Hyperfine structure of S states in muonic deuterium

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On the basis of the quasipotential method in quantum electrodynamics we calculate corrections of orders α^5 and α^6 to the hyperfine structure of S-wave energy levels of muonic deuterium. Relativistic corrections, effects of vacuum polarization in first, second and third orders of perturbation theory, nuclear structure, and recoil corrections are taken into account. The obtained numerical values of hyperfine splitting $\Delta E^{\rm HFS}(1S) = 50.2814$ meV ($1S$ state) and $\Delta E^{\rm HFS}(2S) = 6.2804$ meV ($2S$ state) represent reliable estimates for a comparison with forthcoming experimental data of the CREMA Collaboration. The hyperfine structure interval $\Delta_{12} = 8\Delta E^{\rm HFS}(2S) - \Delta E^{\rm HFS}(1S) = -0.0379$ meV can be used for a precision check of the quantum electrodynamics prediction for muonic deuterium.

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I. INTRODUCTION

In recent years considerable interest in the investigation of fine and hyperfine energy structures of simple atoms has been related to light muonic atoms: muonic hydrogen, muonic deuterium, and ions of muonic helium. This is the result of essential progress achieved by the experimental collaboration CREMA (Charge Radius Experiment with Muonic Atoms) in the study of such muonic atoms [1,2]. Thus, for example, in the measurement of the transition frequency $2P_{3/2}^{F=2} - 2S_{1/2}^{F=1}$ in muonic hydrogen a more precise value of the proton charge radius, $r_p = 0.84087(39)$ fm, has been obtained. The measurement of the transition frequency $2P_{3/2}^{F=1} - 2S_{1/2}^{F=0}$ for the singlet $2S$ state allowed the hyperfine splitting (HFS) of the $2S$ energy level in muonic hydrogen, the value of the Zemach radius $r_Z = 1.082(37)$ fm, and the magnetic radius $r_M = 0.87(6)$ fm to be found. Analogous measurements for muonic deuterium have also been completed and are planned for a publication. It is necessary to point out that experiments of the CREMA Collaboration propose one important task to improve by orders of magnitude the values of charge radii in these simple atoms (protons, deuterons, helions, and α particles) that enter theoretical expressions for different fine structure intervals. For successful implementation of this program, theoretical calculations of different order corrections to the fine structure and the hyperfine structure of muonic atoms have a significant importance [3–8]. Of special note are nuclear structure corrections that can be responsible for the solution of the proton radius puzzle [1,9].

Theoretical investigations of the energy levels of light muonic atoms were carried out many years ago [3–6,8] on the basis of the Dirac equation and nonrelativistic approach by perturbation theory in quantum electrodynamics. Experimental activity in past years generates a need to analyze again previous calculations in order to obtain reliably basic energy intervals: the Lamb shift ($2P_{1/2} - 2S_{1/2}$), the hyperfine structure of the $2S$ state, and the fine structure ($2P_{1/2} - 2P_{3/2}$), which could be measured in the CREMA experiments in the first place. The aim of our work consists of performing new investigations of contributions α^5 and α^6 to the hyperfine structure of muonic deuterium which are determined by the effects of vacuum polarization, recoil, and relativistic and deuteron structure corrections. Modern numerical values of fundamental physical constants are taken from Ref. [10]: the electron mass $m_e = 0.510\,998\,928(11) \times 10^{-3}$ GeV, the muon mass $m_\mu = 0.105\,658\,371\,5(35)$ GeV, the fine structure constant $\alpha^{-1} = 137.035\,999\,074(44)$, the proton mass $m_p = 0.938\,272\,046(21)$ GeV, the deuteron mass $m_2 = 1.875\,612\,859(41)$ GeV, the deuteron magnetic moment $\mu_d = 0.857\,438\,230\,8(72)$ in nuclear magnetons, and the muon anomalous magnetic moment $a_\mu = 1.165\,920\,91(63) \times 10^{-3}$.

In our calculation we use the quasipotential method in quantum electrodynamics as applied to particle bound states [11], where the two-particle bound state is described by the Schrödinger equation. The basic contribution to the muon-deuteron interaction operator in the S state is determined by the Breit Hamiltonian [12]:

$$H_B = H_0 + \Delta V_B^{\rm fs} + \Delta V_B^{\rm HFS}, \quad H_0 = \frac{\mathbf{p}^2}{2\mu} - \frac{Z\alpha}{r}, \tag{1}$$

$$\Delta V_B^{\rm fs} = -\frac{\mathbf{p}^4}{8m_1^3} - \frac{\mathbf{p}^4}{8m_2^3} + \frac{\pi Z\alpha}{2}\left(\frac{1}{m_1^2} + \frac{\delta_I}{m_2^2}\right)\delta(\mathbf{r}) - \frac{Z\alpha}{2m_1m_2r}\left(\mathbf{p}^2 + \frac{\mathbf{r}(\mathbf{r}\mathbf{p})\mathbf{p}}{r^2}\right), \tag{2}$$

$$\Delta V_B^{\rm HFS}(r) = \frac{2\pi\alpha}{3m_1m_p} g_d g_\mu(\mathbf{s}_1\mathbf{s}_2)\delta(\mathbf{r}), \tag{3}$$

where m_1 and m_2 are the muon and deuteron masses, respectively, m_p is the proton mass, and g_d and g_μ are gyromagnetic factors of the deuteron and muon, respectively.

The deuteron factor $\delta_I = 0$, because we use the following definition of the deuteron charge radius: $r_d^2 = -6\frac{dF_C}{dQ^2}|_{Q^2=0}$ [13,14]. Then the basic contribution to the hyperfine splitting of S-wave levels (the Fermi energy) is given by the spin-spin interaction part of the potential (3). Averaging Eq. (3) over the Coulomb wave functions of the $1S$ and $2S$ states,

$$\psi_{100}(r) = \frac{W^{3/2}}{\sqrt{\pi}}e^{-Wr}, \quad W = \mu Z\alpha, \tag{4}$$

$$\psi_{200}(r) = \frac{W^{3/2}}{2\sqrt{2\pi}}e^{-Wr/2}\left(1 - \frac{Wr}{2}\right), \tag{5}$$

we obtain the following results for the leading-order contribution to hyperfine splitting:

$$E_F(nS) = \frac{2\mu^3\alpha^4\mu_d}{m_1 m_p n^3} = \begin{cases} 1S\text{: } 49.0875 \text{ meV} \\ 2S\text{: } 6.1359 \text{ meV} \end{cases}. \tag{6}$$

The Fermi energy (6) does not contain a contribution of the muon anomalous magnetic moment (AMM). The muon AMM correction to hyperfine splitting can be presented separately taking an experimental value of the muon AMM [10]:

$$\Delta E^{\rm HFS}_{a_\mu}(nS) = a_\mu E_F(nS) = \begin{cases} 1S\text{: } 0.0572 \text{ meV} \\ 2S\text{: } 0.0072 \text{ meV} \end{cases}. \tag{7}$$

The numerical value of the relativistic correction of order α^6 to HFS can be obtained by means of known analytical expression from [8,15]:

$$\Delta E^{\rm HFS}_{\rm rel}(nS) = \begin{cases} \frac{3}{2}(Z\alpha)^2 E_F(1S) \\ \frac{17}{8}(Z\alpha)^2 E_F(2S) \end{cases} = \begin{cases} 1S\text{: } 0.0039 \text{ meV} \\ 2S\text{: } 0.0007 \text{ meV} \end{cases}. \tag{8}$$

In what follows we investigate other important contributions to the HFS of S-wave energy levels in order to obtain a reliable total result. Numerical values of different corrections are presented for definiteness with the accuracy 10^{-4} meV.

II. EFFECTS OF ONE- AND TWO-LOOP VACUUM POLARIZATION IN FIRST-ORDER PERTURBATION THEORY

First of all, we should analyze the contribution of one-loop vacuum polarization to the potential, which is determined in coordinate representation as follows [4,12]:

$$\Delta V^{\rm HFS}_{1\gamma,\rm VP}(r) = \frac{4\alpha\mu_d(1+a_\mu)}{3m_1 m_p}(\mathbf{s}_1\mathbf{s}_2)\frac{\alpha}{3\pi} \times \int_1^\infty \rho(s)ds\left(\pi\delta(\mathbf{r}) - \frac{m_e^2\xi^2}{r}e^{-2m_e\xi r}\right), \tag{9}$$

where the spectral function $\rho(\xi) = \sqrt{\xi^2-1}(2\xi^2+1)/\xi^4$. For its derivation a replacement in the photon propagator is used:

$$\frac{1}{k^2} \to \frac{\alpha}{3\pi}\int_1^\infty \rho(\xi)d\xi\frac{1}{k^2+4m_e^2\xi^2}. \tag{10}$$

We also preserve a factor with the muon AMM that leads to the accounting effectively of a correction of order α^6. Averaging Eq. (9) over wave functions (4) and (5), we obtain the contribution of order α^5 to the hyperfine structure of the $1S$ and $2S$ states:

$$\Delta E^{\rm HFS}_{1\gamma,\rm VP}(1S) = \frac{2\mu^3\alpha^5\mu_d(1+a_\mu)}{3m_1 m_p\pi}\int_1^\infty \rho(\xi)d\xi\left[1 - \frac{m_e^2\xi^2}{W^2}\int_0^\infty xdxe^{-x(1+\frac{m_e\xi}{W})}\right] = 0.1039 \text{ meV}, \tag{11}$$

$$\Delta E^{\rm HFS}_{1\gamma,\rm VP}(2S) = \frac{\mu^3\alpha^5\mu_d(1+a_\mu)}{3m_1 m_p\pi}\int_1^\infty \rho(\xi)d\xi\left[1 - \frac{4m_e^2\xi^2}{W^2}\int_0^\infty x\left(1-\frac{x}{2}\right)^2dxe^{-x(1+\frac{2m_e\xi}{W})}\right] = 0.0134 \text{ meV}. \tag{12}$$

Changing the electron mass m_e to muon mass m_1 in Eqs. (11) and (12), the muon vacuum polarization contribution to HFS can be found: 0.0009 meV ($1S$) and 0.0001 meV ($2S$). It has a higher-order α^6 because the ratio $W/m_1 \ll 1$ and is included in Table I. The same order α^6 contribution is given also by two-loop vacuum polarization diagrams [see Figs. 1(b)–1(d)] (the Källen and Sabry potential [16]). In order to obtain the interaction operator for the amplitude with two sequential loops [Fig. 1(b)], it is necessary to use twice the replacement (10). Thus in coordinate space the potential takes the following form:

$$\Delta V^{\rm HFS}_{1\gamma,\rm VP-VP}(r) = \frac{8\pi\alpha\mu_d(1+a_\mu)}{3m_1 m_p}(\mathbf{s}_1\mathbf{s}_2)\left(\frac{\alpha}{3\pi}\right)^2\int_1^\infty \rho(\xi)d\xi\int_1^\infty \rho(\eta)d\eta\left[\delta(\mathbf{r}) - \frac{m_e^2}{\pi r(\eta^2-\xi^2)}(\eta^4 e^{-2m_e\eta r} - \xi^4 e^{-2m_e\xi r})\right]. \tag{13}$$

The corresponding correction to the HFS of levels $1S$ and $2S$ can be presented first in integral form over coordinate r and spectral parameters ξ and η. After that the integration over r can be done analytically and two other integrations can be done numerically with the use of the system *Mathematica*. The two-loop vacuum polarization correction of order α^6 in Figs. 1(c) and 1(d) can be calculated similarly. In this case the potential of the muon-deutron interaction is determined by a more complicated expression:

$$\Delta V^{\rm HFS}_{1\gamma,2-\rm loop\ VP}(r) = \frac{8\alpha^3\mu_d(1+a_\mu)}{9\pi^2 m_1 m_p}(\mathbf{s}_1\mathbf{s}_2)\int_0^1 \frac{f(v)dv}{1-v^2}\left[\pi\delta(\mathbf{r}) - \frac{m_e^2}{r(1-v^2)}e^{-\frac{2m_e r}{\sqrt{1-v^2}}}\right], \tag{14}$$

where the two-loop spectral function

$$f(v) = v\left\{(3-v^2)(1+v^2)\left[Li_2\left(-\frac{1-v}{1+v}\right) + 2Li_2\left(\frac{1-v}{1+v}\right) + \frac{3}{2}\ln\frac{1+v}{1-v}\ln\frac{1+v}{2} - \ln\frac{1+v}{1-v}\ln v\right] + \left[\frac{11}{16}(3-v^2)(1+v^2) + \frac{v^4}{4}\right]\ln\frac{1+v}{1-v} + \left[\frac{3}{2}v(3-v^2)\ln\frac{1-v^2}{4} - 2v(3-v^2)\ln v\right] + \frac{3}{8}v(5-3v^2)\right\}. \tag{15}$$

TABLE I. Hyperfine structure of S states of muonic deuterium.

Contribution to hyperfine splitting	$1S$ (meV)	$2S$ (meV)	Equation, reference
Contribution of order α^4, the Fermi energy	49.0875	6.1359	(6), [3]
Muon AMM contribution	0.0572	0.0072	(7), [3]
Relativistic correction of order α^6	0.0039	0.0007	(8), [15], [3]
Vacuum polarization contribution of order α^5	0.3095	0.0341	(11), (12), (22), (23), [3]
Vacuum polarization contribution of order α^6	0.0048	0.0005	(16), (27)–(32)
Vacuum polarization contribution of order α^6 in third order PT	0.0005	0.00004	(33)
Nuclear structure correction of order α^5	-0.9305 ± 0.0090	-0.1163 ± 0.0010	(46)
Nuclear structure and vacuum polarization correction of order α^6	0.0152 ± 0.0001	0.0019 ± 0.00001	(47)
Nuclear structure and muon vacuum polarization correction of order α^6	0.0015	0.0002	(48)
Hadron vacuum polarization contribution of order α^6	0.0018	0.0002	(50), [22]
Nuclear structure correction of order α^6 in 1γ interaction	0.0082	0.0008	(55)
Nuclear structure correction in second order perturbation theory	-0.0555	-0.0069	(58)–(59)
Radiative nuclear finite size correction of order α^6	-0.0039	-0.0005	(71)–(74)
Deuteron polarizability contribution of order α^5	1.6972 ± 0.0340	0.2121 ± 0.0042	[5]
Deuteron internal polarizability contribution of order α^5	0.0840 ± 0.0210	0.0105 ± 0.0025	[19]
Weak interaction contribution	0	0	[8,30]
Summary contribution	50.2814 ± 0.0410	6.2804 ± 0.0050	

Here $Li_2(z)$ is the Euler dilogarithm. Numerical corrections of the operator (14) to the energy spectrum are evaluated as in the case of Eq. (13). Summary corrections from the potentials (13) and (14) are equal

$$\Delta E^{\rm HFS}_{1\gamma,\rm VP,VP}(nS) = \begin{cases} 1S\text{: } 0.0005 \text{ meV} \\ 2S\text{: } 0.00006 \text{ meV}\end{cases}. \tag{16}$$

One-loop and two-loop contributions of orders α^5 and α^6 to the HFS should be considered also in second-order perturbation theory.

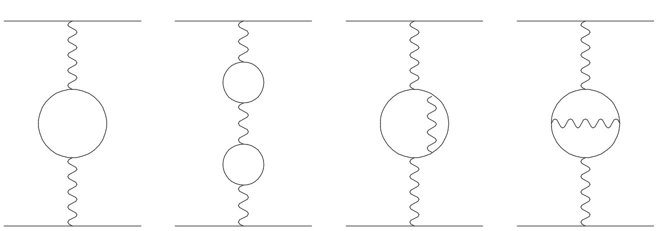


FIG. 1. Effects of one- and two-loop vacuum polarization in a one-photon interaction.

III. EFFECTS OF ONE- AND TWO-LOOP VACUUM POLARIZATION IN SECOND- AND THIRD-ORDER PERTURBATION THEORY

The second-order perturbation theory (PT) corrections to the energy spectrum are determined by the reduced Coulomb Green's function $\tilde{G}$, which has the following partial expansion:

$$\tilde{G}_n(\mathbf{r},\mathbf{r}') = \sum_{l,m} \tilde{g}_{nl}(r,r')Y_{lm}(\mathbf{n})Y^*_{lm}(\mathbf{n}'). \tag{17}$$

The radial function $\tilde{g}_{nl}(r,r')$ was obtained in Ref. [17] in the form of the Sturm expansion in the Laguerre polynomials. The main contribution of the electron vacuum polarization to the HFS in second-order PT has the following form [see Fig. 2(a)]:

$$\Delta E^{\rm HFS}_{\rm SOPTVP1} = 2\langle\psi|\Delta V^C_{\rm VP}\cdot\tilde{G}\cdot\Delta V^{\rm HFS}_B|\psi\rangle, \tag{18}$$

with the modified Coulomb potential

$$\Delta V^C_{\rm VP}(r) = \frac{\alpha}{3\pi}\int_1^\infty \rho(\xi)d\xi\left(-\frac{Z\alpha}{r}\right)e^{-2m_e\xi r}. \tag{19}$$

Since $\Delta V^{\rm HFS}_B(r)$ is proportional to $\delta(\mathbf{r})$, it is necessary to use the reduced Coulomb Green's function with one zero argument. For this case it was obtained on the basis of the Hostler representation after a subtraction of the pole term and

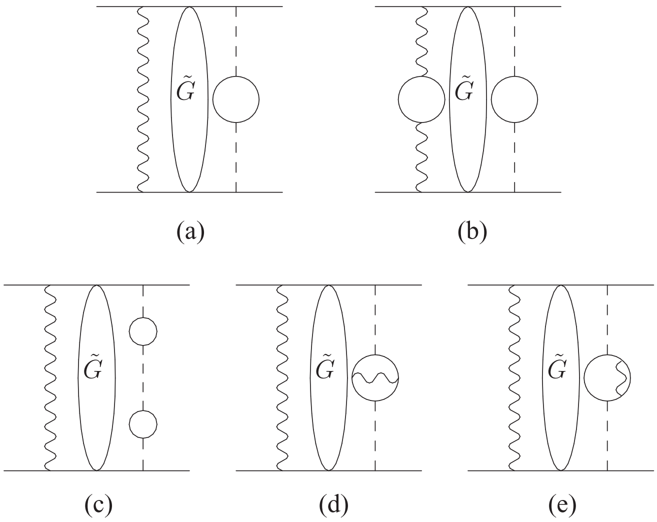


FIG. 2. Effects of one- and two-loop vacuum polarization in second-order perturbation theory. The dashed lines denote the Coulomb photons. $\tilde{G}$ is the reduced Coulomb Green's function.

has the form [17]

$$\tilde{G}_{1S}(\mathbf{r},0) = \frac{Z\alpha\mu^2}{4\pi}\frac{e^{-x}}{x}g_{1S}(x),$$
$$g_{1S}(x) = [4x(\ln 2x + C) + 4x^2 - 10x - 2], \quad (20)$$
$$\tilde{G}_{2S}(\mathbf{r},0) = -\frac{Z\alpha\mu^2}{4\pi}\frac{e^{-x/2}}{2x}g_{2S}(x),$$
$$g_{2S}(x) = [4x(x-2)(\ln x + C) + x^3 - 13x^2 + 6x + 4], \quad (21)$$

where $C = 0.5772\ldots$ is the Euler constant and $x = Wr$. As a result necessary corrections to the HFS of (μd) can be presented as follows:

$$\Delta E^{\rm HFS}_{\rm VP1}(1S) = -E_F(1S)\frac{\alpha}{3\pi}(1+a_\mu)\int_1^\infty \rho(\xi)d\xi \int_0^\infty e^{-x(1+\frac{m_e\xi}{W})} \times g_{1S}(x)dx = 0.2056 \text{ meV}, \quad (22)$$

$$\Delta E^{\rm HFS}_{\rm VP1}(2S) = E_F(2S)\frac{\alpha}{3\pi}(1+a_\mu)\int_1^\infty \rho(\xi)d\xi \int_0^\infty e^{-x(1+\frac{2m_e\xi}{W})} \times g_{2S}(x)\left(1-\frac{x}{2}\right)dx = 0.0207 \text{ meV}. \quad (23)$$

The factor $(1 + a_\mu)$ is included in Eqs. (22) and (23); therefore these expressions contain corrections of orders α^5 and α^6. Changing $m_e \to m_1$ in Eqs. (22) and (23) we calculate the one-loop muon vacuum polarization contributions in second-order PT of order α^6: 0.0009 meV ($1S$) and 0.0001 meV ($2S$). These are included also in Table I together with similar contributions in first-order PT. The two-loop corrections in Figs. 2(b)–2(e) are of order α^6. Let us consider first the contribution which is related to potentials (9) and (19), reduced Coulomb Green's functions (20) and (21), and reduced Coulomb Green's function with nonzero arguments. The general structure of this contribution takes the following form:

$$\Delta E^{\rm HFS}_{\rm SOPTVP2} = 2\langle\psi|\Delta V^{\rm HFS}_{1\gamma,{\rm VP}}\cdot\tilde{G}\cdot\Delta V^C_{\rm VP}|\psi\rangle. \quad (24)$$

The convenient representation for the reduced Coulomb Green's function with nonzero arguments was obtained in Ref. [17]:

$$\tilde{G}_{1S}(r,r') = -\frac{Z\alpha\mu^2}{\pi}e^{-(x_1+x_2)}g_{1S}(x_1,x_2),$$
$$g_{1S}(x_1,x_2) = \frac{1}{2x_<} - \ln 2x_> - \ln 2x_< + Ei(2x_<) + \frac{7}{2} - 2C - (x_1+x_2) + \frac{1-e^{2x_<}}{2x_<}, \quad (25)$$
$$\tilde{G}_{2S}(r,r') = -\frac{Z\alpha\mu^2}{16\pi x_1 x_2}e^{-(x_1+x_2)}g_{2S}(x_1,x_2),$$
$$g_{2S}(x_1,x_2) = 8x_< - 4x_<^2 + 8x_> + 12x_<x_> - 26x_<^2x_> + 2x_<^3x_> - 4x_>^2 - 26x_<x_>^2 + 23x_<^2x_>^2 - x_<^3x_>^2 + 2x_<x_>^3 - x_<^2x_>^3 + 4e^x(1-x_<)(x_>-2)x_> + 4(x_<-2)x_<(x_>-2)x_> \times[-2C + Ei(x_<) - \ln(x_<) - \ln(x_>)]. \quad (26)$$

The substitution of Eqs. (9), (19), (25), and (26) into Eq. (24) provides two contributions for each $1S$ and $2S$ level in integral form:

$$\Delta E^{\rm HFS}_{\rm VP21}(1S) = -\frac{2\alpha^6\mu^3\mu_d(1+a_\mu)}{9\pi^2 m_1 m_p}\int_1^\infty \rho(\xi)d\xi\int_1^\infty \rho(\eta)d\eta\int_0^\infty dx e^{-x(1+\frac{m_e\xi}{W})}g_{1S}(x), \quad (27)$$

$$\Delta E^{\rm HFS}_{\rm VP22}(1S) = -\frac{4\alpha^6\mu^3\mu_d(1+a_\mu)m_e^2}{9\pi^2 m_1 m_p W^2}\int_1^\infty \rho(\xi)d\xi\int_1^\infty \rho(\eta)\eta^2 d\eta\int_0^\infty x_1dx_1 e^{-x_1(1+\frac{m_e\xi}{W})}\int_0^\infty x_2dx_2e^{-x_2(1+\frac{m_e\xi}{W})}g_{1S}(x_1,x_2), \quad (28)$$

$$\Delta E^{\rm HFS}_{\rm VP21}(2S) = \frac{\alpha^6\mu^3\mu_d(1+a_\mu)}{36\pi^2 m_1 m_p}\int_1^\infty \rho(\xi)d\xi\int_1^\infty \rho(\eta)d\eta\int_0^\infty\left(1-\frac{x}{2}\right)dx e^{-x(1+\frac{2m_e\xi}{W})}g_{2S}(x), \quad (29)$$

$$\Delta E^{\rm HFS}_{\rm VP22}(2S) = -\frac{\alpha^6\mu^3\mu_d(1+a_\mu)m_e^2}{18\pi^2 m_1 m_p W^2}\int_1^\infty \rho(\xi)d\xi\int_1^\infty \rho(\eta)\eta^2 d\eta\int_0^\infty\left(1-\frac{x_1}{2}\right)dx_1 e^{-x_1(1+\frac{2m_e\xi}{W})} \times\int_0^\infty\left(1-\frac{x_2}{2}\right)dx_2e^{-x_2(1+\frac{2m_e\xi}{W})}g_{2S}(x_1,x_2). \quad (30)$$

Separately, the contributions (27) and (28) and contributions (29) and (30) are divergent but their sum is finite. Corresponding numerical values are presented in Table I. The contributions of two other diagrams to the HFS can be calculated by means of Eq. (18), where the replacement of the potential (19) on the following potentials should be made [7]:

$$\Delta V^C_{\text{VP-VP}}(r) = \left(\frac{\alpha}{3\pi}\right)^2 \int_1^\infty \rho(\xi)d\xi \int_1^\infty \rho(\eta)d\eta \left(-\frac{Z\alpha}{r}\right) \times \frac{1}{\xi^2-\eta^2}(\xi^2 e^{-2m_e\xi r} - \eta^2 e^{-2m_e\eta r}), \quad (31)$$

$$\Delta V^C_{\text{2-loopVP}}(r) = -\frac{2Z\alpha^3}{3\pi^2 r}\int_0^1 \frac{f(v)dv}{(1-v^2)} e^{-\frac{2m_e r}{\sqrt{1-v^2}}}. \quad (32)$$

Omitting further intermediate expressions we include in Table I numerical values of corrections from potentials (29) and (30). Besides vacuum polarization corrections there are a number of nuclear structure and recoil contributions playing essential roles in hyperfine splitting.

In third-order PT there is a correction of order α^6 to the hyperfine splitting which is represented symbolically in Fig. 3 (see Ref. [18]). The initial expression for it is the following:

$$\Delta E^{\text{HFS}}_{\text{TOPT}} = \langle\psi_n|\Delta V^C_{\text{VP}}\cdot\tilde G\cdot\Delta V^{\text{HFS}}\cdot\tilde G\cdot\Delta V^C_{\text{VP}}|\psi_n\rangle + 2\langle\psi_n|\Delta V^C_{\text{VP}}\cdot\tilde G\cdot\Delta V^C_{\text{VP}}\cdot\tilde G\cdot\Delta V^{\text{HFS}}|\psi_n\rangle - \langle\psi_n|\Delta V^{\text{HFS}}|\psi_n\rangle\langle\psi_n|\Delta V^C_{\text{VP}}\cdot\tilde G\cdot\tilde G\cdot\Delta V^C_{VP}|\psi_n\rangle - 2\langle\psi_n|\Delta V^C_{\text{VP}}|\psi_n\rangle\langle\psi_n|\Delta V^C_{\text{VP}}\cdot\tilde G\cdot\tilde G\cdot\Delta V^{\text{HFS}}|\psi_n\rangle. \quad (33)$$

The integration over one coordinate in Eq. (33) is performed analytically, but the second coordinate integration and two integrations over spectral parameters are calculated numerically. Numerically contribution (33) is essentially smaller than other corrections of order α^6 (see Table I where it is written on a separate line).

IV. NUCLEAR STRUCTURE AND RECOIL CORRECTIONS

The basic nuclear structure contribution to the HFS of S states is determined by two-photon exchange diagrams (see Fig. 4). The deuteron electromagnetic current parametrization takes the following form:

$$J^\mu_d(p_2,q_2) = \varepsilon^*_\rho(q_2)\left\{\frac{(p_2+q_2)_\mu}{2m_2}g_{\rho\sigma}F_1(k^2) - \frac{(p_2+q_2)_\mu}{2m_2}\times\frac{k_\rho k_\sigma}{2m_2^2}F_2(k^2) - \Sigma^{\mu\nu}_{\rho\sigma}\frac{k^\nu}{2m_2}F_3(k^2)\right\}\varepsilon_\sigma(p_2), \quad (34)$$

where p_2 and q_2 are four-momenta of the deuteron in the initial and final states, and $k = q_2 - p_2$. The deuteron polarization vector ε_μ satisfies the following conditions:

$$\varepsilon^*_\mu(\mathbf{k},\lambda)\varepsilon^\mu(\mathbf{k},\lambda') = -\delta_{\lambda\lambda'}, \quad k_\mu\varepsilon^\mu(\mathbf{k},\lambda) = 0,$$
$$\sum_\lambda \varepsilon^*_\mu(\mathbf{k},\lambda)\varepsilon_\nu(\mathbf{k},\lambda) = -g_{\mu\nu} + \frac{k_\mu k_\nu}{m_2^2}. \quad (35)$$

The generator of infinitesimal Lorentz transformations is given by

$$\Sigma^{\mu\nu}_{\rho\sigma} = g^\mu_\rho g^\nu_\sigma - g^\mu_\sigma g^\nu_\rho. \quad (36)$$

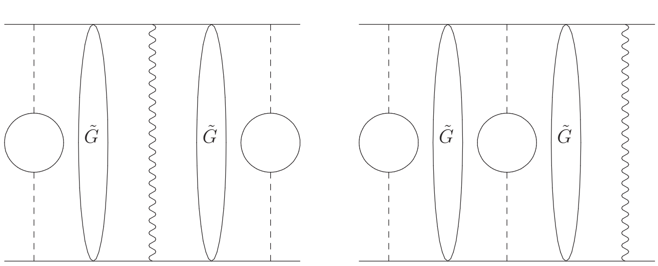
The deuteron electromagnetic form factors $F_i(k^2)$ are functions of the square of the photon four-momentum. They are related to the charge F_C, magnetic F_M, and quadrupole F_Q form factors for the deuteron by the equations

$$F_C = F_1 + \frac{2}{3}\eta[F_1 + (1+\eta)F_2 - F_3], \quad F_M = F_3,$$
$$F_Q = F_1 + (1+\eta)F_2 - F_3, \; \eta = -\frac{k^2}{4m_2^2}. \quad (37)$$

The muon electromagnetic current has the form

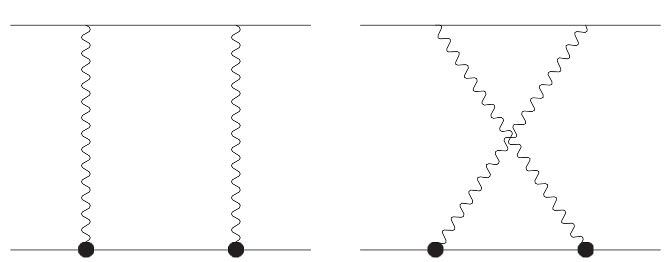
$$J^\mu_l(p_1,q_1) = \bar u(q_1)\left[\frac{(p_1+q_1)^\mu}{2m_1} - (1+a_\mu)\sigma^{\mu\nu}\frac{k_\nu}{2m_1}\right]u(p_1), \quad (38)$$

where p_1 and q_1 are the initial and final muon four-momenta, and $\sigma^{\mu\nu} = (\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)/2$. The amplitudes describing the virtual Compton scattering of a muon and a deuteron are defined by direct and crossed two-photon diagrams in the



(a) (b)

FIG. 3. Effects of vacuum polarization in third-order PT. The dashed lines denote the Coulomb photons. $\tilde G$ is the reduced Coulomb Green's function.



(a) (b)

FIG. 4. Nuclear structure effects of order α^5. The bold points denote the deuteron vertex functions.

following forms [19]:

$$M^{(l)}_{\mu\nu}=\bar u(q_1)\left[\gamma_\mu\frac{\hat p_1+\hat k+m_1}{(p_1+k)^2-m_1^2}\gamma_\nu+\gamma_\nu\frac{\hat p_1-\hat k+m_1}{(p_1-k)^2-m_1^2}\gamma_\mu\right]u(p_1), \quad (39)$$

$$M^{(d)}_{\mu\nu}=\varepsilon^*_\rho(q_2)\left[\frac{(q_2+p_2-k)_\mu}{2m_2}g_{\rho\lambda}F_1-\frac{(q_2+p_2-k)_\mu}{2m_2}\frac{k_\rho k_\lambda}{2m_2^2}F_2-\Sigma^{\mu\alpha}_{\rho\lambda}\frac{k_\alpha}{2m_2}F_3\right]$$
$$\times\frac{-g_{\lambda\omega}+\frac{(p_2-k)_\lambda(p_2-k)_\omega}{m_2^2}}{(p_2-k)^2-m_2^2}\left[\frac{(p_2+q_2-k)_\nu}{2m_2}g_{\omega\sigma}F_1-\frac{(p_2+q_2-k)_\nu}{2m_2}\frac{k_\omega k_\sigma}{2m_2^2}F_2+\Sigma^{\nu\beta}_{\omega\sigma}\frac{k_\beta}{2m_2}F_3\right]\varepsilon_\sigma(p_2). \quad (40)$$

To construct the quasipotential of hyperfine splitting we introduce the special spin projection operators $\hat\pi_{\mu,3/2}$ and $\hat\pi_{\mu,1/2}$ in states with the muon-deuteron pair spin 3/2 and 1/2:

$$\hat\Pi_{\mu,3/2}=[u(p_1)\epsilon_\mu(p_2)]_{3/2}=\Psi_\mu(P),\ \ \hat\Pi_{\mu,1/2}=\frac{i}{\sqrt3}\gamma_5\left(\gamma_\mu-v_{1,\mu}\right)\Psi(P), \quad (41)$$

$$\sum_\lambda\Psi^\lambda_\mu(P)\bar\Psi^\lambda_\nu(P)=-\frac{\hat v_1+1}{2}\left(g_{\mu\nu}-\frac13\gamma_\mu\gamma_\nu-\frac23v_{1,\mu}v_{1,\nu}+\frac13(v_{1,\mu}\gamma_\nu-v_{1,\nu}\gamma_\mu)\right), \quad (42)$$

where the spin-vector $\Psi_\mu(P)$ and the spinor $\Psi(P)$ describe bound states of the muon and deuteron with spins 3/2 and 1/2, $v_{1,\mu}=P_\mu/M$, $P=p_1+p_2$, and $M=m_1+m_2$. Multiplying amplitudes (39) and (40) and introducing projection operators (41), we obtain by means of the package FORM [20] the expression for the HFS part of the potential of the two-photon interaction in the Coulomb gauge for exchanged photons [19]:

$$V^{\rm HFS}_{2\gamma,{\rm str}}=(Z\alpha)^2\int\frac{id^4k}{\pi^2}\frac{1}{(k^2)^2}\frac{1}{k^4-4k_0^2m_1^2}\frac{1}{k^4-4k_0^2m_2^2}$$
$$\times\left\{2F_1F_3k^6\left(\frac{k^2}{m_2^2}-\frac{\mathbf{k}^2}{m_2^2}-4\right)+2F_2F_3\frac{k^4}{m_2^2}\left(4k_0^4+\mathbf{k}^4-4\mathbf{k}^2k_0^2-\frac{k^6}{m_2^2}\right)+2F_3^2k^2\mathbf{k}^2\left(k_0^2+\frac{k^4}{m_2^2}\right)\right\}. \quad (43)$$

The infrared divergence in Eq. (43) at $k\to0$ is related to the term $\sim F_1F_3k^2$. It can be eliminated by means of the iteration term of the quasipotential:

$$\Delta V^{\rm HFS}_{\rm iter}=[V_{1\gamma}G^fV_{1\gamma}]^{\rm HFS}=E_F\frac{16\mu\alpha}{3\pi n^3}(\mathbf{S}_1\mathbf{S}_2)\int_0^\infty\frac{dk}{k^2}F_1F_3. \quad (44)$$

The angle integration in Eq. (43) in the Euclidean momentum space can be carried out analytically. As a result the contribution of two-photon exchange amplitudes to the HFS of S levels can be written in the form of a one-dimensional integral:

$$E^{\rm HFS}_{2\gamma}=E_F\alpha\int_0^\infty\frac{dk}{k^2}V_{2\gamma}(k)$$
$$=\frac{E_F\alpha}{16\pi m_1^3m_2^5(m_1^2-m_2^2)}\int_0^\infty\frac{dk}{k^2}\{4m_1^2m_2^2F_1F_3[k^5(m_2^2-m_1^2)+8k^2m_1^2m_2^2(h_2-h_1)$$
$$+16m_1^2m_2^4(h_2-h_1)-32m_1^2m_2^4(m_2-m_1)+k^4(m_1^2h_2-m_2^2h_1)]+F_2F_3k^2[k^5(m_2^4-m_1^4)+6k^3m_1^2m_2^2(m_1^2-m_2^2)$$
$$+8k^2m_1^2m_2^2[m_1^2(h_2-2h_1)+m_2^2h_1]+16m_1^4m_2^4(h_2-h_1)+k^4(m_1^4h_2-m_2^4h_1)]$$
$$+F_3^2k^2m_2^2[k^3(m_1^2-m_2^2)(5m_1^2+m_2^2)+k^2(-5m_1^4h_2+m_2^4h_1+4m_1^2m_2^2h_1)+6km_1^2m_2^2(m_1^2-m_2^2)]\}, \quad (45)$$

where the subtraction of the iteration term of the potential (44) is performed and $h_{1,2}=\sqrt{k^2+4m^2_{1,2}}$. Moreover, we remove the factor $F_3(0)=m_2\mu_d/Zm_p$ from the form factor F_3 in Eq. (45) as in Eq. (44). Numerical integration in Eq. (45) performed with the use of a known parametrization for the deuteron form factors [21] gives the following result:

$$\Delta E^{\rm HFS}_{2\gamma,{\rm str}}(nS)=\begin{cases}1S:\ -0.9305\ {\rm meV}\\ 2S:\ -0.1163\ {\rm meV}\end{cases}. \quad (46)$$

An expression, Eq. (45), can be used for the evaluation of the nuclear structure and vacuum polarization corrections shown in Fig. 5. A change of the potential in this case as compared with Eq. (45) can be derived after the replacement (10) and insertion of the factor 2. The corresponding contribution to the HFS of S levels is represented by the following form:

$$E^{\rm HFS}_{2\gamma,{\rm VP}}=-E_F\frac{2\alpha^2}{3\pi}\int_1^\infty\rho(\xi)d\xi\int_0^\infty V_{2\gamma}(k)\frac{dk}{k^2+4m_e^2\xi^2}=\begin{cases}1S:\ 0.0152\ {\rm meV}\\ 2S:\ 0.0019\ {\rm meV}\end{cases}. \quad (47)$$

The muon vacuum polarization contribution is also calculated by formula (47) after the replacement

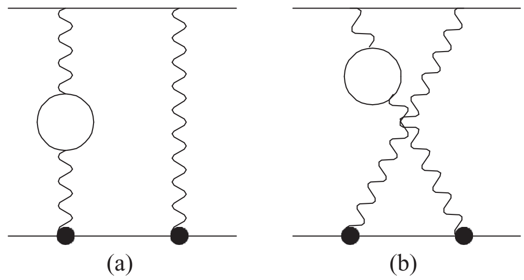


FIG. 5. Two-photon exchange amplitudes accounting for the effects of vacuum polarization and nuclear structure. The wavy lines denote the photons. The bold points denote the deuteron vertex functions.

$m_e \to m_1$:

$$E^{\rm HFS}_{2\gamma,\rm MVP} = \begin{cases} 1S: 0.0015 \text{ meV} \\ 2S: 0.0002 \text{ meV} \end{cases}. \quad (48)$$

To increase the calculation accuracy we consider the hadron vacuum polarization (HVP) contribution. A potential $V_{2\gamma}$ accounting for nuclear structure corrections is applicable for this aim. A replacement in the photon propagator for the HVP correction takes the form

$$\frac{1}{k^2} \to \left(\frac{\alpha}{\pi}\right) \int_{s_{\rm th}}^{\infty} \frac{\rho_{\rm had}(s) ds}{k^2+s} \quad (49)$$

and leads to the following expression for the HVP correction:

$$E^{\rm HFS}_{2\gamma,\rm HVP} = -E_F \frac{2\alpha^2}{\pi} \int_1^{\infty} \rho_{\rm had}(s) ds \int_0^{\infty} V_{2\gamma}(k) \frac{dk}{k^2+s}. \quad (50)$$

The basic contribution to the hadron spectral function $\rho_{\rm had}(s)$ is determined by the pion form factor $F_\pi(s)$ for the energy interval $4m_\pi^2 \div 0.81$ GeV^2:

$$\rho_{\rm had}(s) = \frac{\left(s-4m_\pi^2\right)^{3/2}}{12 s^{5/2}} |F_\pi(s)|^2. \quad (51)$$

The contributions of resonances $J^{\rm PC} = 1^{--}$ of J/ψ and Υ families and other nonresonance energy s intervals are calculated as in our previous works [22]. The Total HVP contribution is presented in Table I.

Sixth-order α corrections are also shown in Fig. 6. To evaluate the contribution of the amplitude in Fig. 6(a) we use an expansion of the deuteron magnetic form factor,

$$G_M(k^2) = \frac{m_2}{Z m_p} \mu_d \left(1 - \frac{1}{6} r_M^2 \mathbf{k}^2\right), \quad (52)$$

which leads to the following potential in momentum space,

$$\Delta V^{\rm HFS}(k) = -\frac{4\pi\alpha\mu_d}{9 m_1 m_p} r_M^2 (\mathbf{s}_1 \mathbf{s}_2) \mathbf{k}^2, \quad (53)$$

and to the following potential in coordinate representation,

$$\Delta V^{\rm HFS}(k) = \frac{4\pi\alpha\mu_d}{9 m_1 m_p} r_M^2 (\mathbf{s}_1 \mathbf{s}_2) \nabla^2 \delta(\mathbf{r}). \quad (54)$$

Averaging Eq. (54) over the Coulomb wave functions, we obtain an analytical expression for a correction to HFS and its numerical values for the levels $1S$ and $2S$:

$$\Delta E^{\rm HFS}_{1\gamma,\rm str} = \frac{2}{3} \mu^2 \alpha^2 r_M^2 \frac{3n^2+1}{n^2} E_F = \begin{cases} 1S: 0.0082 \text{ meV} \\ 2S: 0.0008 \text{ meV} \end{cases}. \quad (55)$$

Another nuclear structure correction of order α^6 to the HFS of muonic deuterium is determined in second-order PT by the amplitude presented in Fig. 6(b). Each of the perturbation potentials in this diagram is proportional to $\delta(\mathbf{r})$ if we use an expansion over small transfer momentum. As a result a contribution of second-order PT is proportional to the divergent expression $\tilde G(0,0)$. To avoid an appearance of $\tilde G(0,0)$ we use the nuclear structure perturbation potential in the following form:

$$\Delta V^C_{{\rm str},1\gamma}(k) = -\frac{Z\alpha}{\mathbf{k}^2} \left[\frac{1}{\left(1+\frac{k^2}{\Lambda^2}\right)^2} - 1\right] = \frac{Z\alpha}{\Lambda^2} \frac{\left(2+\frac{k^2}{\Lambda^2}\right)}{\left(1+\frac{k^2}{\Lambda^2}\right)^2}, \quad (56)$$

$$\Lambda = \frac{\sqrt{12}}{r_d}.$$

A convenience of the dipole parametrization for the deuteron charge form factor in this case instead of a parametrization from Ref. [21] is related to the fact that in the coordinate representation we obtain under such conditions sufficiently simple expressions:

$$\Delta V^C_{{\rm str},1\gamma}(r) = \frac{Z\alpha(2+\Lambda r)}{8\pi r} e^{-\Lambda r}. \quad (57)$$

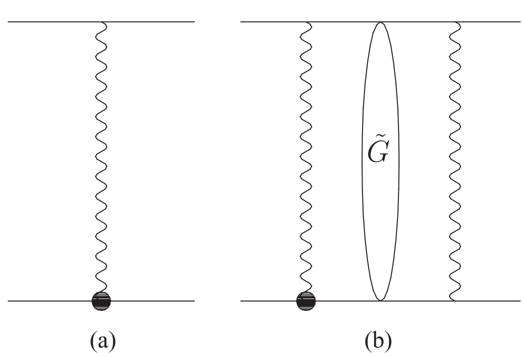


FIG. 6. Nuclear structure effects in one-photon interaction and in second-order perturbation theory. $\tilde G$ is the reduced Coulomb Green's function.

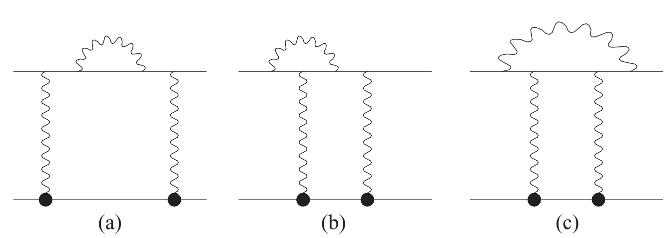


FIG. 7. Direct two-photon exchange amplitudes with radiative corrections to the muon line giving contributions of order $\alpha(Z\alpha)^5$ to the HFS. The wavy lines on the diagrams denote the photons. The bold points on the diagrams denote the vertex operators of the protons or the deuterons.

Using the Green's functions (20) and (21) the analytical integration in second-order PT can be performed. It gives the following result:

$$\Delta E^{\rm HFS}_{\rm str,SOPT}(1S) = E_F(1S)\frac{\mu\alpha}{2\pi\Lambda(1+\frac{2W}{\Lambda})^4}\Bigg\{-2\frac{W}{\Lambda}\left[4\frac{W}{\Lambda}\left(5+3\frac{W}{\Lambda}\right)+13\right] - 16\frac{W}{\Lambda}\left(1+\frac{W}{\Lambda}\right)\left(1+\frac{2W}{\Lambda}\right)\coth^{-1}\left(1+\frac{4W}{\Lambda}\right)-3\Bigg\} = -0.0555 \text{ meV}, \quad (58)$$

$$\Delta E^{\rm HFS}_{\rm str,SOPT}(2S) = -E_F(2S)\frac{\mu\alpha}{8\pi\Lambda(1+\frac{W}{\Lambda})^6}\Bigg(\frac{W}{\Lambda}\left\{\left[\frac{W}{\Lambda}\left(14+3\frac{W}{\Lambda}\right)+31\right]\frac{W^2}{\Lambda^2}+16\right\} + 8\frac{W}{\Lambda}\left(1+\frac{W}{\Lambda}\right)\left[\left(3+\frac{W}{\Lambda}\right)\frac{W^2}{\Lambda^2}+4\right]\coth^{-1}\left(1+\frac{2W}{\Lambda}\right)+6\Bigg) = -0.0069 \text{ meV}. \quad (59)$$

V. RADIATIVE CORRECTIONS TO TWO-PHOTON EXCHANGE DIAGRAMS

The lepton line radiative corrections to two-photon exchange amplitudes are of order $\alpha(Z\alpha)^5$. Such corrections to the HFS of muonium were studied in detail in Ref. [23]. The total integral expression for all radiative corrections in Fig. 7 to the HFS of order $\alpha(Z\alpha)^5$ including recoil effects was constructed in Ref. [24] in the Fried-Yennie gauge [25]. The advantage of the Fried-Yennie gauge in the calculation of corrections in Fig. 7 is that it leads to infrared finite renormalizable integral expressions for the muon self-energy operator, the vertex function, and the lepton tensor describing the diagram with the spanning photon [26]. Using such general expressions an analytical calculation of corrections $\alpha(Z\alpha)^5$ to the HFS in the pointlike nucleus approximation can be performed. If the effects of nuclear structure are taken into account then these expressions allow us to obtain numerical values of the diagrams in Figs. 7(a)–7(c) separately. The muon-deuteron scattering amplitude corresponding to direct two-photon exchange diagrams with radiative insertions in the muon line can be presented in the following form:

$$\mathcal{M} = \frac{-i(Z\alpha)^2}{\pi^2}\int d^4k[\bar u(q_1)L_{\mu\nu}u(p_1)]D_{\mu\omega}(k)D_{\nu\lambda}(k)[\epsilon^*_\rho(q_2)\Gamma_{\omega,\rho\beta}(q_2,p_2+k)\mathcal{D}_{\beta\tau}(p_2+k)\Gamma_{\lambda,\tau\alpha}(p_2+k,p_2)\epsilon_\alpha(p_2)], \quad (60)$$

where $\epsilon_\rho(q)$ is the wave function of the free deuteron with spin 1, $p_{1,2}$ and $q_{1,2}$ are the four-momenta of the muon and the deuteron in the initial and final states: $p_{1,2}\approx q_{1,2}$. The vertex operator describing the photon-deuteron interaction is determined by three form factors as follows:

$$\Gamma_{\omega,\rho\sigma}(q_2,p_2+k) = \frac{(2p_2+k)_\omega}{2m_2}g_{\rho\sigma}F_1(k) - \frac{(2p_2+k)_\omega}{2m_2}\frac{k_\rho k_\sigma}{2m_2^2}F_2(k) - (g_{\rho\gamma}g_{\sigma\omega}-g_{\rho\omega}g_{\sigma\gamma})\frac{k_\gamma}{2m_2}F_3(k). \quad (61)$$

The deuteron propagator and the photon propagator in the Coulomb gauge are equal to

$$\mathcal{D}_{\alpha\beta}(p) = \frac{-g_{\alpha\beta}+\frac{p_\alpha p_\beta}{m_2^2}}{(p^2-m_2^2+i0)},\quad D_{\lambda\sigma}(k) = \frac{1}{k^2}\left[g_{\lambda\sigma}+\frac{k_\lambda k_\sigma - k_0k_\lambda g_{\sigma0}-k_0k_\sigma g_{\lambda0}}{\mathbf{k}^2}\right]. \quad (62)$$

The lepton tensor $L_{\mu\nu}$ has a definite form for three amplitudes in Fig. 7. Using the package FEYNCALC [27] we perform independent construction of lepton tensors corresponding to the muon self-energy, vertex corrections, and the diagram with the spanning photon:

$$L^{\rm se}_{\mu\nu} = -\frac{3\alpha}{4\pi}\gamma_\mu\left(\hat p_1-\hat k\right)\gamma_\nu\int_0^1\frac{(1-x)dx}{(1-x)m_1^2+x\mathbf{k}^2}, \quad (63)$$

$$L^{\rm vertex}_{\mu\nu} = 2\frac{\alpha}{4\pi}\int_0^1 dz\int_0^1 dx\gamma_\mu\frac{\hat p_1-\hat k+m_1}{(p_1-k)^2-m_1^2+i0}\left[F^{(1)}_\nu+\frac{F^{(2)}_\nu}{\Delta}+\frac{F^{(3)}_\nu}{\Delta^2}\right], \quad (64)$$

$$F^{(1)}_\nu = -6x\gamma_\nu\ln\frac{m_1^2x+\mathbf{k}^2z(1-xz)}{m_1^2x},\quad F^{(3)}_\nu = 2x^3(1-x)\hat Q\left(\hat p_1-\hat k+m_1\right)\gamma_\nu(\hat p_1+m_1)\hat Q, \quad (65)$$

$$F^{(2)}_\nu = -x^3\left(2\gamma_\nu Q^2-2\hat Q\gamma_\nu\hat Q\right) - x^2[\gamma_\alpha\hat Q\gamma_\nu(\hat p_1+m_1)\gamma_\alpha+\gamma_\alpha(\hat p_1-\hat k+m_1)\gamma_\nu\hat Q\gamma_\alpha+2\gamma_\nu(\hat p_1+m_1)\hat Q + 2\hat Q(\hat p_1-\hat k+m_1)\gamma_\nu] - x(2-x)\gamma_\alpha(\hat p_1-\hat k+m_1)\gamma_\nu(\hat p_1+m_1)\gamma_\alpha, \quad (66)$$

$$Q = -p_1+kz,\quad \Delta = x^2m_1^2 - xz(1-xz)k^2+2kp_1xz(1-x),$$

$$L^{\rm jellyfish}_{\mu\nu} = \frac{\alpha}{4\pi}\int_0^1 dz\int_0^1 dx\left(\frac{F^{(1)}_{\mu\nu}}{\Delta}+\frac{F^{(2)}_{\mu\nu}}{\Delta^2}+\frac{F^{(3)}_{\mu\nu}}{\Delta^3}\right). \quad (67)$$

An explicit form of the tensors $F^{(i)}_{\mu\nu}$ is presented in Ref. [28]. For a construction of hyperfine potential respective to the amplitude (60) we use the projection operators (41). The insertion of Eq. (41) into Eq. (60) allows us to jump to the trace calculation and a contraction over the Lorentz indices by means of the system FORM [20]. Hence, a general structure of potentials contributing to the energy shifts for states with the angular momenta 1/2 and 3/2 is the following:

$$N_{1/2} = \frac{1}{6}\mathrm{Tr}\left\{\sum_\sigma \Psi^\sigma(P)\bar{\Psi}^\sigma(P)(\gamma_\rho - v_{1,\rho})\gamma_5(1+\hat{v}_1)L_{\mu\nu}(1+\hat{v}_1)\gamma_5(\gamma_\alpha - v_{1,\alpha})\right\} \times \Gamma_{\omega,\rho\beta}(q_2,p_2+k)\mathcal{D}_{\beta\tau}(p_2+k)\Gamma_{\lambda,\tau\alpha}(p_2+k,p_2)D_{\mu\omega}(k)D_{\nu\lambda}(k), \quad (68)$$

$$N_{3/2} = \frac{1}{4}\mathrm{Tr}\left\{\sum_\sigma \Psi^\sigma_\alpha(P)\bar{\Psi}^\sigma_\rho(P)(1+\hat{v}_1)L_{\mu\nu}(1+\hat{v}_1)\right\}\Gamma_{\omega,\rho\beta}(q_2,p_2+k)\mathcal{D}_{\beta\tau}(p_2+k)\Gamma_{\lambda,\tau\alpha}(p_2+k,p_2)D_{\mu\omega}(k)D_{\nu\lambda}(k). \quad (69)$$

Neglecting the recoil effects in the denominator of the deuteron propagator we obtain $1/[(p_2+k)^2 - m_2^2 + i0] \approx 1/(k^2+2kp_2+i0) \approx 1/(2k_0m_2+i0)$. The contribution of crossed two-photon exchange diagrams is also determined by Eqs. (60)–(67) with the replacement $k \to -k$ in the deuteron propagator. Then a summary contribution is proportional to $\delta(k_0)$:

$$\frac{1}{2m_2k_0+i0} + \frac{1}{-2m_2k_0+i0} = -\frac{i\pi}{m_2}\delta(k_0). \quad (70)$$

As a result three types of corrections of order $\alpha(Z\alpha)^5$ to the HFS of muonic deuterium are expressed in integral form over the loop momentum $\mathbf{k}$ and the Feynman parameters:

$$\Delta E^{\rm HFS}_{\rm se} = E_F 6\frac{\alpha(Z\alpha)}{\pi^2}\int_0^1 x dx \int_0^\infty \frac{F_1(k^2)F_3(k^2)dk}{x+(1-x)k^2}, \quad (71)$$

$$\Delta E^{\rm HFS}_{\text{vertex-1}} = -E_F 24\frac{\alpha(Z\alpha)}{\pi^2}\int_0^1 dz\int_0^1 x dx\int_0^\infty \frac{F_1(k^2)F_3(k^2)\ln\left[\frac{x+k^2z(1-xz)}{x}\right]dk}{k^2}, \quad (72)$$

$$\Delta E^{\rm HFS}_{\text{vertex-2}} = E_F 8\frac{\alpha(Z\alpha)}{\pi^2}\int_0^1 dz\int_0^1 dx\int_0^\infty \frac{dk}{k^2}\left\{\frac{F_1(k^2)F_3(k^2)}{[x+k^2z(1-xz)]^2}[-2xz^2(1-xz)k^4 + zk^2[3x^3z - x^2(9z+1)+x(4z+7)-4]+x^2(5-x)] - \frac{1}{2}\right\}, \quad (73)$$

$$\Delta E^{\rm HFS}_{\rm jellyfish} = E_F 4\frac{\alpha(Z\alpha)}{\pi^2}\int_0^1(1-z)dz\int_0^1(1-x)dx\int_0^\infty\frac{F_1(k^2)F_3(k^2)dk}{[x+(1-x)k^2]^3} \times [6x+6x^2-6x^2z+2x^3-12x^3z-12x^4z+k^2(-6z+18xz+4xz^2+7x^2z-30x^2z^2 - 2x^2z^3-36x^3z^2+12x^3z^3+24x^4z^3)+k^4(9xz^2-31x^2z^3+34x^3z^4-12x^4z^5)]. \quad (74)$$

The contribution of the form factor $F_2(k^2)$ in Eqs. (71)–(74) is omitted because the terms $F_2(k^2)F_3(k^2)$ are suppressed by powers of the mass m_2. The term 1/2 in curly brackets in Eq. (73) is related to the subtraction term of the quasipotential. All corrections (71), (72), (73), and (74) are expressed through the convergent integrals. It is necessary to point out that expressions for the vertex function and the lepton tensor with the spanning photon were obtained in Ref. [26] in a slightly different form. Nevertheless, in the case of the pointlike nucleus they lead to contributions to the HFS of S states which coincide with Eqs. (71)–(74) [29]. In numerical evaluation of finite size corrections (71)–(74) we use the deuteron form factor parametrization from Ref. [21]. Numerical results are presented in Table I.

VI. NUMERICAL RESULTS AND DISCUSSION

In this work we calculate QED corrections, nuclear structure, and recoil corrections of orders α^5 and α^6 to the HFS of $1S$ and $2S$ energy levels in muonic deuterium. In contrast to earlier performed investigations of the energy spectra of light muonic atoms in Ref. [3] we use a three-dimensional quasipotential approach for the description of the muon-deuteron bound state. All considered corrections due to effects of vacuum polarization and deuteron structure are presented in integral form and calculated numerically. Numerical values of studied corrections are exhibited in Table I. We present in Table I relevant references to equations which allow us to analyze again the sources of corresponding corrections. In line 5 of Table I we present a sum of corrections of order α^6 which includes two-loop electron vacuum polarization corrections in first and second orders of perturbation theory and muon one-loop vacuum polarization corrections in first and second orders of perturbation theory.

As pointed out above the hyperfine structure of muonic atoms was investigated in Refs. [3,4]. In these papers the transition frequencies between the energy levels $2P$ and $2S$ were obtained in the case of muonic hydrogen, muonic deuterium, and ions of muonic helium. The only detailed calculation of $2S$-state HFS in muonic deuterium is presented in Ref. [3]. The splitting formula obtained in this paper,

$$\Delta E_{2s} = \frac{3}{2}\beta_D(1 + a_\mu)(1 + \epsilon_{VP} + \epsilon_{\rm vertex} + \epsilon_{\rm Breit} + \epsilon_{\rm Zemach}) = 6.0584(7)\ {\rm meV}, \tag{75}$$

contains basic corrections to the Fermi energy: the vacuum polarization, the relativistic correction, the vertex correction, and the Zemach contribution. Note that the Zemach correction [−0.1177(7) meV] for the $2S$ state in muonic deuterium from Ref. [3] is slightly different from our value (−0.1163 meV) that may be related with recoil effects. One-loop electron vacuum polarization corrections in first-order $\epsilon_{\rm VP1} = 0.00218$ and in second-order PT $\epsilon_{\rm VP2} = 0.00337$ for the $2S$ state from Ref. [3] coincide exactly with our results expressed by Eqs. (12) and (23). As it follows from our Table I, the total value of the $2S$-state HFS, 6.0683 meV, without an account of the deuteron polarizability contribution obtained in Ref. [5] is in good agreement with the result 6.0584 meV obtained in Ref. [3]. A small difference in the results is conditioned first of all by nuclear recoil, structure, and polarizability effects in two-photon exchange diagrams which we calculate using modern experimental data on deuteron electromagnetic form factors and by nuclear structure corrections of order α^6. We include in Table I the numerical value of the deuteron polarizability contribution to HFS in muonic deuterium evaluated by means of analytical expressions obtained in Ref. [5] for electronic deuterium in the zero range approximation. This is the main part of the polarizability correction. The estimate of the internal deuteron polarizability correction is made in Table I on the basis of results for muonic hydrogen. This contribution is now under consideration as in our work [19].

Assuming that the deuteron electromagnetic form factor parametrizations were obtained with an uncertainty near 0.5% at small values of photon momentum squared Q^2 we obtain that theoretical error in the calculation of the basic nuclear structure contribution of order $(Z\alpha)^5$ which is determined by a product of two electromagnetic form factors cannot be less than 1% or ±0.0010 meV for the $2S$ state. Another source of the error is related to recoil effects of order $\alpha(Z\alpha)^5 m_1/m_2$ in amplitudes in Fig. 7, which can amount to the value 0.00002 meV. The error in the determination of the internal polarizability correction is estimated approximately on muonic hydrogen at 0.0025 meV ($2S$ state) (near 25%). The estimate of an uncertainty in the deuteron polarizability correction is given on the basis of results from Ref. [5]. It is approximately equal to 0.0042 meV (2%). The weak interaction contribution is equal to zero in the nonrelativistic approximation as was demonstrated in Refs. [8,30]. Our total theoretical error is estimated at 0.0050 meV in the case of the $2S$ state. To obtain this estimate we add the abovementioned uncertainties in quadrature. Main theoretical uncertainties connected with the structure and polarizability corrections are shown explicitly in Table I. Other uncertainties are omitted because they are negligibly small. The HFS interval Δ_{12} does not contain uncertainties with nuclear structure and polarizability. So, the value $\Delta_{12} = -0.0379$ meV obtained in this work can be used for an additional check of QED in the case of muonic deuterium with a precision of 0.001 meV. To construct the quasipotential corresponding to amplitudes in Fig. 4 we develop the method of projection operators on the bound states with definite spins. It allows us to employ different systems of analytical calculations [20,27]. In this approach more complicated corrections, for example, radiative recoil corrections to the HFS of order $\alpha(Z\alpha)^5 m_1/m_2$ can be evaluated if an increase of the accuracy is needed. The results from Table I should be taken into account for a comparison with experimental data [1].

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[1] R. Pohl, A. Antognini, F. Nez *et al.*, Nature (London) **466**, 213 (2010); A. Antognini, F. Biraben, J. M. R. Cardoso *et al.*, Can. J. Phys. **89**, 47 (2011); A. Antognini *et al.*, Science **339**, 417 (2013); R. Pohl, R. Gilman, G. A. Miller, and K. Pachucki, Annu. Rev. Nucl. Part. Sci. **63**, 175 (2013).

[2] T. Nebel, F. D. Amaro, A. Antognini *et al.*, Hyperfine Interact. **212**, 195 (2012); A. Antognini, F. Kottmann, F. Biraben *et al.*, Ann. Phys. **331**, 127 (2013).

[3] E. Borie and G. A. Rinker, Rev. Mod. Phys. **54**, 67 (1982); E. Borie, Z. Phys. A **278**, 127 (1976); Phys. Rev. A **72**, 052511 (2005); **71**, 032508 (2005); Ann. Phys. (NY) **327**, 733 (2012).

[4] K. Pachucki, Phys. Rev. A **54**, 1994 (1996); **60**, 3593 (1999); Phys. Rev. Lett. **106**, 193007 (2011); A. Veitia and K. Pachucki, Phys. Rev. A **69**, 042501 (2004).

[5] I. B. Khriplovich and A. I. Milstein, J. Exp. Theor. Phys. **98**, 181 (2004); A. I. Milstein, I. B. Khriplovich, and S. S. Petrosyan, *ibid.* **82**, 616 (1996).

[6] J. L. Friar, J. Martorell, and D. W. L. Sprung, Phys. Rev. A **56**, 4579 (1997); U. D. Jentschura, Ann. Phys. **326**, 500 (2011); Phys. Rev. A **84**, 012505 (2011); C. E. Carlson, V. Nazaryan, and K. Griffioen, *ibid.* **83**, 042509 (2011); E. Yu. Korzinin, V. G. Ivanov, and S. G. Karshenboim, Phys. Rev. D **88**, 125019 (2013); J. L. Friar, Phys. Rev. C **88**, 034003 (2013); C. E. Carlson, M. Gorchtein, and M. Vanderhaeghen, Phys. Rev. A **89**, 022504 (2014); P. Indelicato, *ibid.* **87**, 022501 (2013); S. G. Karshenboim, V. G. Ivanov, E. Yu. Korzinin, and V. A. Shelyuto, *ibid.* **81**, 060501 (2010).

[7] A. P. Martynenko, Phys. Rev. A **76**, 012505 (2007); J. Exp. Theor. Phys. **106**, 690 (2008); Phys. Atom. Nucl. **69**, 1309 (2006); **71**, 125 (2008); A. A. Krutov and A. P. Martynenko, Phys. Rev. A **84**, 052514 (2011).

[8] M. I. Eides, H. Grotch, and V. A. Shelyuto, Phys. Rep. **342**, 63 (2001); *Theory of Light Hydrogenic Bound States*, Springer Tracts in Modern Physics, Vol. 222, edited by A. Fujimori, J. Kühn, Th. Müller, F. Steiner, J. Trümper, C. Varma, and P. Wülfle (Springer, Berlin, Heidelbeg, New York, 2007).

[9] G. A. Miller, A. W. Thomas, J. D. Carroll, and J. Rafelski, Phys. Rev. A **84**, 020101(R) (2011); A. De Rujula, Phys. Lett. B **697**, 26 (2011); **693**, 555 (2010); I. C. Cloët and G. A. Miller, Phys. Rev. C **83**, 012201 (2011); A. Walker-Loud, C. E. Carlson, and G. A. Miller, Phys. Rev. Lett. **108**, 232301 (2012); R. J. Hill and G. Paz, *ibid.* **107**, 160402 (2011); M. O. Distler, J. C. Bernauer, and T. Walcher, Phys. Lett. B **696**, 343 (2011).

[10] P. J. Mohr and B. N. Taylor, Rev. Mod. Phys. **84**, 1527 (2012).

[11] R. N. Faustov and A. P. Martynenko, J. Exp. Theor. Phys. **88**, 672 (1999); **98**, 39 (2004); Phys. Lett. B **541**, 135 (2002); A. P. Martynenko, Phys. Rev. A **71**, 022506 (2005); J. Exp. Theor. Phys. **101**, 1021 (2005).

[12] V. B. Berestetskii, E. M. Lifshits, and L. P. Pitaevskii, Quantum Electrodynamics, M. Nauka, (1980).

[13] K. Pachucki and S. G. Karshenboim, J. Phys. B **28**, L221 (1995).

[14] I. B. Khriplovich, A. I. Milstein, and R. A. Sen'kov, Phys. Lett. A **221**, 370 (1996).

[15] G. Breit, Phys. Rev. **35**, 1447 (1930).

[16] G. Källen and A. Sabry, Mat.-Fis. Medd.-K. Dan. Vidensk. Selsk. **29**, 17 (1955).

[17] H. F. Hameka, J. Chem. Phys. **47**, 2728 (1967).

[18] S. G. Karshenboim, E. Yu. Korzinin, and V. G. Ivanov, JETP Lett. **88**, 641 (2008).

[19] R. N. Faustov and A. P. Martynenko, Phys. Rev. A **67**, 052506 (2003); Phys. At. Nucl. **67**, 457 (2004) [Yad. Fiz. **67**, 477 (2004)]; E. V. Cherednikova, R. N. Faustov, and A. P. Martynenko, Nucl. Phys. A **703**, 365 (2002).

[20] J. A. M. Vermaseren, arXiv:math-ph/0010025.

[21] D. Abbott, A. Ahmidouch, and H. Anklin *et al.*, Eur. Phys. J. A **7**, 421 (2000).

[22] R. N. Faustov and A. P. Martynenko, Phys. Atom. Nucl. **61**, 471 (1998); Eur. Phys. J. direct C **1**, 6 (1999); R. N. Faustov, A. Karimkhodzhaev, and A. P. Martynenko, Phys. Rev. A **59**, 2498 (1999); Phys. At. Nucl. **62**, 2103 (1999).

[23] S. G. Karshenboim, V. A. Shelyuto, and M. I. Eides, Sov. J. Nucl. Phys. **47**, 287 (1988); M. I. Eides, S. G. Karshenboim, and V. A. Shelyuto, Zh. Eksp. Teor. Fiz. **94**, 42 (1988) [Sov. Phys. JETP **67**, 671 (1988)]; Zh. Eksp. Teor. Fiz. **92**, 1188 (1987) [Sov. Phys. JETP **65**, 664 (1988)]; Sov. J. Nucl. Phys. **44**, 723 (1986); Ann. Phys. (NY) **205**, 231 (1991).

[24] M. I. Eides, S. G. Karshenboim, and V. A. Shelyuto, Phys. Lett. B **202**, 572 (1988); V. Yu. Brook, M. I. Eides, S. G. Karshenboim, and V. A. Shelyuto, *ibid.* **216**, 401 (1989).

[25] H. M. Fried and D. R. Yennie, Phys. Rev. **112**, 1391 (1958); A. A. Abrikosov, Zh. Eksp. Teor. Fiz. **30**, 96 (1956) [Sov. Phys. JETP **3**, 71 (1956)]; Y. Tomozawa, Ann. Phys. **128**, 491 (1980); J. R. Sapirstein, E. A. Terray, and D. R. Yennie, Phys. Rev. D **29**, 2290 (1984); G. S. Adkins, *ibid.* **39**, 3798 (1989).

[26] M. I. Eides, H. Grotch, and V. A. Shelyuto, Phys. Rev. A **63**, 052509 (2001); M. I. Eides and V. A. Shelyuto, Eur. Phys. J. C **21**, 489 (2001).

[27] R. Mertig, M. Böhm, and A. Denner, Comput. Phys. Commun. **64**, 345 (1991).

[28] R. N. Faustov, A. P. Martynenko, G. A. Martynenko, and V. V. Sorokin, Phys. Lett. B **733**, 354 (2014).

[29] N. M. Kroll and F. Pollock, Phys. Rev. **86**, 876 (1952); **84**, 594 (1951).

[30] M. I. Eides, Phys. Rev. A **53**, 2953 (1996); **85**, 034503 (2012).