesses can represent a successful strategy when dealing with nonfinite Hilbert space systems, such as a harmonic oscillator, where state tomography is feasible only for Gaussian states [13].

Extended entanglement witness. An entanglement witness is a Hermitian operator W that detects an entangled state ρ if it has a negative expectation value for this state $\mathrm{Tr}[W\rho] < 0$, while $\mathrm{Tr}[W\sigma] \geqslant 0$ for all separable states σ [3]. In this paper we adopt the construction proposed in Ref. [6], where a class of entanglement witnesses was introduced based on two-point spin correlation functions. These functions define the spin structure factors [14] of an ensemble of N particles via the expectation value of the operator

$$S^{\alpha\beta}(q) = \sum_{n<m} e^{iq(x_n - x_m)} S_n^\alpha S_m^\beta, \tag{1}$$

where S_n^α is the α component of the nth particle spin operator, x_n is its position, which is assumed to be perfectly known, and q is the scattering wave vector. For the sake of simplicity, we have assumed a one-dimensional system. However, all the following discussions can be straightforwardly generalized to the three-dimensional case. The witness operator for a general N-spin system is defined as

$$W_{CL}(q) = \mathbb{1} - \tfrac{1}{2}[\bar{\Sigma}(q) + \bar{\Sigma}(-q)], \tag{2}$$

where $\mathbb{1}$ is the identity operator and

$$\bar{\Sigma}(q) = \frac{1}{B(N,2)}[c_x S^{xx}(q) + c_y S^{yy}(q) + c_z S^{zz}(q)]. \tag{3}$$

Here $B(N,2)$ is the standard binomial factor and $c_\alpha \in \mathbb{R}, |c_\alpha| \leqslant 1$ for $\alpha = x,y,z$. By means of scattering measurements, it is possible to detect (multipartite) entangled states of many-particle systems [6]. We stress that in Eq. (2) one assumes completely deterministic knowledge of the scatterer's positions. For each constituent the motional degree of freedom is treated as a classical variable and as such it does not affect the measurement statistics. However, in some systems, such as ion traps, not only does this condition not apply, but also entanglement between the fictitious spin $\frac{1}{2}$ of each ion and the collective vibronic state can be routinely generated [15]. Hence, in the following we promote the scatterer's positions to operators. In order to generalize the entanglement witness of Eq. (2) to this case, let us write the spin-density operator $S^\alpha(x)$ along the direction α of an N-particle system as [16]

$$S^\alpha(x) = \sum_{n=1}^{N} S_n^\alpha \otimes \delta(x - \hat{x}_n). \tag{4}$$

In Eq. (4) the Dirac $\delta(\cdot)$ is meant to be an operator in the x representation and $\hat{x}_n$ is the position operator of the nth particle. It is straightforward to show by Fourier transformation that Eq. (1) in this case reads

$$S^{\alpha\beta}(q) = \sum_{n<m} S_n^\alpha S_m^\beta e^{iq(\hat{x}_n - \hat{x}_m)}. \tag{5}$$

This formula defines the structure factor $S^{\alpha\beta}(q)$ of Eq. (1), with the crucial difference that now all the positions are quantized. The generalized entanglement witness $W_Q(q)$ is defined in the same way as in Eqs. (2) and (3), but it is now a function of both spin and position operators. We show that this alternative definition still meets the criteria for an entanglement witness such as the one introduced in [6]. That is, for any state of the composite-system spin position of the form $\sigma^N \otimes \rho$, with $\sigma^N = \sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_N$ and ρ the spatial state of the system, we can see that

$$\begin{aligned} &|\langle \Sigma(q)\rangle_{\sigma^N\otimes\rho}| \\ &\quad = \frac{1}{B(N,2)}\left|\sum_{n<m}\left(\sum_{\alpha=x,y,z} c_\alpha \left\langle S_n^\alpha S_m^\alpha\right\rangle_{\sigma^N}\right)\langle \cos[q(\hat{x}_n-\hat{x}_m)]\rangle_\rho\right| \\ &\quad \leqslant \frac{1}{B(N,2)}\sum_{n<m}\left|\sum_{\alpha=x,y,z} c_\alpha \left\langle S_n^\alpha S_m^\alpha\right\rangle_{\sigma^N}\right| \leqslant 1. \end{aligned} \tag{6}$$

The first inequality above comes from the fact that, whenever we deal with a composite state of the form $\sigma^N \otimes \rho$, the bound $|\langle \cos[q(\hat{x}_n-\hat{x}_m)]\rangle_\rho| \leqslant 1$ holds for any ρ. Hence, the considered witness rules out states of the form $\sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_N \otimes \rho$, namely, states fully separable in the spin and biseparable with respect to the cut spin position. Thus, this witness can detect not only entanglement among spins, but also multipartite entanglement in the composite spin-position system and hybrid entanglement between spins and positions. Unfortunately, position-position entanglement cannot be revealed.

An experimental implementation of the witness (2) via Bragg scattering was proposed in [7] for trapped ions in optical cavities. There the authors showed that by tuning the laser wave vector it is possible to measure $W_{CL}(q)$ for several values of the scattered momentum q. This scheme can be perfectly generalized to the case of ions with quantum mechanical motion with both the collective spin and motional state being encoded in the cavity mode's state.

Spins in a harmonic potential. Let us begin by considering a simple system consisting of two spin-$\frac{1}{2}$ particles (qubit) trapped in a double-well harmonic potential with minima centered in x_A, x_B. Each particle is described by a state of the form $|\uparrow\rangle|f_A\rangle$, where the first ket refers to the spin state (in this case, e.g., the state $|\uparrow\rangle = \sigma^z|\uparrow\rangle$) and the second describes position. For the sake of simplicity, the latter will be represented by the ground state of each harmonic oscillator, that is, a Gaussian wave function centered in $x_{A(B)}$. We investigate how the continuous degrees of freedom affect the detection of the qubit-qubit entanglement and whether the generalized witness can detect any hybrid entanglement between spins and harmonic oscillators. We consider the following states:

$$\begin{aligned} |\psi_1\rangle &= \frac{1}{\sqrt{2}}(|\uparrow,\downarrow\rangle - |\downarrow,\uparrow\rangle), \\ |\psi_2\rangle &= \frac{1}{\sqrt{2}}(|\uparrow,\downarrow\rangle - |\downarrow,\uparrow\rangle)\otimes|f_A,f_B\rangle, \\ |\psi_3\rangle &= \frac{1}{\sqrt{2}}(|\uparrow,\downarrow,f_A,f_B\rangle - |\downarrow,\uparrow,f_B,f_A\rangle), \end{aligned} \tag{7}$$

where $|f_J\rangle = \int dx \sqrt{f_J(x)}|x\rangle$ with $f_J(x) = e^{-(x-x_J)^2/2\sigma^2}/\sqrt{2\pi\sigma^2}$ and $J = A, B$. The above states are representative of several situations we can encounter. In state $|\psi_1\rangle$ the particles are classically localized at a distance $x_A - x_B \equiv r$. Thus, it will serve as a reference state for a comparison with states $|\psi_2\rangle$ and $|\psi_3\rangle$. For state $|\psi_2\rangle$ the spatial part of the wave function is separable from the spin state and thus we expect just a modulation of the expectation value of the witness. Finally, for state $|\psi_3\rangle$ all the degrees of freedom are involved in a nontrivial way. Figure 1 (top) shows the behavior of $\langle W_{CL}\rangle_1$, $\langle W_Q\rangle_2$, and $\langle W_Q\rangle_3$ as a function of the rescaled scattered momentum $\theta \equiv qr$ for the three states above and for $y \equiv r/\sigma = 1.2$. In all of the three cases $c_x = c_y = c_z = -1$. Notice that with this choice of y there is a significant overlap between the two Gaussian states. We see that in all of the three cases some entanglement is present in the system as the witness takes negative values. This can be either entanglement between the two qubits or hybrid entanglement. In particular, $\langle W_Q\rangle_2$ is significantly more negative than $\langle W_Q\rangle_3$. The quantum nature of particle positions results in smearing out the reference curve $\langle W_{CL}\rangle_1$, making spin–spin-entanglement detection harder. For state $|\psi_3\rangle$ the entanglement is instead distributed among all the different degrees of freedom. In order to further study the behavior of $\langle W_Q\rangle_3$ we consider the two extreme cases $y \ll 1$ and $y \gg 1$. Whenever $y \ll 1$ holds the state $|\psi_3\rangle$ approaches $|\psi_2\rangle$ with $|\langle f_A|f_B\rangle| \approx 1$ and thus $\langle W_Q\rangle_3$ tends to $\langle W_Q\rangle_2$. On the other hand, as soon as $y \gg 1$, $\langle W_Q\rangle_2$ will converge to the $\langle W_{CL}\rangle_1$, while the $\langle W_Q\rangle_3$ will become positive for any value of θ, preventing any entanglement detection [see Fig. 1 (bottom)]. Notice that the two-qubit reduced state of $|\psi_3\rangle$ is entangled (even though the amount of entanglement would decrease exponentially in y). Therefore, in this case W_Q fails to reveal both the entanglement between

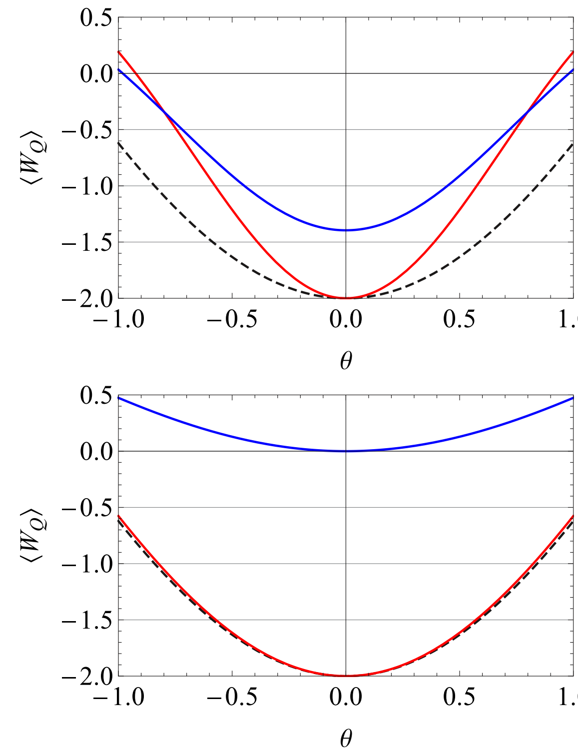

FIG. 1. (Color online) Expectation values of W_{CL} and W_Q as functions of θ for $y = 1.2$ (top) and in the limit $y \gg 1$ (bottom). The dashed black line denotes $\langle W_{CL}\rangle_1$, the red solid line $\langle W_Q\rangle_2$, and the blue solid line $\langle W_Q\rangle_3$.

the two qubits and hybrid entanglement (with the state $|\psi_3\rangle$ being Greenberger-Horne-Zeilinger (GHZ)-like in the limit $|\langle f_A|f_B\rangle| \approx 0$).

Entanglement witnessing in trapped-ion systems. In this section we will derive the specific form of the generalized witness W_Q of Eq. (2), defined via Eq. (5), for trapped-ion systems. This is of great interest for basic studies of entanglement in trapped-ion strings as well as Wigner crystals [17]. Although the following approach is completely general, we restrict our attention to a one-dimensional system. Let us thus imagine that we have a string of N ions of mass m, harmonically confined in a Paul trap and interacting via Coulomb repulsion. In this stable spatial configuration the ions fluctuate around their equilibrium positions. We assume the ion-ion equilibrium distance to be constant, namely, a. This corresponds to considering the central segment of a linear Coulomb crystal [18]. For sufficiently low temperature, such a system can be approximated by a chain of interacting quantum harmonic oscillators whose equilibrium positions can be analytically ($N \leqslant 3$) and numerically computed ($N > 3$) [19]. Furthermore, we can map this interacting system onto a noninteracting one by standard normal-mode transformation. If $\hat{x}_n$ is the position operator of the nth particle then [20]

$$\hat{x}_n \approx an + \sum_{k>0}\sum_{\mu=\pm}\sqrt{\frac{\hbar}{Nm\omega_k}}R_{(n,k,\mu)}(\hat{a}_{k,\mu} + \hat{a}^\dagger_{k,\mu}), \tag{8}$$

where an is the equilibrium position of the nth particle, $\hat{a}_{k,\mu}$ ($\hat{a}^\dagger_{k,\mu}$) is the annihilation (creation) for the kth normal mode with parity $\mu = \pm$ at frequency ω_k, and the real coefficients $R_{(n,k,\mu)}$ are the elements of the normal-mode transformation matrix. By using expansion (8) we can reexpress the structure factor (5) as

$$S^{\alpha\beta}(\theta) = \sum_{n<m} S_n^\alpha S_m^\beta e^{i\theta(n-m)} D[i\theta\vec{\phi}(n,m)], \tag{9}$$

where $D[\alpha] = \exp[\alpha\hat{a}^\dagger - \alpha^*\hat{a}]$ is the displacement operator, $\bigotimes_{k,\mu} D[i\theta\phi_{(k,\mu)}(n,m)] \equiv D[i\theta\vec{\phi}(n,m)]$, $\theta = qa$, and $\phi_{(k,\mu)}(n,m) \equiv \sqrt{\hbar/Nm\omega_k}(R_{(n,k,\mu)} - R_{(m,k,\mu)})$. The witness operator (2) in this case is constructed upon using (9). Whenever the state of the composite system is $\sigma \otimes \rho$, with σ and ρ being the states of the internal degrees of freedom of the ion chain and the external (vibrational) degrees of freedom, respectively, the expectation value of the witness reads

$$\langle W_Q(\theta)\rangle = 1 - \frac{1}{B(N,2)}\sum_{n<m}\sum_{\alpha=x,y,z} c_\alpha\langle\sigma_n^\alpha\sigma_m^\alpha\rangle_\sigma \times \mathrm{Re}\{e^{i\theta(n-m)}\langle D[i\theta\vec{\phi}(n,m)]\rangle_\rho\}. \tag{10}$$

Thus, whenever the collective state of the spins and modes is factorized, the expectation value of the witness W_Q is just modulated according to the characteristic function $\chi_\rho(\alpha) = \langle D(\alpha)\rangle_\rho$ [21] of the phononic bath, as sampled in some specific points of the phase space. Hence, in this scenario the vibrational degrees of freedom simply act as nonclassical noise affecting the spin–spin-entanglement detection. As a very simple but meaningful example let us consider the state

$$\rho = \tfrac{1}{2}(|\uparrow,\downarrow\rangle + |\downarrow,\uparrow\rangle)(\langle\uparrow,\downarrow| + \langle\downarrow,\uparrow|) \otimes \rho_{\pi/a,T}, \tag{11}$$

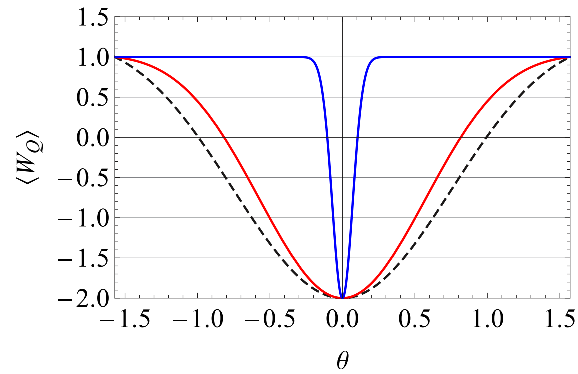

FIG. 2. (Color online) Expectation value of W_Q as a function of θ for $\eta = 1$ and $\Delta = 100$, 1, and 0.01 (dashed black, solid red, and solid blue lines, respectively).

where $\rho_{\pi/a,T}$ is a thermal state at temperature T for the mode at $k = \pi/a$. We study the limit to the entanglement detection caused by the temperature of the vibrational collective mode. The expectation value of $W_Q(\theta)$ is

$$\langle W_Q(\theta)\rangle = 1 - e^{-(1/2)\theta^2\eta^2\coth(\Delta/2)}(c_x + c_y - c_z)\cos\theta, \tag{12}$$

where $\eta = a^{-1/4}(\hbar/Q\sqrt{8m})^{1/2}$, with Q being the atom's electric charge, $\Delta = \hbar\omega_{\pi/a}/k_BT$, and $\omega_{\pi/a}$ is the frequency of the $k = \pi/a$ mode.

The expectation value of the witness in this case is shown in Fig. 2 as a function of the rescaled momentum θ for different values of the energy-temperature ratio $\Delta = \hbar\omega_{\pi/a}/k_BT$. In this plot the values of the parameters are the same as in the experiment reported in [22,23], corresponding to $a \approx 33\ \mu$m for $^{24}Mg^+$ atoms. Figure 2 shows that increasing the temperature results in limiting the entanglement detection of the spin state as the θ range corresponding to $\langle W_Q\rangle < 0$ further shrinks. Notice, furthermore, that from Eq. (12), by taking the limit $T \to 0$ (or, equivalently, $\Delta \gg 1$), we reobtain exactly the expectation value of the witness W_{CL} over the state $\frac{1}{\sqrt{2}}(|\uparrow,\downarrow\rangle + |\downarrow,\uparrow\rangle)$ without vibronic degrees of freedom.

Detection of hybrid entanglement. In this section we study the behavior of the witness W_Q for some relevant quantum states that can be laser generated in trapped ion [15,24]. For the sake of simplicity, we restrict our analysis to a system consisting of two ion's spin $\frac{1}{2}$ coupled to the collective vibrational mode. We show that even in such a simple system hybrid entanglement can be detected for several specific states, at least in principle. We denote by $|n\rangle$ the Fock state with n excitations of the $k = \pi/a$ vibrational mode and by $|\alpha\rangle = e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle$ a coherent state. Let us thus consider the following states of the composite system:

$$\begin{aligned}
|\phi_1\rangle &= \frac{1}{\sqrt{2}}(|\uparrow,\downarrow,\alpha\rangle + |\downarrow,\uparrow,-\alpha\rangle),\\
|\phi_2\rangle &= \frac{1}{\sqrt{2}}(|\uparrow,\downarrow,0\rangle + |\downarrow,\uparrow,1\rangle),\\
\phi_3 &= p|\Psi^+\rangle\langle\Psi^+| \otimes |0\rangle\langle 0| + (1-p)|\downarrow,\downarrow\rangle\\
&\quad\times\ \langle\downarrow,\downarrow| \otimes |1\rangle\langle 1|,
\end{aligned} \tag{13}$$

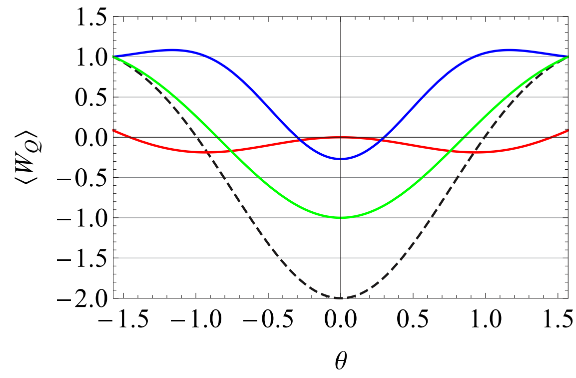

FIG. 3. (Color online) Expectation value of W_Q in terms of θ for the states in Eq. (13): $\langle W_Q\rangle_1$ (with $\alpha = 0$) (dashed black line), $\langle W_Q\rangle_1$ (with $\alpha = 1$) (solid blue line), $\langle W_Q\rangle_2$ (solid red line), and $\langle W_Q\rangle_3$ (solid green line).

where $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|\uparrow,\downarrow\rangle + |\downarrow,\uparrow\rangle)$. The first two are entangled in all the degrees of freedom, while the last state usually results when the spin excitation spontaneously decoheres to the bath. The expectation values of the entanglement witness for the optimal parameters $c_x = c_y = 1$ and $c_z = -1$ are shown in Fig. 3 as functions of the reduced wave vector θ. The dashed black line represents $\langle W_Q\rangle_1$, i.e., the expectation value of W_Q on the state $|\phi_1\rangle$ with $\alpha = 0$. This state is separable with respect to different degrees of freedom, i.e., no hybrid entanglement is present, and the presence of the extra mode results in a modulation of the expectation value according to the characteristic function of the Fock state $|0\rangle$. The expectation value $\langle W_Q\rangle_1$ for $\alpha = 1$ is instead depicted by the solid blue line. Entanglement is still detected, even if for a smaller range of θ, with the blue curve approaching the dashed black one whenever $|\alpha| \ll 1$. Numerical evidence shows that entanglement detection is no longer guaranteed when $|\alpha| > 1$. Thus, large-α coherent states, corresponding to classical light, completely ruin the detection of the entanglement between the two spins $\frac{1}{2}$. We notice that $|\alpha|$ plays here exactly the same role as y in the previous section and so the considerations we made for the state $|\psi_3\rangle$ of Eq. (7) apply. The main feature displayed in Fig. 3 is nevertheless the unexpected behavior of the witness W_Q for the state $|\phi_2\rangle$. In this case two minima are symmetrically located with respect to $\theta = 0$, which instead appears to be the optimal scattering wave vector for all the other states. We notice that in this case, while the state of the total hybrid system is entangled (as also detected by the witness), the reduced spin state is separable. This indicates that the witness operator captures hybrid entanglement between the spins and the oscillatory mode. Finally, the noisy state ϕ_3, modeling the situation where the single excitation of the spins is lost to the oscillatory mode, is detected as long as p is close enough to 1. In fact, for $\theta = 0$ the noise threshold to have entanglement detection is $p \geqslant 1/2$.

Conclusion. In this work we have generalized the class of entanglement witnesses based on a structure factor to the case when the scatterer's position is quantized. To illustrate the consequences of such a generalization we first studied a toy model of two spins in a double-well harmonic potential. Second, we explicitly adapted the witness operator to the case of a chain of trapped ions. In both case studies, we showed that several properties of hybrid systems can be investigated within this framework, spanning from hybrid entanglement detection to spin–spin-entanglement detection disturbance originating from collective quantum motion. The theory presented here could be experimentally implemented via the method proposed in Ref. [7].

Acknowledgments. M.B. would like to acknowledge the Scottish Doctoral Training Centre in Condensed Matter Physics (EPSRC CM DTC) for financial support.

[1] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).
[2] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
[3] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996); B. M. Terhal, *ibid.* **271**, 319 (2000).
[4] O. Gühne, P. Hyllus, D. Bruß, A. Ekert, M. Lewenstein, C. Macchiavello, and A. Sanpera, Phys. Rev. A **66**, 062305 (2002).
[5] M. Barbieri, F. De Martini, G. Di Nepi, P. Mataloni, G. M. D'Ariano, and C. Macchiavello, Phys. Rev. Lett. **91**, 227901 (2003); M. Bourennane, M. Eibl, C. Kurtsiefer, S. Gaertner, H. Weinfurter, O. Gühne, P. Hyllus, D. Bruß, M. Lewenstein, and A. Sanpera, *ibid.* **92**, 087902 (2004).
[6] P. Krammer, H. Kampermann, D. Bruß, R. A. Bertlmann, L. C. Kwek, and C. Macchiavello, Phys. Rev. Lett. **103**, 100502 (2009).
[7] C. Macchiavello and G. Morigi, Phys. Rev. A **87**, 044301 (2013).
[8] A. Chiuri, G. Vallone, N. Bruno, C. Macchiavello, D. Bruß, and P. Mataloni, Phys. Rev. Lett. **105**, 250501 (2010).
[9] M. Cramer, M. B. Plenio, and H. Wunderlich, Phys. Rev. Lett. **106**, 020401 (2011).
[10] M. Cramer, A. Bernard, N. Fabbri, L. Fallani, C. Fort, S. Rosi, F. Caruso, M. Inguscio, and M. B. Plenio, Nat. Commun. **4**, 2161 (2013).
[11] D. Porras and J. I. Cirac, Phys. Rev. Lett. **92**, 207901 (2004).
[12] D. J. Wineland, C. Monroe, W. M. Itano, D. Leibfried, B. E. King, and D. M. Meekhof, J. Res. Natl. Inst. Stand. Technol. **103**, 259 (1998).
[13] G. Breitenbach, S. Schiller, and J. Mlynek, Nature (London) **387**, 471 (1997).
[14] C. J. Hamer, J. Oitmaa, Z. Weihong, and R. H. McKenzie, Phys. Rev. B **74**, 060402 (2006).
[15] J. D. Jost, J. P. Home, J. M. Amini, D. Hanneke, R. Ozeri, C. Langer, J. J. Bollinger, D. Leibfried, and D. J. Wineland, Nature (London) **459**, 683 (2009).
[16] P. M. Chaikin and T. C. Lubensky, *Principles of Condensed Matter Physics* (Cambridge University Press, Cambridge, 2000).
[17] D. H. E. Dubin and T. M. O'Neil, Rev. Mod. Phys. **71**, 87 (1999).
[18] S. Fishman, G. De Chiara, T. Calarco, and G. Morigi, Phys. Rev. B **77**, 064111 (2008).

[19] D. F. V. James, Appl. Phys. B **66**, 181 (1998).

[20] G. De Chiara, T. Calarco, S. Fishman, and G. Morigi, Phys. Rev. A **78**, 043414 (2008).

[21] G. Christopher and P. Knight, *Introductory Quantum Optics* (Cambridge University Press, Cambridge, 2005).

[22] G. Birkl, S. Kassner, and H. Walther, Nature (London) **357**, 310 (1992).

[23] I. Waki, S. Kassner, G. Birkl, and H. Walther, Phys. Rev. Lett. **68**, 2007 (1992).

[24] D. Leibfried, R. Blatt, C. Monroe, and D. Wineland, Rev. Mod. Phys. **75**, 281 (2003).